

Invariant subspace problemについて

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C. J. Read の結果「non trivial to invariant subspace をもたない Banach 空間上の有界線型作用素が存在する」について紹介する。これはすでに [1] に発表されているが、同論文の末尾に「 ℓ^1 空間上にもこのような作用素 T が作られる」とある。この改良された結果を述べたノートが手に入、たので講演でもこれをもとに紹介を行なった。

以下は、このノートをもとにしたタイプである。(オリジナルのノートは手書きのため読み難く、た) なお、正確に欠けていると思われる点は、筆者の理解する範囲で修正し、[Note] と図を補った。また、内容の順序も一部並べ変えた。このため、最終的に Read にて発表されるものは別の形になっていると思われる。なおこのノートは伊藤隆司先生(武蔵工大)から越先生(北大)を通じて入手した。記してお礼を述べたい。

C. Read's construction of an operator on the Banach space ℓ^1 having no non-trivial invariant subspace.

(Based on a seminar given by Read in Edinburgh on 7/5/84)

Let $\mathcal{P}$ denote the complex vector space of all polynomials in a single variable x with complex coefficients, and let $\mathcal{P}_n$ denote the subspace of polynomials of degree $\leq n$, so that $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. We let p_n denote the natural projection of $\mathcal{P}$ onto $\mathcal{P}_n$, so that $p_n(\sum a_k x^k) = \sum a_k x^k$. We also define a norm $\|\cdot\|_*$ on $\mathcal{P}$ by $\|\sum a_k x^k\|_* = \sum |a_k|$.

Lemma 1. Let n be a positive integer and let $\varepsilon, \delta, M > 0$. Then, there exists $K = K(\varepsilon, \delta, M, n) > 0$ such that if $0 \leq m \leq n$ and $g \in \mathcal{P}_n$ with $\|g\|_* \leq M$ and $\|p_m g\|_* \geq \delta$ then we can find $q \in \mathcal{P}_n$ with $\|p_n(qg) - x^m\|_* < \varepsilon$ and $\|q\|_* \leq K$.

Proof. If $g = \sum_{j=k}^n \alpha_j x^j$ and $\alpha_k \neq 0$, then considering a linear combination of $g, p_n(xg), \dots, p_n(x^n g)$ we find a polynomial $q \in \mathcal{P}_n$ with $p_n(qg) = x^m$ for $k \leq m \leq n$. Since the set $\{g \in \mathcal{P}_n : \|g\|_* \leq M, \|p_m g\|_* \geq \delta\}$ is compact, we need only a finite number of $q_1, \dots, q_\ell$ to have $\|p_n(q_j g) - x^m\|_* < \varepsilon$ for some q_j . So, $K = \max_j \|q_j\|_*$ has the desired property.

Let $1 < a_1 < b_1 < a_2 < b_2 \dots$ be a sequence of positive integers which are required to increase rapidly; in the

statements of subsequent lemmas the condition "provided the sequence $a_1, b_1, \dots$ increases fast enough" will be understood.

We put $a_0 = b_0 = 1$ and let $v_1 = v_0 = 0$, $v_n = (n-1)(a_n + b_n)$ for $n > 1$.

We define a basis $f_0, f_1, \dots$ of $\mathcal{P}$ as follows:

$$f_0 = 1;$$

If $n = r = 1$, or $n \geq 2$ and $1 \leq r \leq n-1$, then

(A) for $ra_n \leq k \leq ra_n + v_{n-r-1}$

$$f_k = a_{n-r}(x^k - x^{k-a_n});$$

(B) for $a_1 + b_1 < k < a_2$, $(r-1)a_n + v_{n-r} < k < ra_n$ ($n \neq 2$)

$$f_k = 2^{[(r - \frac{1}{2})a_n - k]/b_{n-1}} \cdot x^k;$$

(C) for $k = a_1 + b_1$, $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$ ($n \geq 2$)

$$f_k = x^k - b_n x^{k-b_n};$$

(D) for $a_1 < k < a_1 + b_1$, $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$ ($n \geq 2$)

$$f_k = 2^{[(r - \frac{1}{2})b_n - k]/a_n} \cdot x^k.$$

For given $n \geq 1$, formulae (A) - (D) define f_k for $v_{n-1} < k < v_n$. Note that the polynomial f_k always has a non-zero x^k term and no terms of higher order; hence $f_0, \dots, f_n$ span $\mathcal{P}_n$ for each n . (See Fig. 1).

We define a norm $\|\cdot\|_x$ on $\mathcal{P}$ by $\|\sum \alpha_k f_k\|_x = \sum |\alpha_k|$ and let X be the completion of $\mathcal{P}$ with respect to this norm. So,

X is a Banach space isomorphic to ℓ^1 . We also define a linear operator $T : \mathcal{P} \rightarrow \mathcal{P}$ by $Tx^k = x^{k+1}$, i.e., T is multiplication by x . Note that $p(T)q = pq$ for polynomials p, q .

We note two consequences of the definition of the norm $\|\cdot\|_x$. If $1 \leq r \leq n-1$ and $ra_n \leq k \leq ra_n + v_{n-r-1}$, then from (A) we get

$$x^k - x^{k-ra_n} = a_{n-r}^{-1} f_k + a_{n-r+1}^{-1} f_{k-a_n} + \dots + a_{n-1}^{-1} f_{k-(r-1)a_n}$$

So

$$\|x^k - x^{k-ra_n}\|_x = a_{n-r}^{-1} + \dots + a_{n-1}^{-1} \leq 2/a_{n-r} \quad \dots(1)$$

assuming $\{a_n\}$ increases fast enough. Similarly, from (C) we get, for $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$,

$$\|x^k - b_n^r x^{k-rb_n}\|_x = 1 + b_n + \dots + b_n^{r-1} \leq 2b_n^{r-1} \quad \dots(2)$$

[Note: (1) and (2) show the corresponding finite dimensional subspaces are nearly isometric each other with respect to the coordinate x^k and the norm $\|\cdot\|_x$. See Fig. 2].

Lemma 2. $\|Tf_k\|_x \leq 2$ for all k .

