

A convergence property for quasisuperminimizers on metric measure spaces

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§1. Preliminaries

We assume that $X = (X, d, \mu)$ be a complete metric space with a metric d and a positive Borel regular measure μ which is finite on a bounded set.

Let u be a real valued function on X . A nonnegative Borel measurable function g on X is said to be an upper gradient of u if for every rectifiable path γ joining x and y in X ,

$$(1.1) \quad |u(x) - u(y)| \leq \int_{\gamma} g \, ds.$$

The p -modulus of a family Γ of paths in X is defined by

$$\inf_{\rho} \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel measurable functions ρ such that for all rectifiable paths γ in Γ

$$\int_{\gamma} \rho \, ds \geq 1.$$

We say that a property holds for p -almost every path if the family of paths on which the property does not hold is of zero the p -modulus. If (1.1) holds for p -almost every path γ , then we say that g is a p -weak upper gradient of u .

Let $1 < p < \infty$ and $L^p(X)$ be the space of functions f on X such that $|f|^p$ is integrable with respect to the measure μ . A function u belongs the space $\tilde{N}^{1,p}(X)$ if $u \in L^p(X)$ and u has a p -weak upper gradient g such that $g \in L^p(X)$. For a function $u \in \tilde{N}^{1,p}(X)$, we define

$$\|u\|_{\tilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all p -weak upper gradients of u . For functions $u, v \in \tilde{N}^{1,p}(X)$, we define the relation $u \sim v$ if and only if $\|u - v\|_{\tilde{N}^{1,p}(X)} = 0$. We define the Newtonian space $N^{1,p}(X) = \tilde{N}^{1,p}(X) / \sim$ equipped with the norm $\|\cdot\|_{N^{1,p}(X)}$.

Following properties of the Newtonian spaces are known (see [S1]):

- (i) $N^{1,p}(X)$ is a Banach space.
- (ii) Lipschitz functions are dense in $N^{1,p}(X)$.
- (iii) Every $u \in N^{1,p}(X)$ has a unique minimal p -weak upper gradient $g_u \in L^p(X)$ in the sense that for every p -weak upper gradient g of u , $g_u \leq g$ μ -a.e in X .

For a set E in X , the p -capacity of E is defined by

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E , and the Newtonian space with zero boundary values is defined by

$$N_0^{1,p}(E) = \{u \in N^{1,p}(X) \mid C_p(\{x \in X \setminus E \mid u(x) \neq 0\}) = 0\}.$$

Let Ω be an open subset in X . If $u \in N^{1,p}(E)$ for every measurable set $E \Subset \Omega$, we write $u \in N_{\text{loc}}^{1,p}(\Omega)$. For more various properties of Newtonian spaces, see [S1].

In addition, we assume following two conditions:

- (I) The measure μ is doubling, that is, there exists a constant $C > 0$ such that

$$0 < \mu(2B) \leq C \mu(B)$$

whenever $B = B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$ is a ball in X and $\lambda B = B(x_0, \lambda r)$ for $\lambda \in \mathbf{R}$.

- (II) X supports a weak $(1, p)$ -Poincaré inequality, that is, there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all measurable functions f on X and all upper gradients g of f ,

$$\frac{1}{\mu(B)} \int_B |f - f_B| d\mu \leq C(\text{dima } B) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p d\mu \right)^{1/p},$$

where $f_B = \frac{1}{\mu(B)} \int_B f d\mu$.

In [B] there are various examples of spaces equipped with a doubling measure and supporting Poincaré inequality.

§2. Quasisuperminimizers

Let a constant $Q \geq 1$. A function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is said to be a (Q, p) -quasiminimizer in Ω if for all open $\Omega' \Subset \Omega$ and all $\varphi \in N_0^{1,p}(\Omega')$ we have

$$(2.1) \quad \int_{\Omega'} g_u^p d\mu \leq Q \int_{\Omega'} g_{u+\varphi}^p d\mu.$$

A function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is said to be a (Q, p) -quasisuperminimizer in Ω if (2.1) holds for all nonnegative $\varphi \in N_0^{1,p}(\Omega')$. A function u is said to be a (Q, p) -quasisubminimizer if $-u$ is a (Q, p) -quasisuperminimizer. A function u is a (Q, p) -quasiminimizer if and only if u is a (Q, p) -quasisuperminimizer and a (Q, p) -quasisubminimizer.

A (Q, p) -quasiminimizer (respectively, (Q, p) -quasisuperminimizer) has a continuous (respectively, lower semicontinuous) representative (see [KM1; Theorem 5.1], [KM2; Lemma 5.3] and [KS; Proposition 3.3 and Theorem 5.2]). If u is a $(1, p)$ -quasiminimizer (respectively, $(1, p)$ -quasisuperminimizer), we say that u is a minimizer (respectively, superminimizer). A continuous minimizer is said to be p -harmonic. Potential theory for p -harmonic functions on metric measure spaces has been studied in [C], [S2], [KM1], [BBS1] and [BBS2] etc.

If u is a (Q, p) -quasisuperminimizer and $\lambda \geq 0$, τ are constants, then $\lambda u + \tau$ is a (Q, p) -quasisuperminimizer.

§3. A convergence property for quasisuperminimizers

In [KM2; Theorem 6.1] the following convergence result for quasisuperminimizers was established:

Proposition. *Let Ω be an open set in X and let $\{u_n\}$ be a nondecreasing sequence of (Q, p) -quasisuperminimizers in Ω and $u = \lim_{n \rightarrow \infty} u_n$. If either u is locally bounded above or $u \in N_{\text{loc}}^{1,p}(\Omega)$, then u is a (Q, p) -quasisuperminimizer in Ω .*

We can relax the condition in the above proposition as follows.

Theorem. *Let Ω be an open set in X and let $\{u_n\}$ be a nondecreasing sequence of (Q, p) -quasisuperminimizers in Ω . If there is a function $f \in$*

$N_{\text{loc}}^{1,p}(\Omega)$ such that $u_n \leq f$ μ -a.e. for all n , then $u = \lim_{n \rightarrow \infty} u_n$ is a (Q, p) -quaisuperminimizer in Ω .

Let Ω be an open subset of X . A function $u : \Omega \rightarrow \mathbf{R} \cup \{\infty\}$ is said to be (Q, p) -quaisuperharmonic in Ω in the sense of [KM2] if

- (i) u is lower semicontinuous,
- (ii) $u \not\equiv \infty$ in Ω , and
- (ii) there exist an exhaustion $\{\Omega_n\}$ of Ω and a nondecreasing sequence $\{u_n\}$ of (Q, p) -quaisuperminimizers in Ω_n such that $u = \lim_{n \rightarrow \infty} u_n^*$, where $u_n^*(x) = \text{ess} \liminf_{y \rightarrow x} u_n(y)$.

If u is a (Q, p) -quaisuperminimizers, then u has a (Q, p) -quaisuperharmonic representative (see [KM2 ; Proposition 7.2]).

From the above theorem the next corollary follows immediately.

Corollary. *Let Ω be an open set in X and let u be a (Q, p) -quaisuperharmonic function in the sense of [KM2] in Ω . If there is a function $f \in N_{\text{loc}}^{1,p}(\Omega)$ such that $u \leq f$ μ -a.e., then u is a (Q, p) -quaisuperminimizers in Ω .*

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