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Nonlinear analysis with endlessly continuable functions

By

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Abstract

We give estimates for the convolution product of an arbitrary number of endlessly continuable functions. This allows us to deal with nonlinear operations for the corresponding resurgent series, e.g. substitution into a convergent power series.

§ 1. Introduction

This is an announcement of our forthcoming paper [KS], the main subject of which is the ring structure of the space $\mathscr{R}$ of *resurgent* formal series.

Recall that $\mathscr{R}$ is the subspace of the ring of formal series $\mathbb{C}[[z]]$ defined as

$$\mathscr{R} := \mathcal{B}^{-1}(\mathbb{C}\,\delta \oplus \hat{\mathscr{R}}), \tag{1.1}$$

where $\mathcal{B}\colon\ \mathbb{C}[[z]] \to \mathbb{C}\,\delta \oplus \mathbb{C}[[\zeta]]$ is the formal Borel transform, defined by

$$\tilde{\varphi}(z) = \sum_{j=0}^{\infty} \varphi_j z^j \mapsto \mathcal{B}(\tilde{\varphi})(\zeta) = \varphi_0\delta + \hat{\varphi}(\zeta), \ \text{ with } \ \hat{\varphi}(\zeta) := \sum_{j=1}^{\infty} \varphi_j \frac{\zeta^{j-1}}{(j-1)!}, \tag{1.2}$$

and $\hat{\mathscr{R}}$ is the subspace of $\mathbb{C}[[\zeta]]$ consisting of all convergent power series which are "endlessly continuable" in the following sense:

Definition 1.1. A convergent power series $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ is said to be *endlessly continuable* if, for every real $L > 0$, there exists a finite subset F_L of $\mathbb{C}$ such that $\hat{\varphi}$ can be analytically continued along every Lipschitz path $\gamma\colon\ [0,1] \to \mathbb{C}$ of length $< L$ such that $\gamma(0) = 0$ and $\gamma\big((0,1]\big) \subset \mathbb{C} \setminus F_L$.

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Resurgence theory was invented by J. Écalle in the early 1980s [E] and has many applications in the study of holomorphic dynamical systems, analytic differential equations, WKB analysis, etc. Here, we are dealing with a simplified version, inspired by [CNP], which is sufficient for most applications (our definition of endlessly continuable functions is slightly more restrictive than the one in [CNP], which is itself less general than the definition of "functions without a cut" on which is based [E] – see also [OD]).

We are interested in nonlinear operations in the space of formal series, like substitution of one or several series without constant term $\tilde{\varphi}_1, \dots, \tilde{\varphi}_r$ into a power series $F(w_1, \dots, w_r)$, defined as

$$F(\tilde{\varphi}_1, \dots, \tilde{\varphi}_r) := \sum_{k \in \mathbb{N}^r} c_k \, \tilde{\varphi}_1^{k_1} \cdots \tilde{\varphi}_r^{k_r} \tag{1.3}$$

for $F = \sum_{k \in \mathbb{N}^r} c_k \, w_1^{k_1} \cdots w_r^{k_r}$. One of the main results of [KS] is

Theorem 1.2. *Let $r \geq 1$ be an integer. Then, for any convergent power series $F(w_1, \dots, w_r) \in \mathbb{C}\{w_1, \dots, w_r\}$ and for any resurgent series $\tilde{\varphi}_1, \dots, \tilde{\varphi}_r \in \mathscr{R}$ without constant term,*

$$F(\tilde{\varphi}_1, \dots, \tilde{\varphi}_r) \in \mathscr{R}.$$

It is the aim of this announcement to outline the idea of the proof, based on the notion of "discrete filtered sets" and quantitative estimates for the convolution of endless continuable functions.

