

GEOMETRY OF POLAR ACTIONS ON COMPLEX HYPERBOLIC SPACES

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ABSTRACT. Polar actions on complex hyperbolic spaces without singular orbits have been classified by Berndt and Díaz-Ramos. In this paper, we will focus on geometry of their orbits, and mention classifications of the minimal ones and Ricci soliton ones.

1. INTRODUCTION

A submanifold of a complex hyperbolic space $\mathbb{C}H^n$ is said to be *homogeneous* if it is an orbit of an isometric action on $\mathbb{C}H^n$. Note that by an isometric action we always mean an action of a connected closed subgroup of the isometry group. Homogeneous submanifolds of $\mathbb{C}H^n$ have studied very actively and deeply (refer to, for instance, [1, 6, 11, 12, 15] and references therein). They are of particular interest from the viewpoint not only of submanifold geometry of $\mathbb{C}H^n$, but also of studies of homogeneous submanifolds in other symmetric spaces of noncompact type.

In this paper, we consider the case where isometric actions on $\mathbb{C}H^n$ are polar, and study the geometry of the orbits of such actions. An isometric action on a Riemannian manifold M is said to be *polar* if there exists a connected complete submanifold Σ of M such that Σ meets each orbit of the action and is perpendicular to each orbit at every intersection point. Note that such a submanifold Σ is called a *section* of the action, and it is always a totally geodesic submanifold (for instance, see [3, Theorem 3.2.1]). For more details of polar actions, for instance, refer to [2, 4, 5, 9, 13, 14] and references therein.

We moreover restrict to the case where polar actions on $\mathbb{C}H^n$ induce *homogeneous polar foliations*, that is, have no singular orbits. In this case, the actions have been classified by Berndt and Díaz-Ramos in [4]. They have proved that there exist exactly $2n - 1$ actions which induce nontrivial homogeneous polar foliations of $\mathbb{C}H^n$ up to orbit equivalence. Recall that a homogeneous foliation of $\mathbb{C}H^n$ is said to be *trivial* if the leaves are points in $\mathbb{C}H^n$ or the leaf coincides with $\mathbb{C}H^n$. Moreover, the author have classified orbits of such polar actions up to isometric congruence, and have given explicit expressions of the orbits in [15]. The subject of this paper is to review the above results on geometry of the orbits, and also to announce our recent work on Ricci solitons (Subsection 3.2).

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Remark 1.1. Our results in this paper include the some known results in the case of cohomogeneity one actions on $\mathbb{CH}^n$ in [1, 6, 12]. See Remark 3.3 and 3.12 for more details.

This paper is organized as follows. In Section 2, we recall the solvable model of a complex hyperbolic space $\mathbb{CH}^n$, and review the classifications of homogeneous polar foliations of $\mathbb{CH}^n$ and the orbits of such actions. In Section 3, we mention the curvature properties of the orbits in $\mathbb{CH}^n$. Firstly, we mention the minimality of the orbit and the classification of the minimal orbits, which are obtained in [15]. Secondly, we announce our recent result on homogeneous Ricci soliton submanifolds of $\mathbb{CH}^n$. In particular, we classify the orbits which are Ricci solitons.

2. PRELIMINARIES

2.1. The solvable models of complex hyperbolic spaces. In this subsection, we prepare the solvable models of complex hyperbolic spaces $\mathbb{CH}^n$ with $n \geq 2$, which we need in the following sections. We refer mainly to [8], [11].

Definition 2.1. We call a triple $(\mathfrak{s}, \langle, \rangle, J)$ the *solvable model* of $\mathbb{CH}^n$ if

- (1) $\mathfrak{s} := \text{span}_{\mathbb{R}}\{A_0, X_1, Y_1, \dots, X_{n-1}, Y_{n-1}, Z_0\}$ is a Lie algebra whose bracket relations are defined by

$$[A_0, X_i] = (1/2)X_i, [A_0, Y_i] = (1/2)Y_i, [A_0, Z_0] = Z_0, [X_i, Y_i] = Z_0,$$

- (2) $\langle, \rangle$ is an inner product on $\mathfrak{s}$ such that the above basis is orthonormal,
 (3) J is a complex structure on $\mathfrak{s}$ defined by

$$J(A_0) = Z_0, J(Z_0) = -A_0, J(X_i) = Y_i, J(Y_i) = -X_i.$$

Let S be the simply-connected Lie group with Lie algebra $\mathfrak{s}$. One knows the expression of the complex hyperbolic space as a homogeneous space, that is,

$$\mathbb{CH}^n = \text{SU}(1, n)/\text{S}(\text{U}(1) \times \text{U}(n)).$$

The Lie group S coincides with the solvable part of the Iwasawa decomposition of $\text{SU}(1, n)$. Furthermore, we can naturally identify $\mathbb{CH}^n$ with the Lie group S . More precisely, we have the following.

