

0. INTRODUCTION.

Let $\pi: X \rightarrow Y$ be a morphism of locally noetherian schemes, finite, separable and surjective on all the components of X . Assume X is reduced, Y is normal and let $\mathcal{D}_{X/Y}$ (resp. Y_D) be the discriminant sheaf (resp. scheme) of π (see section 1).

The problem we are interested in is to study the singularities of X in terms of $\mathcal{D}_{X/Y}$ and/or Y_D .

In section 1 we give the definitions of $\mathcal{D}_{X/Y}$ and Y_D , in sections 2 and 3 we study respectively the number of isolated singularities of a projective surface, and the singularities in codimension 1 of a scheme, and finally in section 4 we give two applications.

1. THE DISCRIMINANT SHEAF (SCHEME) OF π .

Assume that $\deg \pi = n$ and that X and Y are integral; then for every affine open subset $U = \text{spec } A_U$ of Y put $\Gamma(U, \mathcal{D}_{X/Y}) =$

$= D_{B'_U / A_U}$, where $\text{spec } B'_U = \pi^{-1}(U)$ and $D_{B'_U / A_U}$ denotes the discriminant of B'_U over A_U .

$\mathcal{D}_{X/Y}$ is a coherent sheaf of $\mathcal{O}_Y$ -ideals (see [5], 3.1), and it is called the discriminant sheaf of π : the corresponding closed subscheme Y_D of Y is called the discriminant scheme of π .

2. ISOLATED SINGULARITIES.

Let X be an irreducible surface of order n in the complex projective 3-space (more generally: over an algebraically closed field of characteristic zero).

Assume that X has only d conical double points as singularities, and look for an upper-bound for d .

In 1906-1907 Basset (see [1],[2]) proved the following two limitations for d :

$$\text{I) } d \leq (2/3)n(n-1)(n-2)$$

$$\text{II) } d \leq (1/2)[n(n-1)^2 - 5 - \sqrt{n(n-1)(3n-14)+25}] \quad (n \geq 5).$$

The technique he used was the following: consider the projection $\pi: X \rightarrow Y$ from a generic point P of $\mathbb{P}^3$ over a projective plane Y , and look at the singularities of the discriminant scheme Y_D of π . Taking for granted that Y_D is an irreducible curve with only plückerian singularities, Basset deduced I) and II) by applying the Plücker's formulas to the characters of Y_D . Basset's proof was not correct, as it is not known whether Y_D has only plückerian singularities, which seems rather unlikely. Nevertheless the limitations I) and II) given by Basset are correct and actually they are the best upper-bound so far obtained for d .

They were recently proved by Stagnaro in [14].

Considering the projection $\pi: X \rightarrow Y$ as before, Stagnaro proved that:

- i) Y_D is an irreducible curve, of order $n(n-1)$ and class $n(n-1)^2 - 2d$,
- ii) the non-linear branches of Y_D have order v and class 1, and are centered at the points $A' = \pi(A)$, where $i(A; \mathcal{C}'_P \cap \mathcal{C}''_P \cap X) = v - 1 > 0$. ($\mathcal{C}'_P$ and $\mathcal{C}''_P$ are respectively the first and the second polar of P with respect to X).

Then he proved I) and II) by applying the generalized Plücker's formulas to the characters of Y_D .

For the sake of completeness we list now the upper-bounds for d given by I) and II), and the results so far obtained in order to construct algebraic projective surfaces of order n having the maximum number of conical double points.

$n =$	3	4	5	6	7	8	..
I) $d \leq$	4	16	40	80	140	224	..
II) $d \leq$	-	-	34	66	114	181	..

(For $n \geq 5$ II) gives better limitations than I)).

The best examples of algebraic projective surfaces of order n having a high number $d(n)$ of conical double points are the following:

$$d(3)=4 \quad ([4]) \quad ; d(4)=16 \quad ([11]) \quad ; d(5)=31^* \quad ([15]) \quad ;$$

$$d(6)=64 \quad ([3], [13]) \quad ; d(7)=90 \quad ([13]) \quad ; d(8)=160 \quad ([9], [10]) \quad ;$$

for $n \geq 9$ see [8], [10].

* I have recently been told that in a preprint by Beauville it is proved that for $n=5$ d cannot exceed 31.

3. SINGULARITIES IN CODIMENSION 1.

Let $\pi: X \rightarrow Y$ be a morphism as in section 1. Let $y \in Y$ be a point of codimension 1, and put:

$$A = \mathcal{O}_{Y,Y}, k \text{ its residue field}, B' = \mathcal{O}_{X,\pi^{-1}(y)},$$

$B = \bar{B}'$, $f = \ell_A(B/\text{rad } B)$, $g = \ell_A(B'/\text{rad } B')$, $D_{B'/A}$ the discriminant of B' over A (remark that $D_{B'/A} = (\mathcal{D}_{X/Y})_y$), v the valuation associated with A .

Assume that $k(m)/k$ is separable for all $m \in \text{Max } B$.

We want to study the singularities of B' , that is the singularities of X in codimension 1; in particular we want to study the normality and the seminormality of B' , which is equivalent to study the normality and the seminormality of the whole X , if we assume X to be S_2 .