Recall that the convolution in $\mathbb{C}[[\zeta]]$, defined as $\hat{\varphi}_1 * \hat{\varphi}_2 := \mathcal{B}\big(\mathcal{B}^{-1}(\hat{\varphi}_1) \cdot \mathcal{B}^{-1}(\hat{\varphi}_2)\big)$ (*i.e.* the mere counterpart via $\mathcal{B}$ of multiplication of formal series without constant term), takes the following form for convergent power series:

$$\hat{\varphi}_1, \hat{\varphi}_2 \in \mathbb{C}\{\zeta\} \quad \Rightarrow \quad \hat{\varphi}_1 * \hat{\varphi}_2(\zeta) = \int_0^\zeta \hat{\varphi}_1(\zeta_1) \hat{\varphi}_2(\zeta - \zeta_1) \, \mathrm{d}\zeta_1$$
$$\text{for } |\zeta| < \min\{\mathrm{RCV}(\hat{\varphi}_1), \mathrm{RCV}(\hat{\varphi}_2)\},$$

where $\mathrm{RCV}(\,\cdot\,)$ is a notation for the radius of convergence of a power series. The symbol δ which appears in (1.1) and (1.2) is nothing but the convolution unit (obtained from $(\mathbb{C}[[\zeta]], *)$ by adjunction of unit); convolution makes $\mathbb{C}\,\delta \oplus \mathbb{C}[[\zeta]]$ a ring (isomorphic to the ring of formal series $\mathbb{C}[[z]]$), of which $\mathbb{C}\,\delta \oplus \mathbb{C}\{\zeta\}$ is a subring.

It is proved in [OD] that the convolution product of two endlessly continuable functions is endlessly continuable, hence $\mathbb{C}\,\delta \oplus \hat{\mathscr{R}}$ is a subring of $\mathbb{C}\,\delta \oplus \mathbb{C}\{\zeta\}$ and $\mathscr{R}$ is a subring of $\mathbb{C}[[z]]$. However, to reach the conclusions of Theorem 1.2, precise estimates on the convolution product of an arbitrary number of endlessly continuable functions are needed, so as to prove the convergence of the series of holomorphic functions $\sum c_k \, \hat{\varphi}_1^{*k_1} * \cdots * \hat{\varphi}_r^{*k_r}$ and to check its endless continuability.

Remark 1.3. Let Ω be a nonempty closed discrete subset of $\mathbb{C}$. If a convergent power series $\hat{\varphi}$ meets the requirement of Definition 1.1 with

$$F_L = \{\, \omega \in \Omega \mid |\omega| \leq L \,\} \quad \text{for each } L > 0, \tag{1.4}$$

then it is said to be Ω-*continuable*. Pulling back this definition by $\mathcal{B}$, we obtain the definition of an Ω-*resurgent series*, which is a particular case of resurgent series. It is proved in [S2] that the space of Ω-continuable functions is closed under convolution if and only if Ω is stable under addition. Under that assumption, the space of Ω-resurgent series is thus a subring of $\mathbb{C}[[z]]$; nonlinear analysis with Ω-resurgent series is dealt with in [S3], where an analogue of Theorem 1.2 is proved for them.

Our study of endlessly continuable functions and our proof of Theorem 1.2 are based on the notion of "Ω-continuable function", where Ω is a "discrete filtered set" in the sense of [OD]; this is a generalization of the situation described in Remark 1.3, so that the meaning of Ω-continuability will now be more extended (Ω will stand for a family of finite sets $(\Omega_L)_{L\geq 0}$ not necessarily of the form (1.4)).

The plan of the paper is as follows. Discrete filtered sets and the corresponding Ω-continuable functions are defined in Section 2, where we give a refined version of the main result, Theorem 2.11. Then, in Section 3, we state Theorem 3.5 which gives precise estimates for the convolution product of an arbitrary number of Ω-continuable functions, and show how this implies Theorem 2.11. Finally, in Section 4, we sketch the main step of the proof of Theorem 3.5.

§ 2. Discrete filtered sets and Ω-resurgent series

§ 2.1. Ω-continuable functions and Ω-resurgent series

We use the notations

$$\mathbb{N} = \{0, 1, 2, \ldots\}, \qquad \mathbb{R}_{\geq 0} = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}.$$

Definition 2.1 (Adapted from [CNP],[OD]). A *discrete filtered set*, or *d.f.s.* for short, is a family $\Omega = (\Omega_L)_{L\in\mathbb{R}_{\geq 0}}$ of subsets of $\mathbb{C}$ such that

i) Ω_L is a finite set for each $L \in \mathbb{R}_{\geq 0}$,

ii) $\Omega_{L_1} \subseteq \Omega_{L_2}$ for $L_1 \leq L_2$,

iii) there exists $\delta > 0$ such that $\Omega_\delta = \emptyset$.