From Lemma 2 it follows that T extends by continuity to a bounded linear operator on X , with $\|T\|_x \leq 2$.

For $m > 1$, we define $\sigma_m = \{k \in \mathbb{N} : \text{for some } n > m \text{ we have } (n-m)a_n \leq k \leq (n-m)a_n + v_{m-1}\}$. [Note: In other words, $\sigma_m = \bigcup_{n>m} \{k \in (A) : \text{with } n = n \text{ and } r = n-m \text{ in the definition of } f_k\}$].

Lemma 3. Suppose $m > 2$, $k > (m-1)a_m$ and $b_m + a_m \leq s \leq b_m + (m-1)a_m$. Then,

- (a) if $k \notin \sigma_m$, we have $\|T^s f_k\|_x \leq 4$.
 (b) if $k \in \sigma_m$ then, writing $k = (n-m)a_n + j$, $0 \leq j \leq v_{m-1}$, we have $\|T^s f_k + a_m x^{j+s}\|_x \leq 1$.

We now define a linear mapping $Q_m : \mathcal{P} \rightarrow \mathcal{P}_{(m-1)a_m}$ for $m > 2$ [see Fig. 3.3 for Q_m with $m = k$] by

$$Q_m(f_k) = \begin{cases} f_k & \text{if } 0 \leq k \leq (m-1)a_m \\ 0 & \text{if } k > (m-1)a_m, k \notin \sigma_m \\ -a_m x^j & \text{if } k \in \sigma_m \text{ (i.e., } k = (n-m)a_n + j, j \leq v_{m-1}). \end{cases}$$

The conclusion of Lemma 3 can be restated as follows:

$$\begin{aligned} m > 2, b_m + a_m \leq s \leq b_m + (m-1)a_m, g \in \mathcal{P} \\ \Rightarrow \|T^s g - T^s Q_m(g)\|_x < 4 \|g\|_x \dots (6) \end{aligned}$$

In fact, to prove (6) it suffices to prove it for $g = f_k$; this is trivial if $k \leq (m-1)a_m$ and follows from Lemma 3 for $k > (m-1)a_m$.

We can find $C_m \geq a_m$, depending only on $a_1, b_1, \dots, a_m$, such that $\|f_j\|_* \leq C_m$ and $\|x^j\|_x \leq C_m$ for $j = 0, \dots, (m-1)a_m$. Then, by the definition of Q_m , we have $\|Q_m(f_k)\|_x \leq a_m C_m$ for all k , so $\|Q_m g\|_x \leq a_m C_m \|g\|_x$ for $g \in \mathcal{P}$, and so Q_m extends continuously to X and (6) holds for $g \in X$.

Note also that $\|Q_m(f_k)\|_* \leq \max(a_m, C_m) \leq C_m$, so

$$\|Q_m(g)\|_* \leq C_m \|g\|_X, \quad g \in X.$$

Lemma 4. Let $g \in X$ with $\|g\|_X = 1$. Suppose that for some $m > 2$ and $1 \leq r < m-2$ we have $\|p_{ra_m} Q_m(g)\|_* \geq 1/a_m$. Then, there is a polynomial ϕ with $\|\phi(T)g - 1\|_X < 3/a_{m-r-1}$.

THEOREM. Let Y be a closed subspace of X with $TY \subseteq Y$. Then, either $Y = \{0\}$ or $Y = X$.

Proof of Lemma 2. We consider k belonging to the four types (A), (B), (C), (D), separately. In the course of the proof we note estimate (3), (4), (5) below for future use.

Case (A): $ra_n \leq k \leq ra_n + v_{n-r-1}$

In this case $Tf_k = a_{n-r}(x^{k+1} - x^{k+1-a_n})$. Now, if $k < ra_n + v_{n-r-1}$, we get similarly $Tf_k = f_{k+1}$, so $\|Tf_k\|_X = 1$. This leaves the case $k = ra_n + v_{n-r-1}$. If $n = r = 1$ or $n=2$ and $r=1$, then $k+1 = a_n+1 \in (D)$ and $k+1-a_n = 1 \in (B)$, so

$$\|x^{k+1}\|_X = 2^{(a_n+1 - \frac{1}{2}b_n)/a_n} < 1/b_n; \text{ and}$$

$$\|x^{k+1-a_n}\|_X = 2^{(1 - \frac{1}{2}a_1)} < 1/a_1^2.$$

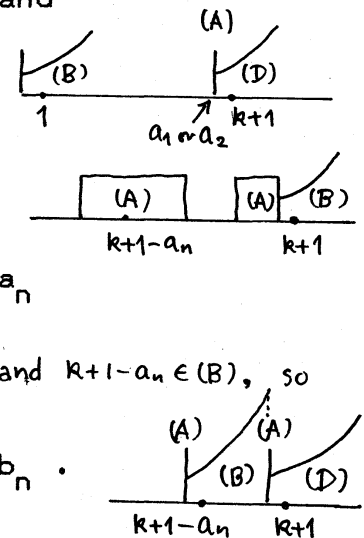
So, $\|Tf_k\|_X \leq 1/a_{n-r}$.

If $n > 2$ and $r < n-1$, then $k+1 \in (B)$, so

$$\|x^{k+1}\|_X = 2^{(v_{n-r-1}+1 - \frac{1}{2}a_n)/b_{n-1}} < 1/a_n$$

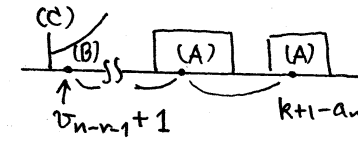
and if $n > 2$ and $r = n-1$, then $k+1 \in (D)$ and $k+1-a_n \in (B)$, so

$$\|x^{k+1}\|_X = 2^{[(n-1)a_n+1 - \frac{1}{2}b_n]/na_n} < 1/b_n.$$



$$\|x^{k+1-a_n}\|_x = 2^{(-\frac{1}{2}a_n+1)/b_{n-1}} < 1/a_n$$

Also, if $\begin{cases} r < n-1 \\ n > 2 \end{cases}$, then $v_{n-r-1}+1 \in (B)$, and using (1),

$$\begin{aligned} \|x^{k+1-a_n}\|_x &= \|x^{(r-1)a_n+v_{n-r-1}+1}\|_x \\ &\leq 2/a_{n-r+1} + \|x^{v_{n-r-1}+1}\|_x \\ &= 2/a_{n-r+1} + 2^{(v_{n-r-1}+1-\frac{1}{2}a_{n-r})/b_{n-r-1}} \end{aligned}$$


