

Stochastic complex Ginzburg-Landau equation with space-time white noise

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Abstract

We study the stochastic complex Ginzburg-Landau equation with complex-valued space-time white noise on the three dimensional torus. This nonlinear equation is so singular that it can only be understood in a renormalized sense. We prove local well-posedness of it in the framework of paracontrolled distribution theory. This article is an announcement of the authors' full paper with the same title.

1 Introduction

In this article, we report local well-posedness of the stochastic complex Ginzburg-Landau equation (CGL) with complex-valued space-time white noise ξ in the three-dimensional torus $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$

$$(P) \quad \begin{cases} \partial_t u = (i + \mu)\Delta u + \nu(1 - |u|^2)u + \xi & \text{on } (0, \infty) \times \mathbf{T}^3, \\ u(0, \cdot) = u_0(\cdot). \end{cases}$$

Here, $i = \sqrt{-1}$, μ is a positive constant and ν is a complex constant. There are a lot of preceding results on CGL; for example, [Hai02], [BS04b], [BS04a], [KS04], [Yan04], [Oda06], [PG11].

First of all, we explain difficulty of this problem. We rewrite (P) as $\mathcal{L}^1 u = \nu(1 - |u|^2)u + u + \xi$ and consider a stationary solution to the linear equation $\mathcal{L}^1 Z = \xi$ on $(0, \infty) \times \mathbf{T}^3$, where $\mathcal{L}^1 = \partial_t - \{(i + \mu)\Delta - 1\}$. Then, by setting $P_t^1 = e^{t\{(i + \mu)\Delta - 1\}}$ and $I(u)_t = \int_{-\infty}^t P_{t-s}^1 u_s ds$ for distribution-valued functions u on $[0, \infty)$, we see that the solution is given by $Z_t = I(\xi)_t$ formally and it is not a function but a distribution with respect to the space variable in dimension three. More precisely, Z_t belongs to $C^{-\frac{1}{2}-\kappa}$ for any $\kappa > 0$, where C^α is the Hölder-Besov space with the Hölder exponent $\alpha \in \mathbf{R}$; see Section 2 for definition. Hence the products Z_t^2 , $Z_t \overline{Z_t}$, $\overline{Z_t}^2$ and so on

are not defined a priori. Since the irregularity of the solution to (P) comes from the white noise, it is natural to guess that the space regularity of u_t is not better than that of Z_t and that the product $|u_t|^2 u_t = u_t^2 \overline{u_t}$ is not defined a priori. Hence, in order to define a notion of solution to (P), it is necessary to define the product in some way.

Hairer [Hai14] and Gubinelli-Imkeller-Perkowski [GIP15] developed great results in order to overcome such difficulty, respectively. Their works are breakthrough in the theory of singular stochastic partial differential equation and a lot of results are shown after the works; for example, [BK16], [FH14], [Hos16a], [ZZ15], [CC13], [MW16], [Hos16b], [GP17], [BB16].

We also use them to obtain local well-posedness of CGL. In the authors' full paper [IHN17], they use the both theories and establish the well-posedness; however, in this article, we only state the result obtained by the theory of paracontrolled distributions developed in [GIP15].

2 Notation

Before starting our discussion, we introduce notations. We denote by $\mathcal{D}$ the space of all smooth functions on $\mathbf{T}^3$ and by $\mathcal{D}'$ its dual. For every $\alpha \in \mathbf{R}$, we denote by $\mathcal{C}^\alpha$ the Hölder-Besov space, which is defined by the completion of the space of smooth functions on $\mathbf{T}^3$ under the Hölder-Besov norm $\|\cdot\|_{\mathcal{C}^\alpha}$. To define the norm, we use the Littlewood-Paley block $\{\Delta_m = \mathcal{F}^{-1} \rho_m \mathcal{F}\}_{m=-1}^\infty$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transformation and its inverse, respectively, and $\{\rho_m\}_{m=-1}^\infty$ is the dyadic partition of unity. The norm is defined by

$$\|f\|_{\mathcal{C}^\alpha} = \sup_{m \geq -1} 2^{m\alpha} \|\Delta_m f\|_{L^\infty}.$$

We denote by $C_T \mathcal{C}^\alpha$ the space of all $\mathcal{C}^\alpha$ -valued continuous functions on $[0, T]$ for every $T > 0$. We define $C_T^\delta \mathcal{C}^\alpha$ by the space of all δ -Hölder continuous functions from $[0, T]$ to $\mathcal{C}^\alpha$ and set $\mathcal{L}_T^{\alpha, \delta} = C_T \mathcal{C}^\alpha \cap C_T^\delta \mathcal{C}^{\alpha-2\delta}$.