Given a d.f.s. Ω, we set

$$\mathcal{S}_\Omega := \big\{(\lambda,\omega) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \omega \in \Omega_\lambda\big\}$$

and define $\overline{\mathcal{S}}_\Omega$ as the closure of $\mathcal{S}_\Omega$ in $\mathbb{R}_{\geq 0} \times \mathbb{C}$. We then call

$$\mathcal{M}_\Omega := \big(\mathbb{R}_{\geq 0} \times \mathbb{C}\big) \setminus \overline{\mathcal{S}}_\Omega \quad \text{(open subset of } \mathbb{R}_{\geq 0} \times \mathbb{C})$$

the *allowed open set* associated with Ω.

Definition 2.2. We denote by Π the set of all Lipschitz paths $\gamma\colon [0,t_*] \to \mathbb{C}$ such that $\gamma(0) = 0$, with some real $t_* \geq 0$ depending on γ. We then denote by $\gamma_{|t} \in \Pi$ the restriction of γ to the interval $[0,t]$ for any $t \in [0,t_*]$, and by $L(\gamma)$ the total length of γ. Given a d.f.s. Ω, we call Ω-*allowed path* any $\gamma \in \Pi$ such that

$$\big(L(\gamma_{|t}), \gamma(t)\big) \in \mathcal{M}_\Omega \ \text{ for all } t.$$

We denote by Π_Ω the set of all Ω-allowed paths.

Definition 2.3. Given a d.f.s. Ω, we call Ω-*continuable function* a holomorphic germ $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ which can be analytically continued along any path $\gamma \in \Pi_\Omega$. We denote by $\hat{\mathscr{R}}_\Omega$ the set of all Ω-continuable functions and define

$$\mathscr{R}_\Omega := \mathcal{B}^{-1}\big(\mathbb{C}\,\delta \oplus \hat{\mathscr{R}}_\Omega\big)$$

to be the set of Ω-*resurgent* series.

Example 2.4. Given a closed discrete subset Ω of $\mathbb{C}$, the formula $\tilde{\Omega}_L := \{\,\omega \in \Omega \mid |\omega| \leq L\,\}$ for $L \in \mathbb{R}_{\geq 0}$ defines a d.f.s. $\tilde{\Omega}$. Then the notion of Ω-continuability defined in Remark 1.3 agrees with the notion of $\tilde{\Omega}$-continuability of Definition 2.3. Thus we can identify the set Ω and the d.f.s. $\tilde{\Omega}$.

The relation between the Ω-continuable functions or the Ω-resurgent series of Definition 2.3 and the resurgent series or the endlessly continuable functions of Section 1 is as follows:

Proposition 2.5. *A formal series $\tilde{\varphi} \in \mathbb{C}[[z]]$ is resurgent if and only if there exists a d.f.s. Ω such that $\tilde{\varphi}$ is Ω-resurgent. Correspondingly,*

$$\hat{\mathscr{R}} = \bigcup_{\Omega \in \{\,d.f.s.\,\}} \hat{\mathscr{R}}_\Omega. \tag{2.1}$$

The proof of Proposition 2.5 is outlined in Section 2.3.

Remark 2.6. Observe that

$$\Omega \subset \Omega' \quad \Rightarrow \quad \mathscr{R}_\Omega \subset \mathscr{R}_{\Omega'},$$

where the symbol $\subset$ in the left-hand side stands for the partial order defined by $\Omega_L \subset \Omega'_L$ for each L. Indeed, $\Omega \subset \Omega'$ implies $\mathcal{S}_\Omega \subset \mathcal{S}_{\Omega'}$, hence $\mathcal{M}_{\Omega'} \subset \mathcal{M}_\Omega$ and $\Pi_{\Omega'} \subset \Pi_\Omega$.