Proposition 2.2. Denote by the same symbols $\langle, \rangle$ and J the induced left-invariant Riemannian metric and the complex structure on S , respectively. Then, $(S, \langle, \rangle, J)$ is holomorphically isometric to $\mathbb{CH}^n$ with constant holomorphic sectional curvature -1 .

2.2. Classifications of polar actions on complex hyperbolic spaces and their orbits. In this subsection, we recall the classification of polar actions on $\mathbb{CH}^n$ without singular orbits by Berndt and Díaz-Ramos. And then, we recall the classification of the orbits of such actions by the author.

First of all, we introduce Lie subgroups $S_b(\varphi)$ of S , which play essential roles in the study of polar actions on $\mathbb{CH}^n$ without singular orbits. For $\varphi \in [0, \pi/2]$, let us define

$$\xi_0 := \cos(\varphi)X_1 + \sin(\varphi)A_0.$$

We mean always by $\ominus$ the orthogonal complement with respect to $\langle, \rangle$.

Definition 2.3. For $b \in \{1, \dots, n\}$ and $\varphi \in [0, \pi/2]$, we define $\mathfrak{s}_b(\varphi)$ as follows:

(1) if $\varphi \in [0, \pi/2)$, then set

$$\mathfrak{s}_b(\varphi) := \mathfrak{s} \ominus \text{span}_{\mathbb{R}}\{\xi_0, X_2, \dots, X_b\},$$

where $b \in \{1, \dots, n-1\}$,

(2) if $\varphi = \pi/2$, then set

$$\mathfrak{s}_b(\pi/2) := \mathfrak{s} \ominus \text{span}_{\mathbb{R}}\{X_1, X_2, \dots, X_b\},$$

where $b \in \{1, \dots, n\}$.

One can easily check that $\mathfrak{s}_b(\varphi)$ is a Lie subalgebra of $\mathfrak{s}$ of codimension b . Note that $\mathfrak{s}_b(\pi/2)$ is nilpotent, whereas $\mathfrak{s}_b(\varphi)$ is solvable but is not nilpotent for $\varphi \in [0, \pi/2)$.

Denote by $S_b(\varphi)$ the connected Lie subgroup of S with Lie algebra $\mathfrak{s}_b(\varphi)$. Then, one can show that the action of $S_b(\varphi)$ on $\mathbb{CH}^n$ induces a homogeneous polar foliation of cohomogeneity b , and its section is a totally geodesic real hyperbolic space $\mathbb{RH}^b$. Furthermore, we have the following classification result.

Theorem 2.4 ([4]). *An isometric action on $\mathbb{CH}^n$ induces a nontrivial homogeneous polar foliation of $\mathbb{CH}^n$ if and only if it is orbit equivalent to one of the following:*

- (1) *the action of $S_b(0)$, where $b \in \{1, \dots, n-1\}$,*
- (2) *the action of $S_b(\pi/2)$, where $b \in \{1, \dots, n\}$.*

We next mention the classification of the orbits of polar actions on $\mathbb{CH}^n$ without singular orbits, up to isometric congruence. Denote by o the origin of $\mathbb{CH}^n$.

Theorem 2.5 ([15]). *Every orbit of polar actions on $\mathbb{CH}^n$ without singular orbits is isometrically congruent to one of the following:*

- (1) *the orbit $S_b(\varphi).o$, where $b \in \{1, \dots, n-1\}$ and $\varphi \in [0, \pi/2)$,*
- (2) *the orbit $S_b(\pi/2).o$, where $b \in \{1, \dots, n\}$.*

Remark 2.6. For $p \in \mathbb{CH}^n$, the orbit $S_b(0).p$ is isometrically congruent to the orbit $S_b(\varphi).o$ for some $\varphi \in [0, \pi/2)$ ([15, Proposition 4.5]). In particular, we note that φ is explicitly given by

$$\varphi = \arcsin(\tanh(t_0/2)),$$

where t_0 is the distance between the origin o and the orbit $S_b(0).p$. On the other hand, the action of $S_b(\pi/2)$ has congruency of orbits ([15, Theorem 5.1]). Namely, for each $b \in \{1, \dots, n\}$, every orbit of $S_b(\pi/2)$ is isometrically congruent to the orbit $S_b(\pi/2).o$. Refer to [16] for more details of the congruency of orbits.

3. CURVATURE PROPERTIES

In this section, we study the curvature properties of the orbit $S_b(\varphi).o$.

Let us recall that $\mathbb{CH}^n$ can be identified with the Lie group S . One thus can identify the submanifold $S_b(\varphi).o$ with the Lie subgroup $S_b(\varphi)$ equipped with the induced left-invariant metric. Throughout this section, therefore, we shall express the curvatures in terms of the metric Lie algebra $\mathfrak{s}_b(\varphi)$.