THEOREM 1. (Characterization of normality). ([7], 1.3).

$$i) \ v(D_{B'/A}) \geq n-g.$$

$$ii) \ v(D_{B'/A}) = n-g \text{ iff } B' \text{ is normal and tamely ramified over } A.$$

The proof of this theorem relies on the following facts:

a) B is tamely ramified over A iff the different $\delta_{B/A}$ of B over A is equal $\prod_i m_i^{e_i-1}$ where $m_i \in \text{Max } B$ and e_i is its ramification index for all i (see [12], prop. 13, p. 67);

$$b) \ D_{B/A} = N(\delta_{B/A}), N \text{ denoting the norm (see [12], prop. 6, p. 60);}$$

$$c) \ v(D_{B'/A}) = 2\ell_A(B/B') + v(D_{B/A}) \quad ([7], 1.1).$$

THEOREM 2. (A sufficient condition for normality). ([7], 1.8).

If B' is normal and tamely ramified over A , then $v(D_{B'/A}) \leq n-1$.

The converse holds if either:

- i) $n=2$, or
- ii) B' is local, or
- iii) there exists a finite group G of automorphisms of B' such that $B'^G = A$.

THEOREM 3. (Characterization of seminormality). ([7], 2.3).

- i) $v(D_{B'/A}) \geq n+f-2g$.
- ii) $v(D_{B'/A}) = n+f-2g$ iff B' is seminormal and B is tamely ramified over A .

The proof of this theorem relies on the following facts:

- a) B' is seminormal iff $\ell_A(B/B') = f-g$ ([7], 2.1) ;
- b) $v(D_{B'/A}) = 2\ell_A(B/B') + v(D_{B/A})$;
- c) theorem 1.

THEOREM 4. (A sufficient condition for seminormality). ([7], 2.8 and 3.1).

Assume that B is tamely ramified over A . If B' is seminormal (resp. seminormal and Gorenstein), then $v(D_{B'/A}) \leq n+f-1$ (resp. $v(D_{B'/A}) \leq n$).

The converse holds if either:

- i) $n=2$, or
- ii) B' is local, or
- iii) there exists a finite group G of automorphisms of B' such that $B'^G = A$.

THEOREM 5. (The monogenic case). ([6], and [7] section 3).

Suppose $B' = A[x]$ and let $x^n - a$ ($a \in A$, $n \geq 3$) be the characteristic polynomial of x : assume that either $\text{char } k = 0$ or $\text{char } k > n$.

Then the following are equivalent:

- i) B' is seminormal.
- ii) B' is normal.
- iii) $v(D_{B'/A}) \leq n$.
- iv) $v(a) \leq 1$.

4. TWO APPLICATIONS.

(In the following examples, for the sake of simplicity, we shall frequently denote by the same symbol a surface and its equation).

EXAMPLE 1.

Let X be an irreducible surface (not a cone) of order n in the projective 3-space over a field k algebraically closed, of characteristic $\neq 2$.

Assume that X has equation $x_0^2 a + 2x_0 b + c = 0$, where $a, b, c \in k[x_1, x_2, x_3]$ are forms of degree $n-2, n-1, n$ respectively, and (x_0, x_1, x_2, x_3) are the coordinates in $\mathbb{P}^3(k)$.

The point $P(1, 0, 0, 0)$ is $(n-2)$ -fold for X and $a=0$ is the tangent cone to X at P : assume that it has no multiple generatrices, and let Δ be the curve of the plane $x_0=0$ given by $b^2 - ac = 0$.

We have: X is normal (resp. seminormal) iff Δ does not have multiple components (resp. Δ has at most double components).

Indeed: put $V = X - (X \cap a)$, $W = Y - (Y \cap a)$ (where Y denotes the plane $X_0 = 0$) and let $\pi: V \rightarrow W$ be the projection from P ; clearly π is a finite, separable, surjective morphism of degree 2, having $W_D = \Delta - (\Delta \cap a)$ as discriminant scheme. Therefore, from theorem 2 (resp. theorem 4) it follows that V is normal (resp. seminormal) iff W_D has no multiple components (resp. W_D has at most double components).

Moreover it can be proved that, under our assumptions, $X - V$ has only normal points and that $\Delta - W_D$ has no multiple components, and from this our claim follows.

EXAMPLE 2.

Let X be an irreducible hypersurface of order $n \geq 3$ in $\mathbb{P}^r(k)$, where k is an algebraically closed field of characteristic either 0 or $> n$.

Assume that X has equation $x_0^n = h(x_1, \dots, x_r)$, where $h \in k[x_1, \dots, x_r]$ is a form of degree n and $(x_0, \dots, x_r)$ are the coordinates in $\mathbb{P}^r(k)$.

We have: X is normal iff X is seminormal iff the polynomial h has no multiple factors.

X is normal (resp. seminormal) iff X is normal (resp. seminormal) on all the charts of an affine covering; therefore we may assume X affine. Let $Z^n = h(V_1, \dots, V_{r-1})$ be its equation, and consider

the projection $\pi: X \rightarrow Y$ from the point $P(1,0,\dots,0)$ on the hyperplane Y having equation $Z=0$: π is the finite, separable, surjective morphism of degree n , which corresponds to the canonical ring homomorphism $R=k[V_1,\dots,V_{r-1}] \rightarrow \{R[Z]/(Z^n-h)\} \cong R[z]$.

In codimension 1 we have the following situation: $A=R_p$, $B'=R_p[z]$ ($\text{ht } p = 1$), and therefore, by applying theorem 5 we can prove our claim.

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