Remark 2.7. Obviously, entire functions are always Ω-continuable: $\mathscr{O}(\mathbb{C}) \subset \hat{\mathscr{R}}_\Omega$ for all d.f.s. Ω (e.g. because $\mathscr{O}(\mathbb{C}) = \hat{\mathscr{R}}_\emptyset$, denoting by $\emptyset$ the trivial d.f.s.). The inclusion is not necessarily strict for a non-trivial d.f.s. In fact, one can show that

$$\hat{\mathscr{R}}_\Omega = \mathscr{O}(\mathbb{C}) \quad \Leftrightarrow \quad \forall L > 0,\ \exists L' > L \text{ such that } \Omega_{L'} \subset \{\, \omega \in \mathbb{C} \mid |\omega| < L \,\}.$$

A simple example where this happens is when $\Omega_L = \emptyset$ for $0 \le L < 2$ and $\Omega_L = \{1\}$ for $L \ge 2$.

§ 2.2. Sums of discrete filtered sets

The proof of the following lemma is easy and left to the reader.

Lemma 2.8. *Let Ω and Ω' be two d.f.s. Then the formula*

$$(\Omega * \Omega')_L := \{\, \omega_1 + \omega_2 \mid \omega_1 \in \Omega_{L_1}, \omega_2 \in \Omega'_{L_2}, L_1 + L_2 = L \,\} \cup \Omega_L \cup \Omega'_L \quad \textit{for } L \in \mathbb{R}_{\ge 0}$$

defines a d.f.s. $\Omega * \Omega'$. The law $*$ on the set of all d.f.s. is commutative and associative. The formula $\Omega^{*n} := \underbrace{\Omega * \cdots * \Omega}_{n \text{ times}}$ (for $n \ge 1$) defines an inductive system and

$$\Omega^{*\infty} := \varinjlim_n \ \Omega^{*n}$$

is a d.f.s.

We call $\Omega * \Omega'$ the *sum* of the d.f.s. Ω and Ω'. In [OD] the following is claimed:

Theorem 2.9 ([OD]). *Assume that Ω and Ω' are d.f.s. and $\tilde{\varphi} \in \mathscr{R}_\Omega$, $\tilde{\psi} \in \mathscr{R}_{\Omega'}$. Then the product series $\tilde{\varphi} \cdot \tilde{\psi}$ is $\Omega * \Omega'$-resurgent.*

In view of Proposition 2.5, a direct consequence of Theorem 2.9 is

Corollary 2.10. *The space of resurgent formal series $\mathscr{R}$ is a subring of the ring of formal series $\mathbb{C}[[z]]$.*

Similarly, in view of Proposition 2.5, Theorem 1.2 is a direct consequence of

Theorem 2.11. *Let $r \geq 1$ be integer and let $\Omega_1, \ldots, \Omega_r$ be d.f.s. Then for any convergent power series $F(w_1, \ldots, w_r) \in \mathbb{C}\{w_1, \ldots, w_r\}$ and for any $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r \in \mathbb{C}[[z]]$ without constant term, one has*

$$\tilde{\varphi}_1 \in \mathscr{R}_{\Omega_1}, \ldots, \tilde{\varphi}_r \in \mathscr{R}_{\Omega_r} \quad \Rightarrow \quad F(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) \in \mathscr{R}_{\Omega^*},$$

where $\Omega^* := (\Omega_1 * \cdots * \Omega_r)^{*\infty}$.

The proof of Theorem 2.11 is outlined in Sections 3 and 4.

Theorem 2.9 may be viewed as a particular case of Theorem 2.11 (by taking $F(w_1, w_2) = w_1 w_2$). The proof of the former theorem consists in checking that, for $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$ and $\hat{\psi} \in \hat{\mathscr{R}}_{\Omega'}$, the convolution product $\hat{\varphi} * \hat{\psi}$ can be analytically continued along the paths of $\Pi_{\Omega * \Omega'}$ and thus belongs to $\hat{\mathscr{R}}_{\Omega * \Omega'}$. In the situation of Theorem 2.11, with the notation (1.3), we have $\hat{\psi}_k := c_k \, \hat{\varphi}_1^{*k_1} * \cdots * \hat{\varphi}_r^{*k_r} \in \hat{\mathscr{R}}_{\Omega^*}$ for each nonzero $k \in \mathbb{N}^r$, but some analysis is required to prove the convergence of the series $\sum \hat{\psi}_k$ of Ω^*-continuable functions in $\hat{\mathscr{R}}_{\Omega^*}$; what we need is a precise estimate for the convolution product of an arbitrary number of endlessly continuable functions, and this will be the content of Theorem 3.5.