3.1. The minimality. In this subsection, we mention the minimality of the orbit $S_b(\varphi).o$ obtained in [15].

Proposition 3.1 ([15]). *The mean curvature vector $\mathcal{H}$ of $S_b(\varphi)$ is given by*

$$\mathcal{H} = (1/2)(2n - b + 1) \sin(\varphi) \xi_0,$$

Hence, the orbit $S_b(\varphi).o$ is minimal if and only if $\varphi = 0$.

Recall that $\varphi = \arcsin(\tanh(t_0/2))$, where t_0 denotes the distance between o and $S_b(0).p$. It hence follows from the monotonicity of this function that $\varphi = 0$ if and only if $t_0 = 0$. Altogether, we have the following result.

Theorem 3.2. *For $\varphi \in [0, \pi/2)$, the action of $S_b(0)$ has the unique minimal orbit $S_b(0).o$, whereas the action of $S_b(\pi/2)$ has no minimal orbits.*

Remark 3.3. In the case of cohomogeneity one, namely, $b = 1$, the actions of $S_1(\pi/2)$ and $S_1(0)$ on $\mathbb{CH}^n$ are well-known. Especially, Theorem 3.2 has been proved in [1] (also refer to [6]).

- (1) The action of $S_1(\pi/2)$ induces the so-called solvable foliation, and the orbit $S_1(0).o$ is a unique minimal orbit. In fact, $S_1(0).o$ is known as the homogeneous ruled minimal hypersurface, and the other orbits are equidistant hypersurfaces to $S_1(0).o$.
- (2) The action of $S_1(\pi/2)$, which is the nilpotent part of the Iwasawa decomposition of $SU(1, n)$, induces the so-called horosphere foliation, and every orbit is known as a horosphere of $\mathbb{CH}^n$. In this case, the action of $S_1(\pi/2)$ has congruency of orbits, and has no minimal orbits.

The mean curvature vector is said to be *parallel* if $\nabla_X^\perp \mathcal{H} = 0$ holds for any $X \in \mathfrak{s}_b(\varphi)$, where $\nabla^\perp$ denotes the normal connection of $S_b(\varphi)$. One can obtain the following by direct calculations.

Proposition 3.4. *The mean curvature vector $\mathcal{H}$ of $S_b(\varphi)$ is always parallel.*

Remark 3.5. We note that the proposition above can be shown by the general theory of polar actions. As we mentioned before, the action of $S_b(\varphi)$ is polar, and the orbit $S_b(\varphi).o$ is a principal orbit of the action. Therefore, it follows from [3, Corollary 3.2.5] that the mean curvature vector field on $S_b(\varphi).o$ is parallel with respect to $\nabla^\perp$.

3.2. Ricci solitons. In this subsection, we announce our recent result on homogeneous Ricci soliton submanifolds of complex hyperbolic spaces $\mathbb{CH}^n$. In particular, we classify the orbits $S_b(\varphi)$ which are Ricci solitons.

First of all, let us recall the notion of Ricci solitons.

Definition 3.6. A Riemannian manifold (M, g) is called a *Ricci soliton* if its Ricci curvature ric satisfies

$$\text{ric} = cg + \mathcal{L}_X g$$

for some $c \in \mathbb{R}$ and some vector field $X \in \mathfrak{X}(M)$, where $\mathcal{L}_X$ denotes the usual Lie derivative.

It is easy to see that the notion of Ricci solitons is a generalization of Einstein manifolds. We now also recall the notion of algebraic Ricci solitons, which essentially have been introduced by Lauret (see [17, 18]).

Definition 3.7. A metric Lie algebra $(\mathfrak{g}, \langle, \rangle)$ is called an *algebraic Ricci soliton* if its Ricci operator $\text{Ric} : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfies

$$(3.1) \quad \text{Ric} = c \cdot \text{id}_{\mathfrak{g}} + D$$

for some $c \in \mathbb{R}$ and some $D \in \text{Der}(\mathfrak{g})$, where $\text{Der}(\mathfrak{g})$ denotes the Lie algebra of derivations on $\mathfrak{g}$.

Remark 3.8. Let (G, g) be a simply-connected Lie group with a left-invariant metric, and $(\mathfrak{g}, \langle, \rangle)$ be the corresponding metric Lie algebra. Then, one knows that (G, g) is a Ricci soliton if $(\mathfrak{g}, \langle, \rangle)$ is an algebraic Ricci soliton. In addition, if G is *completely solvable*, which means that the eigenvalues of any ad_X are all real, then the converse also holds. See [18] for more details.

By direct calculations, one can see that $S_b(\varphi)$ is completely solvable. Therefore, we have only to study whether $\mathfrak{s}_b(\varphi)$ is an algebraic Ricci soliton for our goal.