[Note: The above estimate is true even if $r=1$]. So

$$\begin{aligned} \|Tf_{ra_n+v_{n-r-1}}\|_x &\leq a_{n-r} (4/a_{n-r+1} + 2^{(v_{n-r-1}+1-\frac{1}{2}a_{n-r})/b_{n-r-1}}) \\ &\leq 1/a_{n-r} \quad \dots\dots\dots(3) \end{aligned}$$

provided $\{a_n, b_n\}$ increase fast enough. So, in case (A),

$$\|Tf_k\|_x \leq 1 \text{ always.}$$

Case (B): $(r-1)a_n + v_{n-r} < k < ra_n$

$$\text{In this case } Tf_k = 2^{[(r-\frac{1}{2})a_n - k]/b_{n-1}} x^{k+1}.$$

Now if $k < ra_n - 1$, we get $Tf_k = 2^{1/b_{n-1}} f_{k+1}$, so

$$\|Tf_k\|_x = 2^{1/b_{n-1}} < 2.$$

This leaves the case $k = ra_n - 1$, then $Tf_k = 2^{(1-\frac{1}{2}a_n)/b_{n-1}} x^{ra_n}$

and $\|x^{ra_n}\|_x \leq 1 + 2/a_{n-r}$ by (1), so provided a_n is large enough,

$$\|Tf_{ra_n-1}\|_x \leq 1/a_n \quad \dots\dots\dots(4)$$

The same proof works for $a_1 + b_1 < k < a_2$ with $n = 2$ and $r = 1$.

Case (C) : $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$

In this case $Tf_k = x^{k+1} - b_n x^{k+1-b_n}$.

If $k < (n-1)a_n + rb_n$, then $Tf_k = f_{k+1}$ so $\|Tf_k\|_x = 1$.

This leaves the cases $k = a_1 + b_1$ ($n = r = 1$) and $k = (n-1)a_n + rb_n$. If $r < n-1$, then by (D)

$$Tf_k = 2^{\lfloor (n-1)a_n + 1 - \frac{1}{2}b_n \rfloor / na_n} (f_{k+1} - b_n f_{k+1-b_n}).$$

If $r = n-1$ or $k = a_1 + b_1$, then $k+1 \in (B)$ and $k+1-b_n \in (D)$, so

$$Tf_k = 2^{k+1 - \frac{1}{2}a_{n+1}} \cdot f_{k+1} - 2^{\lfloor (n-1)a_n + 1 - \frac{1}{2}b_n \rfloor / na_n} \cdot b_n f_{k+1-b_n}$$

or

$$Tf_k = 2^{k+1 - \frac{1}{2}a_2} f_{k+1} - 2^{\lfloor a_1 + 1 - \frac{1}{2}b_1 \rfloor / a_1} b_1 f_{k+1-b_1},$$

respectively. So,

$$\|Tf_k\|_x = (1+b_n) \cdot 2^{\lfloor (n-1)a_n + 1 - \frac{1}{2}b_n \rfloor / na_n} \leq 1$$

if b_n and a_{n+1} large enough.

Case (D) : $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$

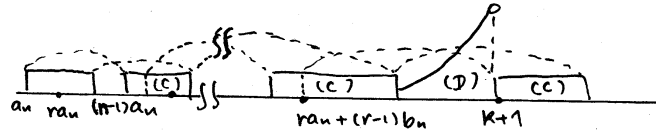
In this case $Tf_k = 2^{\lfloor (r-1)b_n - k \rfloor / na_n} x^{k+1}$.

If $k < r(a_n + b_n) - 1$, then $Tf_k = 2^{1/na_n} f_{k+1}$, so $\|Tf_k\|_x < 2$.

This leaves the case $k = r(a_n + b_n) - 1$; then,

$$Tf_k = 2^{(1-ra_n - \frac{1}{2}b_n)/na_n} \cdot x^{r(a_n+b_n)}$$

and by (2) and (1) we get



$$\|x^{r(a_n+b_n)}\|_x \leq 2b_n^{r-1} + \|x^{ra_n}\|_x \leq 2b_n^{r-1} + 2/a_{n-r} + 1$$

so if b_n is large enough, we get

$$\|Tf_k\|_x < 1/b_n \quad \text{when } k = r(a_n+b_n)-1 \quad \dots\dots(5)$$

The same proof works for $a_1 < k < a_1 + b_1$. This completes the proof.

Proof of Lemma 3. Again we consider separately the cases

(A), (B), (C), (D). Case (b) of the statement is covered by case (A)(i) below; the remaining cases cover (a).

Case (A) $ra_n \leq k \leq ra_n + v_{n-r-1}$

In this case $Tf_k = a_{n-r}(x^{k+s} - x^{k+s-a_n})$ and

$$a_m + b_m + ra_n \leq k+s \leq (m-1)a_m + b_m + ra_n + v_{n-r-1} \quad \dots(*A)$$

The hypothesis $k > (m-1)a_m$ implies $n > m$. We consider three such cases:

(i) $r = n - m$ ($\Leftrightarrow k \in \sigma_m$):