In the particular case of a closed discrete subset Ω of $\mathbb{C}$ assumed to be stable under the addition and viewed as a d.f.s. as in Example 2.4, we have $\Omega^{*\infty} = \Omega$ and Theorem 3.5 was proved for that case in [S3]. One of the main purposes of [KS] is to extend the techniques of [S3] to the more general setting of endlessly continuable functions.

§ 2.3. Upper closure of a d.f.s. and sketch of proof of Proposition 2.5

The proof of Proposition 2.5 makes use of the notion of "upper closure" of a d.f.s., which allows to simplify a bit the definition of Ω-allowedness for a path, and thus of Ω-continuability for a holomorphic germ.

Definition 2.12. We call *upper closure* of a d.f.s. Ω the family of sets $\tilde{\Omega} = (\tilde{\Omega})_{L \geq 0}$ defined by

$$\tilde{\Omega}_L := \bigcap_{\varepsilon > 0} \Omega_{L+\varepsilon} \quad \text{for every } L \in \mathbb{R}_{\geq 0}.$$

Notice that $\Omega \subset \tilde{\Omega}$.

Lemma 2.13. *Let Ω be a d.f.s. Then its upper closure $\tilde{\Omega}$ is a d.f.s., and there exists a real sequence $(L_n)_{n \geq 0}$ such that $0 = L_0 < L_1 < L_2 < \cdots$ and*

$$L_n < L < L_{n+1} \quad \Rightarrow \quad \tilde{\Omega}_{L_n} = \tilde{\Omega}_L = \Omega_L$$

for every integer $n \geq 0$.

Lemma 2.14. *Let Ω be a d.f.s.. Then*

$$\overline{\mathcal{S}}_\Omega = \mathcal{S}_{\tilde{\Omega}}. \tag{2.2}$$

Consequently, Ω-allowedness admits the following characterization: for a path $\gamma \in \Pi$,

$$\begin{aligned} \gamma \in \Pi_\Omega \quad &\Leftrightarrow \quad \textit{for all } t,\ \gamma(t) \in \mathbb{C} \setminus \tilde{\Omega}_{L(\gamma_{|t})} \\ &\Leftrightarrow \quad \textit{for all } t,\ \exists n \textit{ such that } L(\gamma_{|t}) < L_{n+1} \textit{ and } \gamma(t) \in \mathbb{C} \setminus \tilde{\Omega}_{L_n} \end{aligned}$$

(with the notation of Lemma 2.13).

The proofs of Lemma 2.13 and Lemma 2.14 are easy; see [KS] for the details.

Notice that, given Ω and $\gamma \in \Pi_\Omega$, it may be impossible to find *one* real $L \geq L(\gamma)$ such that $\gamma(t) \in \mathbb{C} \setminus \tilde{\Omega}_L$ for *all* $t > 0$. Take for instance Ω defined by $\Omega_L := \emptyset$ for $0 \leq L < 2$ and $\Omega_L := \{1, 2\}$ for $L \geq 2$ (so $\Omega = \tilde{\Omega}$ in that case), and $\gamma \in \Pi_\Omega$ following the line segment $[0, 3/2]$, then winding once around 2 and ending at $3/2$: no "uniform" L can be found for that path. Observe that, in that example, there exists $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$ with an analytic continuation along γ such that the resulting holomorphic germ at $3/2$ is singular at 1 (but 1 is not singular for the principal branch of $\hat{\varphi}$). This is why the proof of Proposition 2.5 requires a bit of work.

Sketch of the proof of Proposition 2.5. It is sufficient to prove (2.1). Suppose $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$ for a certain d.f.s. Ω and let $L > 0$. Then $\hat{\varphi}$ meets the requirement of Definition 1.1 with $F_L = \tilde{\Omega}_L$, hence $\hat{\varphi} \in \hat{\mathscr{R}}$. Thus $\hat{\mathscr{R}}_\Omega \subset \hat{\mathscr{R}}$.