Firstly, we consider the case of $\mathfrak{s}_b(\pi/2)$, that is, the nilpotent case. It is easy to show that a direct sum as metric Lie algebras of an algebraic Ricci soliton and an abelian Lie algebra is also an algebraic Ricci soliton. Hence, one has the following.

Proposition 3.9. *For each $b \in \{1, \dots, n\}$, the metric Lie algebra $\mathfrak{s}_b(\pi/2)$ is an algebraic Ricci soliton.*

In the other cases, the structure theorem for algebraic Ricci solitons [18, Theorem. 4.8] yields the following.

Proposition 3.10. *For $b \in \{1, \dots, n-1\}$ and $\varphi \in [0, \pi/2)$, the metric Lie algebra $\mathfrak{s}_b(\varphi)$ is an algebraic Ricci soliton if and only if $b = n-1$ and $\varphi = 0$.*

Altogether, we have the following result.

Theorem 3.11. *The orbit of $S_b(\varphi).o$ in $\mathbb{CH}^n$ is a Ricci soliton if and only if*

- (1) $b = n-1$ and $\varphi = 0$, or
- (2) $\varphi = \pi/2$.

Remark 3.12. In the case of cohomogeneity one, Theorem 3.11 has been proved in [12]. Note that the proofs in [12], which are based on the explicit formulas for the Ricci operators, are direct and elementary.

REFERENCES

- [1] J. Berndt, Homogeneous hypersurfaces in hyperbolic spaces, *Math. Z.*, **229** (1998), no. 4, 589–600.
- [2] J. Berndt, Polar actions on symmetric spaces, in: *Proceedings of the Fifteenth International Workshop on Diff. Geom.*, **15** (2011), 1–10.
- [3] J. Berndt, S. Console & C. Olmos, Submanifolds and holonomy, Chapman & Hall/CRC Research Notes in Mathematics, **434**. Chapman & Hall/CRC, Boca Raton, FL, 2003.
- [4] J. Berndt & J. C. Díaz-Ramos, Homogeneous polar foliations of complex hyperbolic spaces, *Comm. Anal. Geom.*, **20** (2012), no. 3, 435–454.
- [5] J. Berndt, J. C. Díaz-Ramos & H. Tamaru, Hyperpolar homogeneous foliations on symmetric spaces of noncompact type, *J. Differential Geom.*, **86** (2010), no. 2, 191–235.
- [6] J. Berndt & H. Tamaru, Homogeneous codimension one foliations on noncompact symmetric spaces, *J. Differential Geom.*, **63** (2003), no. 1, 1–40.

- [7] J. Berndt & H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces of rank one, *Trans. Amer. Math. Soc.*, **359** (2007), no. 7, 3425–3438.
- [8] J. Berndt, F. Tricerri & L. Vanhecke, Generalized Heisenberg groups and Damek-Ricci harmonic spaces, *Lecture Notes in Mathematics*, **1598**. Springer-Verlag, Berlin, 1995.
- [9] J. C. Díaz-Ramos, Polar actions in complex space forms, in: *Proceedings of the Sixteenth International on Diff. Geom.*, **16** (2012), 71–90.
- [10] J. C. Díaz-Ramos, M. Domínguez-Vázquez & A. Kollross, Polar actions on complex hyperbolic spaces. *preprint*, arXiv:1208.2823v2.
- [11] T. Hamada, Y. Hoshikawa & H. Tamaru, Curvature properties of Lie hypersurfaces in the complex hyperbolic space, *J. Geom.*, **103** (2012), no. 2, 247–261.
- [12] T. Hashinaga, A. Kubo & H. Tamaru, Homogeneous Ricci soliton hypersurfaces in the complex hyperbolic spaces, *preprint*, arXiv:1305.6128v1.
- [13] E. Heintze, X. Liu & C. Olmos, Isoparametric submanifolds and a Chevalley-type restriction theorem, in: *Integrable systems, geometry, and topology*, 151–190, AMS/IP Stud. Adv. Math., **36**, Amer. Math. Soc., Providence, RI, 2006.
- [14] E. Heintze, R. S. Palais, C. L. Terng & G. Thorbergsson, Hyperpolar actions on symmetric spaces, *Geometry, topology, & physics*, 214–245, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.
- [15] A. Kubo, Geometry of homogeneous polar foliations of complex hyperbolic spaces, *Hiroshima Math. J.*, to appear.
- [16] A. Kubo & H. Tamaru, A sufficient condition for congruency of orbits of Lie groups and some applications, *Geom. Dedicata*, **167** (2013), 233–238.
- [17] J. Lauret, Ricci soliton homogeneous nilmanifolds, *Math. Ann.*, **319** (2001), 715–733.
- [18] J. Lauret, Ricci soliton solvmanifolds, *J. reine angew. Math.*, **650** (2011), 1–21.

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