Write $k = ra_n + j$, where $0 \leq j \leq v_{m-1}$; then

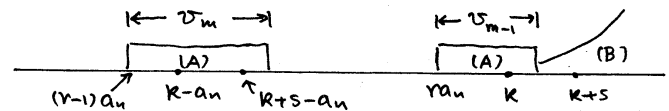
$$b_m < j+s < v_{m-1} + (m-1)a_m + b_m < 2b_m,$$

so $ra_n + v_{n-r} < k+s < (r+1)a_n$, i.e., $k+s \in (B)$. We get

$$\|x^{k+s}\|_x = 2^{\lfloor j+s - \frac{1}{2}a_n \rfloor / b_{n-1}} \leq 2^{\lfloor 2b_m - \frac{1}{2}a_n \rfloor / b_{n-1}} \leq 1/a_n$$

if a_n is large enough. Also, since $j+s \leq 2b_m < v_m$ (since $m > 2$) [Note: $(r-1)a_n \leq s+j+(r-1)a_n \leq (r-1)a_n + v_m$, $m = n-(r-1)-1$], we get from (1)

$$\|x^{k+s-a_n} - x^{s+j}\|_x = \|x^{s+j+(r-1)a_n} - x^{s+j}\|_x \leq 2/a_{m+1}.$$

So,  $(\Leftarrow r \geq 1 \text{ and } z)$

$$\begin{aligned} \|T^s f_k + a_m x^{s+j}\|_x &\leq a_m \|x^{k+s}\|_x + a_m \|x^{k+s-a_n} - x^{s+j}\|_x \\ &\leq a_m (1/a_n + 2/a_{m+1}) < 1. \end{aligned}$$

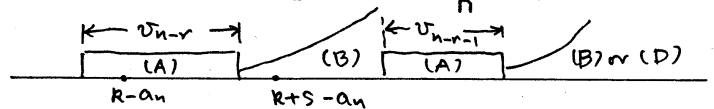
(ii) $r > n - m$:

In this case $s > b_m > v_{n-r}$, so

$$ra_n + v_{n-r-1} < k+s < (r+1)a_n \text{ and } (r-1)a_n + v_{n-r} < k+s-a_n < ra_n.$$

So, $k+s \in (B)$ or (D) and $k+s-a_n \in (B)$. Writing $k = ra_n + j$,

$0 \leq j \leq v_{n-r-1}$, we get



$$\|x^{k+s}\|_x = \|x^{k+s-a_n}\|_x = 2^{\lfloor j+s - \frac{1}{2}a_n \rfloor / b_{n-1}} \leq 2^{\lfloor 2b_m - \frac{1}{2}a_n \rfloor / b_{n-1}},$$

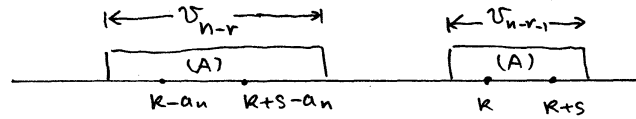
or $2^{(k+s - \frac{1}{2}b_n)/na_n} \leq 2^{(na_n - \frac{1}{2}b_n)/na_n}$

[Note: $(r+1 - \frac{1}{2})a_n - (ra_n + j+s) = (r - \frac{1}{2})a_n - (ra_n + j+s-a_n)$ since $j+s \leq v_{n-r-1} + (m-1)a_m + b_m \leq v_{m-1} + (m-1)a_m + b_m < 2b_m$. So,

$$\|T^s f_k\|_x \leq 2a_{n-r} \cdot 2^{\lfloor 2b_m - \frac{1}{2}a_n \rfloor / b_{n-1}} < 1$$

if a_n is large enough.

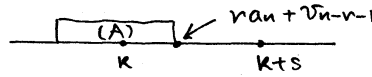
(iii) $r < n - m$:



Write $k = ra_n + j$, $0 \leq j \leq v_{n-r-1}$.

(α) $j+s \leq v_{n-r-1}$: Then [Note: $k+s \in (A)$] $T^s f_k = f_{k+s}$, so $\|T^s f_k\|_x = 1$.

(β) $j+s > v_{n-r-1}$: Then [Note: $T^{v_{n-r-1}-j} f_k = T^{ra_n+v_{n-r-1}-k} f_k = f_{ra_n+v_{n-r-1}}$ as in (α), since $ra_n+v_{n-r-1} \in (A)$] $T^s f_k = T^{j+s-v_{n-r-1}} T f_{ra_n+v_{n-r-1}}$. Using (3) and $\|T\|_x \leq 2$, we get



$$\|T^s f_k\|_x \leq 2^{j+s-v_{n-r-1}-1} / a_{n-r} < 2^s / a_{n-r} < 2^{2b_m} / a_{n-r} < 1$$

if a_{n-r} is large enough.

Case (B): $(r-1)a_n + v_{n-r} < k < ra_n$

$$a_m + b_m + (r-1)a_n + v_{n-r} < k+s < (m-1)a_m + b_m + ra_n \dots (*B)$$

The hypothesis $k > (m-1)a_m$ implies $n > m$. [Note: Hence, $n \neq 2$].

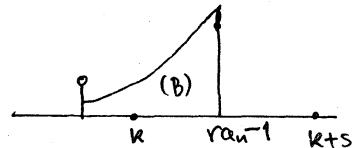
(i) $k+s < ra_n$: Then, $k+s \in (B)$. Hence,

$$T^s f_k = 2^{[(r-\frac{1}{2})a_n - k]/b_{n-1}} \cdot x^{k+s} = 2^{s/b_{n-1}} f_{k+s}.$$

So, $\|T^s f_k\|_x = 2^{s/b_{n-1}} \leq 2^{s/b_m} \leq 4$, since $s \leq 2b_m$.

(ii) $k+s \geq ra_n$: Since $ra_n-1 \in (B)$, as in (i), we have

$$T^s f_k = 2^{(ra_n-1-k)/b_{n-1}} \cdot T^{k+s-ra_n} \cdot T f_{ra_n-1}.$$



So, using (4) and $\|T\|_x \leq 2$ [Note: and $ra_n-k \leq s \leq 2b_m$], we

get

$$\begin{aligned} \|T^s f_k\|_x &\leq 2^{s/b_{n-1}} \cdot 2^{k+s-ra_n} / a_n \\ &\leq 2^{s/b_m} \cdot 2^s / a_n \leq 4 \cdot 2^{2b_m/a_n} < 1 \end{aligned}$$

if a_n is large enough, since $n > m$.