Next we introduce the notion of paradifferential calculus. For every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, we define the resonance $f \odot g$ and the paraproduct $f \otimes g$. They give the decomposition $fg = f \otimes g + f \odot g + f \circ g$. The paraproduct $f \otimes g$ can be defined for any $\alpha, \beta \in \mathbf{R}$, but the resonance $f \odot g$ can be defined for $\alpha + \beta > 0$. Hence, in order to define products fg , it is necessary that $\alpha + \beta > 0$ holds.

For more information about the Hölder-Besov spaces and the paradifferential calculus, we consult [BCD11].

3 Main result

In this section, we state our main result and give a sketch of the proof.

We define a solution to (P) as a limit of solutions to renormalized equations. To introduce the renormalized equations, we explain how to mollify the white noise. Roughly speaking, we define smeared noise ξ^ϵ for a parameter $0 < \epsilon < 1$ by cutting off high frequency part of the Fourier transform of ξ . Let χ be a smooth function defined on $\mathbf{R}^3$ such that (1) $\text{supp } \chi \subset B(0, 1)$, where $B(x, r)$ denotes the open ball of radius $r > 0$ and center $x \in \mathbf{R}^3$, (2) $\chi(0) = 1$. We set $\chi^\epsilon(k) = \chi(\epsilon k)$ for every $k \in \mathbf{Z}^3$. Define $\mathbf{e}_k(x) = e^{-2\pi i k \cdot x}$ for every $k \in \mathbf{Z}^3$ and $x \in \mathbf{T}^3$. Here, the dot $\cdot$ denotes the usual inner product. We define ξ^ϵ by

$$\xi^\epsilon = \sum_{k \in \mathbf{Z}^3} \chi^\epsilon(k) \hat{\xi}(k) \mathbf{e}_k.$$

Here, $\{\hat{\xi}(k)\}_{k \in \mathbf{Z}^3}$ denotes the Fourier transform of ξ and it has the same law with independent copies of complex-valued white noise on $\mathbf{R}$. We see that $\xi^\epsilon \rightarrow \xi$ in an appropriate topology. For such smeared noise ξ^ϵ , we consider the renormalized equation

$$(P') \quad \begin{cases} \partial_t u^\epsilon = (i + \mu) \Delta u^\epsilon + \nu(1 - |u^\epsilon|^2) u^\epsilon + \nu \mathbf{c}^\epsilon u^\epsilon + \xi^\epsilon, & \text{on } (0, \infty) \times \mathbf{T}^3, \\ u(0, \cdot) = u_0(\cdot). \end{cases}$$

Here $\mathbf{c}^\epsilon$ is a complex constant defined by $\mathbf{c}^\epsilon = 2(\mathbf{c}_1^\epsilon - \overline{\nu \mathbf{c}_{2,1}^\epsilon} - 2\nu \mathbf{c}_{2,2}^\epsilon)$, where $\mathbf{c}_1^\epsilon$, $\mathbf{c}_{2,1}^\epsilon$ and $\mathbf{c}_{2,2}^\epsilon$ are complex constants specified later. We note that $|\mathbf{c}^\epsilon| \rightarrow \infty$ as $\epsilon \downarrow 0$. We can make sense of a solution to (P) as the limit of solutions to (P'). The next is our main result:

Theorem 1. *Let $u_0 \in C^{-\frac{2}{3}+\kappa'}$ for $0 < \kappa' \ll 1$. Consider (P'). There exist a unique process u^ϵ and a random time T_*^ϵ such that*

- u^ϵ solves (P') on $[0, T_*^\epsilon) \times \mathbf{T}^3$,
- T_*^ϵ converges to some a.s. positive random time T_* in probability,
- u^ϵ converges to some process u defined on $[0, T_*) \times \mathbf{T}^3$ in the sense that $\sup_{0 \leq s \leq T_*/2} \|u_s^\epsilon - u_s\|_{C^{-\frac{2}{3}+\kappa'}} \rightarrow 0$ as $\epsilon \rightarrow 0$ in probability. Furthermore, u is independent of the choice of ξ^ϵ .

The proof of this theorem consists of a deterministic part and a probabilistic part. In the next subsections, we explain them and show the theorem.

3.1 Deterministic part

In the deterministic part, we construct the solution map of (P) from the space $\mathcal{X}_{T_*}^\kappa$ of driving vectors to the space $\mathcal{D}_{T_*}^{\kappa, \kappa'}$ of solutions, where T_* is a life time of a

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