

Factorization of graphs with common transversal sets

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Abstract. In this paper, S_k -factorization algorithm of an edge-disjoint sum of two K_k -factors of K_v is given.

1. Introduction

Let G and H be graphs. A spanning subgraph F of G is called an H -factor if and only if each component of F is isomorphic to H . If G is expressible as an edge-disjoint sum of H -factors, then this sum is called an H -factorization of G . Let S_k be a star with k vertices. Let K_k and K_v be a complete graph with k vertices and v vertices, respectively.

2. Common transversal sets

We use common transversal sets on S_k -factorization of an edge-disjoint sum of two K_k -factors of K_v .

Consider a set N of v elements, where $v=kt$. Divide N into t subsets $A_1, A_2, \dots, A_t$ so that they are mutually disjoint subsets of same size k . And divide N into another t subsets $B_1, B_2, \dots, B_t$ so that they are mutually disjoint subsets of same size k . Let T be a t -element subsets of N . Then T is called a common transversal set of $\{A_1, A_2, \dots, A_t\}$ and $\{B_1, B_2, \dots, B_t\}$ when it holds that $|T \cap A_j| = |T \cap B_j| = 1$, $1 \leq j \leq t$.

Lemma 1. Let $\{A_1, A_2, \dots, A_t\}$ be a mutually disjoint partition of N and $\{B_1, B_2, \dots, B_t\}$ be another mutually disjoint partition of N . Then there exists a common transversal set T of $\{A_1, A_2, \dots, A_t\}$ and $\{B_1, B_2, \dots, B_t\}$.

3. S_k -factorization algorithm of an edge-disjoint sum of two K_k -factors of K_v

For $k \geq 3$, we have the following:

Lemma 2. *An edge-disjoint sum of two K_k -factors of K_v can be factorized into k S_k -factors.*

Proof. Let F_1 and F_2 be edge-disjoint K_k -factors of K_v . And let G be a sum of F_1 and F_2 . Then $V(F_1)=V(F_2)=V(G)=V(K_v)$. Put $F_1=K_k^{(1)} \cup K_k^{(2)} \cup \dots \cup K_k^{(t)}$ and $F_2=K_k^{(t+1)} \cup K_k^{(t+2)} \cup \dots \cup K_k^{(2t)}$. And let $A_j=V(K_k^{(j)})$ and $B_j=V(K_k^{(t+j)})$, $1 \leq j \leq t$. Then $\{A_1, A_2, \dots, A_t\}$ is a mutually disjoint partition of $V(G)$ and $\{B_1, B_2, \dots, B_t\}$ is another mutually disjoint partition of $V(G)$. Let T_1 be a common transversal set of $\{A_1, A_2, \dots, A_t\}$ and $\{B_1, B_2, \dots, B_t\}$. Let T_j be a common transversal set of $\{A_1-T, A_2-T, \dots, A_t-T\}$ and $\{B_1-T, B_2-T, \dots, B_t-T\}$, where $T=T_1+T_2+\dots+T_{j-1}$ ($2 \leq j \leq t$). Consider $2k-2$ subgraphs G_j of G such as $G_1=F_1$, $G_2=F_2$, $G_j=G_{j-2}-T_{j-2}$ ($3 \leq j \leq 2k-2$), where $T_{k+i}=T_{k-i+1}$ ($1 \leq i \leq k-2$). Consider $2k-2$ subgraphs H_j of G such as $H_j=G_j-E(G_j-T_j)$ ($1 \leq j \leq 2k-4$), $H_{2k-3}=G_{2k-3}$, $H_{2k-2}=G_{2k-2}$.

Note that every component of G_j is a complete graph with $(2k-j+1)/2$ vertices (j :odd) or $(2k-j+2)/2$ vertices (j :even) and that every component of H_j is a star with $(2k-j+1)/2$ vertices (j :odd) or $(2k-j+2)/2$ vertices (j :even).

Then we can construct k edge-disjoint S_k -factors $F_1', F_2', F_3, \dots, F_k$ of G as follows:

$$F_1'=H_1, F_2'=H_2, F_j=H_j \cup H_{2k-j+1} \quad (3 \leq j \leq k).$$

Therefore, it holds that $G=F_1'(+)F_2'(+)F_3(+)\dots(+)F_k$, which is an S_k -factorization.

□

Note 1. The symbol $(+)$ is used to denote the sum of factors.

As a resolvable BIBD($v, b, r, k, \lambda=1$) is just a K_k -factorization of K_v , we have the following lemmas.

Lemma 3. *If there exists a resolvable BIBD($v, b, r, k, \lambda=1$) (r :even), then K_v has an S_k -factorization.*

Lemma 4. *If there exists a resolvable BIBD($v, b, r, k, \lambda=1$) (r :odd, $m=v/k$), then $K(m; k)$ has an S_k -factorization.*

Note 2. $K(m; k)$ is a complete multipartite graph with m partite sets of k vertices each.

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