Case (C): $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$

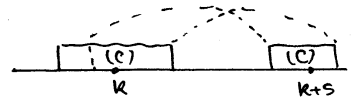
We have $T^s f_k = x^{k+s} - b_n x^{k+s-b_n}$ and

$$b_m + r(a_n + b_n) \leq k+s \leq (m-1)a_m + b_m + (n-1)a_n + rb_n \dots (*C)$$

The hypothesis implies $n \geq m$. We consider separately the cases $n = m$ and $n > m$.

(i) $n = m$:

From (*C), $k+s \geq (r+1)(a_n + b_n)$.



(α) $k+s \leq (n-1)a_n + (r+1)b_n$ (which implies $r \leq n-2$ and $k+s \in$

(C)): We get $T^s f_k = f_{k+s}$, so $\|T^s f_k\|_x = 1$.

(β) $k+s > (n-1)a_n + (r-1)b_n$: Since $k+s < (r+2)(a_n + b_n)$ from

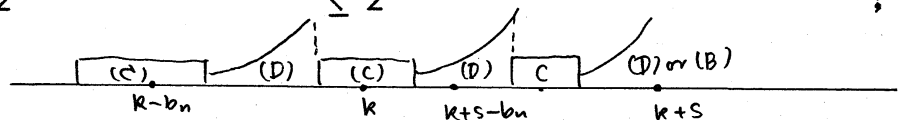
(*C) [Note: if b_n is large enough], we have

if $r < n-2$, then $k+s \in (D)$, so

$$\|x^{k+s}\|_x = 2^{[k+s-(r+\frac{3}{2})b_n]/na_n} \leq 2^{[2(n-1)a_n - \frac{1}{2}b_n]/na_n};$$

if $r \geq n-2$, then $k+s \in (B)$, so

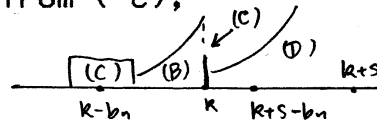
$$\|x^{k+s}\|_x = 2^{[k+s-\frac{1}{2}a_{n+1}]/b_n} \leq 2^{[nb_n+2(n-1)a_n-\frac{1}{2}a_{n+1}]/b_n};$$



similary, since $k+s \leq 2(n-1)a_n + (r+1)b_n$ from (*C),

if $r < n-1$, then $k+s-b_n \in (D)$, so

$$\|x^{k+s-b_n}\|_x = 2^{\lfloor k+s-(r+\frac{3}{2})b_n \rfloor / na_n} \leq 2^{\lfloor 2(n-1)a_n - \frac{1}{2}b_n \rfloor / na_n};$$



if $r = n-1$, then $k+s-b_n \in (B)$, so

$$\|x^{k+s-b_n}\|_x = 2^{\lfloor k+s-b_n - \frac{1}{2}a_{n+1} \rfloor / b_n} \leq 2^{\lfloor 2(n-1)a_n - \frac{1}{2}a_{n+1} \rfloor / b_n}.$$

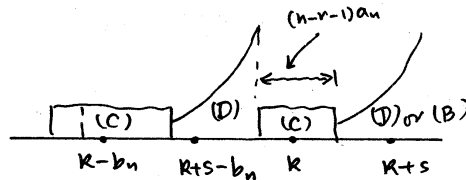
So, if b_n, a_{n+1} are chosen large enough, we get $\|T^s f_k\|_x \leq 1$ in each case.

(ii) $n > m$:

(α) $k+s \leq (n-1)a_n + rb_n$: Then, $k+s \in (C)$. So, $T^s f_k = f_{k+s}$, and $\|T^s f_k\|_x = 1$.

(β) $k+s > (n-1)a_n + rb_n$: If $r < n-1$, we have $k+s < (r+1)(a_n + b_n)$, so $k+s \in (D)$ and

$$\begin{aligned} \|x^{k+s}\|_x &= 2^{\lfloor k+s-(r+\frac{1}{2})b_n \rfloor / na_n} \\ &\leq 2^{\lfloor (n-1)a_n + 2b_n - \frac{1}{2}b_n \rfloor / na_n} \leq 1/b_n^2 \quad [\text{Note: by (*C)}] \end{aligned}$$



and the same estimate holds for $\|x^{k+s-b_n}\|_x$; while if $r =$

$n-1$, then the above estimate holds for $\|x^{k+s-b_n}\|_x$ and, on the other hand, since $k+s \in (B)$,

$$\begin{aligned} \|x^{k+s}\|_x &= 2^{\lfloor k+s - \frac{1}{2}a_{n+1} \rfloor / b_n} \\ &\leq 2^{\lfloor a_n + 2b_n - \frac{1}{2}a_{n+1} \rfloor / b_n} < 1/a_{n+1} \quad [\text{Note: by (*C)}]. \end{aligned}$$

So, in each case we get $\|T^s f_k\|_x \leq 1$.

Case (D): $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$

We have $T^s f_k = 2^{\lfloor (r - \frac{1}{2})b_n - k \rfloor / na_n} \cdot x^{k+s}$ and

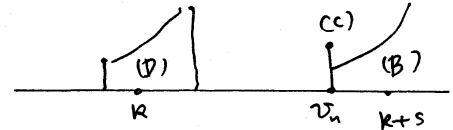
$$a_m + b_m + (n-1)a_n + (r-1)b_n < k+s < (m-1)a_m + b_m + r(a_n + b_n) \dots (*D)$$

and the hypothesis $k > (m-1)a_m$ again implies $n \geq m$ and we consider separately the cases $n = m$ and $n > m$.

(i) $n = m$:

From (*D) we have $k+s > (n-1)a_n + rb_n$. We subdivide this case into four cases (α)-(δ).

(α) $k+s > (n-1)(a_n + b_n)$:



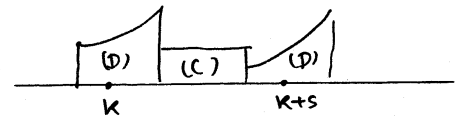
In this case, $k+s \in (B)$ and from (*D)

$$\|x^{k+s}\|_x = 2^{(k+s - \frac{1}{2}a_{n+1})/b_n} \leq 2^{\lfloor (n+1)b_n - \frac{1}{2}a_{n+1} \rfloor / b_n}.$$

So, $\|T^s f_k\|_x \leq 2^{b_n/a_n + \lfloor (n+1)b_n - \frac{1}{2}a_{n+1} \rfloor / b_n} \leq 1$ if a_{n+1} is large enough.

In the remaining three cases (β)-(δ), we always assume $(n-1)a_k + rb_n \leq k+s \leq (n-1)(a_n + b_n)$.

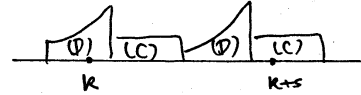
(β) $k+s < (r+1)(a_n + b_n)$, $r < n-1$:



Since $k+s \in (D)$, we have $\|x^{k+s}\|_x = 2^{\lfloor k+s - (r+\frac{1}{2})b_n \rfloor / na_n}$,
so $\|T^s f_k\|_x = 2^{(s-b_n)/na_n} \leq 2$ since $s-b_n \leq (n-1)a_n$.

$$(7) \quad \underline{(r+1)(a_n + b_n) \leq k+s \leq (n-1)a_n + (r+1)b_n, \quad r < n-1 :}$$

In this case, $k+s \in (C)$ and, by (2),



$$\|x^{k+s}\|_x \leq 2b_n^r + b_n^{r+1} \|x^{k+s-(r+1)b_n}\|_x.$$

Now, $k+s-(r+1)b_n \leq (n-1)a_n$, so $\|x^{k+s-(r+1)b_n}\|_x$ is bounded by a function of a_n , so $\|x^{s+k}\|_x \leq b_n^{r+2}$ if b_n is large enough. Also, $k \geq (r+1)(a_n + b_n) - s \geq rb_n - na_n$, so

$$\begin{aligned} \|T_{f_k}^{s,k}\|_x &\leq 2^{\lfloor (r - \frac{1}{2})b_n - k \rfloor} \cdot b_n^{r+2} \\ &\leq 2^{(na_n - \frac{1}{2}b_n)/na_n} \cdot b_n^{r+2} \leq 1 \end{aligned}$$

if b_n is large enough.

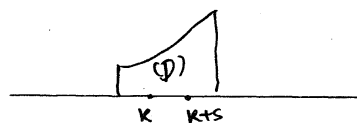
$$(8) \quad \underline{k+s > (n-1)a_n + (r+1)b_n, \quad r < n-2 :}$$

Since $s < 2b_n$, we have $k+s < (r+2)(a_n + b_n)$, i.e., $k+s \in (D)$. So,

$$\begin{aligned} \|x^{k+s}\|_x &= 2^{\lfloor k+s-(r+\frac{1}{2})b_n \rfloor / na_n} \quad \text{[Diagram: A horizontal line with points k and k+s. Above it are five trapezoidal shapes labeled (D), (C), (D), (C), (D) from left to right.] } \\ &\leq 2^{\lfloor k+(n-1)a_n-(r+\frac{1}{2})b_n \rfloor / na_n}. \end{aligned}$$

So, $\|T_{f_k}^{s,k}\|_x \leq 2^{\lfloor (n-1)a_n - b_n \rfloor / na_n} \leq 1$ if b_n is large enough.

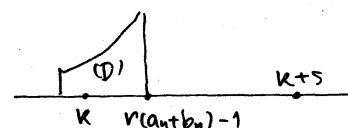
(ii) $n > m$:



(α) $k+s < r(a_n + b_n)$: Then, $s+k \in (D)$, so $x^{k+s} = 2^{[k+s-(r-\frac{1}{2})b_n]/na_n} f_{k+s}$. So, $T^s f_k = 2^{s/na_n} f_{k+s}$ and $\|T^s f_k\|_x = 2^{s/na_n} \leq 2^{2b_m/na_n} \leq 2$ since $n > m$.

(β) $k+s \geq r(a_n + b_n)$: Using $\|T\|_x \leq 2$ and the fact that, by (5), $\|T^{s/r(a_n+b_n)-1} f_k\|_x \leq 1/b_n$, we get

$$\|T^s f_k\|_x \leq 2^s / b_n \leq 2^{2b_m} / b_n \leq 1$$



[Note: $s+k \leq r(a_n + b_n) - 1 + s$] if b_n is large enough, since $m < n$. This completes the proof of Lemma 3.

Proof of Lemma 4. Let $h = Q_m(g)$, so $\|h\|_* \leq C_m$. By Lemma 1, we can find $q \in P_{(m-2)a_m}$ with

$$\|P_{(m-2)a_m}(qh) - x^{ra_m}\|_* \leq 1/a_m C_m ; \text{ and}$$

$$\|q\|_* \leq K = K(1/a_m C_m, 1/a_m C_m, (m-2)a_m).$$

Let $\psi = x^{a_m} q \in P_{(m-1)a_m}$. Then,

$$\|P_{(m-1)a_m}(\psi h) - x^{(r+1)a_m}\|_* \leq 1/a_m C_m ,$$

so

$$\|P_{(m-1)a_m}(\psi h) - x^{(r+1)a_m}\|_x \leq 1/a_m \dots\dots(7)$$

Let $\phi = b_m^{-1} x^{b_m} \psi$. Since $\|x^t - b_m^{-1} x^{t+b_m}\|_x = 1/b_m$ for

$a_m \leq t \leq (m-1)a_m$, we get

$$\begin{aligned} & \| p_{(m-1)a_m+b_m}(\phi h) - p_{(m-1)a_m}(\phi h) \|_x \\ &= \| b_m^{-1} x^{b_m} p_{(m-1)a_m}(\phi h) - p_{(m-1)a_m}(\phi h) \|_x \\ &\leq \| \phi h \|_* / b_m = \| qh \|_* / b_m \leq KC_m / b_m \leq 1/a_m \dots\dots(8) \end{aligned}$$

provided b_m is large enough, as K and C_m depend only on $a_1, b_1, \dots, a_m$. Now the degree of ϕh does not exceed $2(m-1)a_m + b_m \leq 2(a_m + b_m)$, so (D) [Note: the maximum error is at $k=2(m-1)a_m+b_m$ and we have $\| \phi h \|_* \leq \| \phi \|_\infty \| h \|_* \leq \| \phi \|_* \| h \|_*$] gives

$$\begin{aligned} \| \phi h - p_{(m-1)a_m+b_m}(\phi h) \|_x &\leq 2 \frac{[2(m-1)a_m - \frac{1}{2} b_m] / m a_m}{[2(m-1)a_m - \frac{1}{2} b_m] / a_m} \| \phi h \|_* \\ &\leq 2 \frac{[2(m-1)a_m - \frac{1}{2} b_m] / a_m}{[2(m-1)a_m - \frac{1}{2} b_m] / a_m} \cdot KC_m / b_m \\ &\leq 1/a_m \dots\dots(9) \end{aligned}$$

again if b_m is large enough. Now apply (6), noting that

$$\phi(x) = \sum_{s=a_m+b_m}^{(m-1)a_m+b_m} \lambda_s x^s$$

where $\sum |\lambda_s| \leq K/b_m$, and that $T^s Q_m(g) = x^s h$ and $\phi(T)Q_m(g) = \phi h$, we get

$$\begin{aligned} \| \phi(T)g - \phi h \|_x &= \| \phi(T)g - \phi(T)Q_m(g) \|_x \\ &\leq \sum_{s=a_m+b_m}^{(m-1)a_m+b_m} |\lambda_s| \| T^s g - T^s Q_m(g) \|_x \end{aligned}$$

$$\leq K \cdot 4 \|g\|_x / b_m = 4K/b_m < 1/a_m \dots\dots(10)$$

Now from (1), we get $\|x^{(r+1)a_m} - 1\|_x \leq 2/a_{m-r-1}$ and combining this with (7), (8), (9), (10) gives

$$\|\phi(T)g - 1\|_x < 2/a_{m-r-1} + 4/a_m < 3/a_{m-r-1}.$$

This proves the lemma.

Proof of THEOREM. Suppose $Y \neq \{0\}$. Let $g \in Y$ with $\|g\|_x = 1$ and any $\varepsilon > 0$ be given. To show that $Y = X$, it suffices to show that we can find a polynomial ϕ with $\|\phi(T)g - 1\|_x < \varepsilon$. We choose $k > 1$ with $3/a_k < \varepsilon$. In view of Lemma 4, it will be sufficient to find $r \geq 1$ and m with $m-r-1 \geq k$ such that $\|p_{ra_m} Q_m g\|_* \geq 1/a_m$. So, we suppose on the contrary that

$$r \geq 1, m-r-1 \geq k \implies \|p_{ra_m} Q_m g\|_* < 1/a_m \dots\dots(11)$$

and we shall deduce a contradiction.

For each m we can find $D_m > 0$, depending only on $a_1, b_1, \dots, b_m$, such that $\|x^j\|_x \leq D_m, 0 \leq j \leq v_m$.

Now we write $g = \sum_{j=0}^{\infty} \alpha_j f_j$, where $\sum |\alpha_j| = \|g\|_x = 1$ and for $n > 1$ we can write

$$\sum_{j=0}^{(n-1)a_n} \alpha_j f_j = \sum_{j=0}^{(n-1)a_n} \beta_{nj} x^j.$$

We have also, for $n > 2$,

$$Q_n \left(\sum_{j > (n-1)a_n} \alpha_j f_j \right) = \sum_{j=0}^{v_{n-1}} \lambda_{nj} x^j$$

where

$$\lambda_{nj} = -a_n \sum_{m=n+1}^{\infty} \alpha_{j+(m-n)a_m} \dots\dots\dots(12)$$

and so

$$Q_n g = \sum_{j=0}^{(n-1)a_n} \beta_{nj} x^j + \sum_{j=0}^{v_{n-1}} \lambda_{nj} x^j \dots\dots\dots(12)$$

From (11) and (12) we can deduce

$$r \geq 1, m-r-1 \geq k \implies \sum_{j=v_{m-1}+1}^{ra_m} |\beta_{mj}| < 1/a_m \dots\dots(13)$$

From the definition of the polynomial f_j we can see that if $ra_m + v_{m-r-2} < j \leq ra_m + v_{m-r-1}$, then [Note: $j \in (A)$ and no x^j term in f_ℓ for $ra_m + v_{m-r-1} < \ell \leq (m-1)a_m$] $\beta_{mj} = a_{m-r} \alpha_j$, hence from (13) with r replaced by $r+1$ we get

$$r \geq 1, m-r-2 \geq k \implies \sum_{j=ra_m+v_{m-r-2}+1}^{ra_m+v_{m-r-1}} |\alpha_j| < 1/a_{m-r} a_m \dots\dots(14)$$

[Note: (14) covers the shaded area in Fig. 3.1] and combining this with (12) we get

$$\begin{aligned} n \geq k+2 \implies \sum_{j=v_{n-2}+1}^{v_{n-1}} |\lambda_{nj}| &< |a_n| \sum_{m=n+1}^{\infty} \sum_j |\alpha_{j+(m-n)a_m}| \\ &< \sum_{m=n+1}^{\infty} 1/a_m < 2/a_{n+1} \dots\dots(15) \end{aligned}$$

From (11) with $r = 1$ we get, if $n \geq k+2$, using (12)

$$\|p_{a_n} Q_n g\|_x < 1/a_n, \text{ so } \sum_{j=0}^{v_{n-1}} |\beta_{nj} + \lambda_{nj}| < 1/a_n \dots\dots(16)$$

and from (15) and (16) we get

$$\sum_{j=v_{n-2}+1}^{v_{n-1}} |\beta_{nj}| < 2/a_n.$$

Now, if we write

$$\sum_{j=v_{n-2}+1}^{v_{n-1}} \alpha_j f_j = \sum_{j=0}^{v_{n-1}} \beta'_{nj} x^j$$

then the correspondence between $(\alpha_{v_{n-2}+1}, \dots, \alpha_{v_{n-1}})$ and $(\beta'_{v_{n-2}+1}, \dots, \beta'_{v_{n-1}})$ is one-to-one, so there is a constant D'_{n-1} , depending only on $a_1, \dots, b_{n-1}$, such that

$$\sum_{j=v_{n-2}+1}^{v_{n-1}} |\alpha_j| \leq D'_{n-1} \sum_{j=v_{n-2}+1}^{v_{n-1}} |\beta'_{nj}|.$$

Now we observe that if $v_{n-1} < j \leq (n-1)a_n$, then f_j has no x^i term for $v_{n-2} < i \leq v_{n-1}$. Hence, $\beta_j = \beta'_j$ for $v_{n-2} < j \leq v_{n-1}$, whence

$$\sum_{j=v_{n-2}+1}^{v_{n-1}} |\alpha_j| \leq D'_{n-1} \sum_{j=v_{n-2}+1}^{v_{n-1}} |\beta_{nj}| < 2D'_{n-1}/a_n < 1/\sqrt{a_n}$$

if a_n is large enough, as D'_{n-1} depends only on $a_1, \dots, b_{n-1}$. Putting $m = n-1$, we conclude that

$$m \geq k+1 \implies \sum_{j=v_{m-1}+1}^{v_m} |\alpha_j| \leq 1/\sqrt{a_{m+1}} \quad \dots\dots(17)$$

[Note: (17) covers the shaded area in Fig. 3.2]. Using (17) in (12) we find

$$\begin{aligned} n \geq k+1 \implies \sum_{j=0}^{v_{n-1}} |\lambda_{nj}| &\leq a_n \sum_{m=n+1}^{\infty} \sum_{j=0}^{v_{n-1}} |\alpha_{j+(m-n)a_m}| \\ &\leq a_n \sum_{m=n+1}^{\infty} \sum_{j=v_{m-1}}^{v_m} |\alpha_j| \\ &\leq a_n \sum_{m=n+1}^{(n-1)a_n} 1/\sqrt{a_{m+1}} < 1/a_{n+1} \end{aligned}$$

and combining this with (16) gives

$$\begin{aligned} \sum_{j=0}^{v_{n-1}} |\beta_{nj}| &< 2/a_n \text{ for } n \geq k+2 \\ \text{i.e., } \|p_{v_{n-1}} \left(\sum_{j=0}^{(n-1)a_n} \alpha_j f_j \right)\|_* &< 2/a_n \quad \dots\dots(18) \end{aligned}$$

Now from (17) we get

$$\sum_{j=v_{n-1}+1}^{(n-1)a_n} |\alpha_j| \leq 1/\sqrt{a_{n+1}} < 1/a_n, \quad (n \geq k+1)$$

and noting that if $v_{n-1} < j \leq (n-1)a_n$ then $\|p_{v_{n-1}} f_j\|_* \leq a_{n-1}$ [Note: $p_{v_{n-1}} f_j = 0$ ($j \in (B)$, $1 \leq r \leq n-1$), $-a_{n-1}$ ($j \in (A)$, $r = 1$), 0 ($j \in (A)$, $2 \leq r \leq n-1$)], we conclude that

$$\|p_{v_{n-1}} \left(\sum_{j=v_{n-1}+1}^{(n-1)a_n} \alpha_j f_j \right)\|_* \leq \sum_{j=v_{n-1}+1}^{(n-1)a_n} |\alpha_j| \|p_{v_{n-1}} f_j\|_*$$

$$< a_{n-1}/a_n$$

and combining this with (18) gives

$$\left\| \sum_{j=0}^{n-1} \alpha_j f_j \right\|_* < (a_{n-1} + 2)/a_n, \text{ if } n \geq k+2$$

whence

$$\sum_{j=0}^{n-1} |\alpha_j| = \left\| \sum_{j=0}^{n-1} \alpha_j f_j \right\|_x < D_{n-1}(a_{n-1} + 2)/a_n,$$

which tends to zero as $n \rightarrow \infty$ if $\{a_n\}$ increasing fast enough.

Since $\sum |\alpha_j| = 1$, this is the desired contradiction.

References

1. C.J. Read, A solution of the invariant subspace problem on a Banach space, Bull. London Math. Soc. 16 (1984), 337-401.

Case k

$$\begin{aligned}
 f_0 & \begin{bmatrix} 0 \\ \vdots \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} a_1 \\ \vdots \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} a_1 + b_1 \\ \vdots \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} a_2 \\ \vdots \end{bmatrix} \\
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} v_2 \\ \vdots \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} a_3 \\ \vdots \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} 2a_3 \\ \vdots \end{bmatrix} \\
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} a_3 + b_3 \\ \vdots \\ 2a_3 + b_3 \end{bmatrix} \\
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} v_3 \\ \vdots \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}
 \end{aligned}$$

Case k

$$\begin{aligned}
 (C) & \begin{bmatrix} v_{n-1} \\ \vdots \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} a_n \\ a_n + v_{n-2} \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} 2a_n \\ 2a_n + v_{n-3} \end{bmatrix} \\
 & \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \\
 (A) & \begin{bmatrix} (n-4)a_n \\ (n-4)a_n + v_3 \end{bmatrix} \\
 (B) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} (n-3)a_n \\ (n-3)a_n + v_2 \end{bmatrix} \\
 (C) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} (n-2)a_n \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (A) & \begin{bmatrix} (n-1)a_n \\ \vdots \end{bmatrix}
 \end{aligned}$$

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$$\begin{aligned}
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} a_n + b_n \\ (n-1)a_n + b_n \end{bmatrix} \\
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} 2(a_n + b_n) \\ (n-1)a_n + 2b_n \end{bmatrix} \\
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} 3(a_n + b_n) \\ (n-1)a_n + 3b_n \end{bmatrix} \\
 & \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} \\
 (C) & \begin{bmatrix} (n-3)(a_n + b_n) \\ \vdots \\ (n-1)a_n + (n-3)b_n \end{bmatrix} \\
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} (n-2)(a_n + b_n) \\ (n-1)a_n + (n-2)b_n \end{bmatrix} \\
 (D) & \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
 (C) & \begin{bmatrix} v_n \\ \vdots \end{bmatrix}
 \end{aligned}$$

Fig. 1