Suppose now $\hat{\varphi} \in \hat{\mathscr{R}}$. In view of Definition 1.1, we have $\delta := \mathrm{RCV}(\hat{\varphi}) > 0$ and, for each positive integer n, we can choose a finite set F_n such that

(2.3) the germ $\hat{\varphi}$ can be analytically continued along any path $\gamma\colon [0,1] \to \mathbb{C}$ of Π such that $L(\gamma) < (n+1)\delta$ and $\gamma\big((0,1]\big) \subset \mathbb{C} \setminus F_n$.

Let $F_0 := \emptyset$. The property (2.3) holds for $n = 0$ too. For every real $L \geq 0$, we set

$$\Omega_L := \bigcup_{k=0}^{n} F_k \qquad \text{with } n := \lfloor L/\delta \rfloor.$$

One can check that $\Omega := (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ is a d.f.s. which coincides with its upper closure. In [KS], it is shown with the help of Lemma 2.14 that $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$. □

§ 3. The Riemann surface X_Ω – an estimate for the convolution product of several Ω-continuable functions

As announced in Section 2.2, we will now state a theorem from which Theorem 2.11 and thus Theorem 1.2 follow. A few preliminaries are necessary.

In all this section we suppose that Ω is a fixed d.f.s.

Definition 3.1 (Adapted from [OD]). We call Ω-endless Rieman surface any triple $(X, \mathfrak{p}, \underline{0})$ such that X is a connected Riemann surface, $\mathfrak{p}\colon X \to \mathbb{C}$ is a local biholomorphism, $\underline{0} \in \mathfrak{p}^{-1}(0)$, and any path $\gamma\colon [0, t_*] \to \mathbb{C}$ of Π_Ω has a lift $\underline{\gamma}\colon [0, t_*] \to X$ such that $\underline{\gamma}(0) = \underline{0}$.

Notice that, given $\gamma \in \Pi_\Omega$, the lift $\underline{\gamma}$ is unique (because the fibres of $\mathfrak{p}$ are discrete).

It is shown in [KS] how, among all Ω-endless Riemann surfaces, one can construct an object $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ which is cofinal in the following sense:

Proposition 3.2. *There exists an Ω-endless Riemann surface $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ such that, for any Ω-endless Riemann surface $(\tilde{X}, \tilde{\mathfrak{p}}, \underline{\tilde{0}})$, there is a local biholomorphism $\mathfrak{q}\colon X_\Omega \to \tilde{X}$ such that $\mathfrak{q}(\underline{0}_\Omega) = \underline{\tilde{0}}$ and the diagram*

$$\begin{tikzcd} X_\Omega \arrow[rr, "\mathfrak{q}"] \arrow[dr, "\mathfrak{p}_\Omega"'] & & \tilde{X} \arrow[dl, "\tilde{\mathfrak{p}}"] \\ & \mathbb{C} & \end{tikzcd}$$

is commutative. The Ω-endless Riemann surface $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ is unique up to isomorphism and X_Ω is simply connected.

The reader is referred to [KS] for the proof. Notice that in particular, for any d.f.s. Ω' such that $\Omega \subset \Omega'$ as in Remark 2.6, Proposition 3.2 yields a local biholomorphism $\mathfrak{q}\colon X_{\Omega'} \to X_\Omega$.

For an arbitrary connected Riemann surface X, we denote by $\mathscr{O}_X$ the sheaf of holomorphic functions on X. If $\mathfrak{p}\colon X \to \mathbb{C}$ is a local biholomorphism, then there is a natural morphism $\mathfrak{p}^*\colon \mathfrak{p}^{-1}\mathscr{O}_\mathbb{C} \to \mathscr{O}_X$. Recall that $\mathfrak{p}^{-1}\mathscr{O}_\mathbb{C}$ is a sheaf on X, whose stalk at a point $\underline{\zeta}_* \in \mathfrak{p}^{-1}(\zeta_*)$ is $\left(\mathfrak{p}^{-1}\mathscr{O}_\mathbb{C}\right)_{\underline{\zeta}_*} = \mathscr{O}_{\mathbb{C},\mathfrak{p}(\underline{\zeta}_*)} \cong \mathbb{C}\{\zeta - \zeta_*\}$.

Lemma 3.3. *Let $\hat{\varphi} \in \mathbb{C}\{\zeta\} = \mathscr{O}_{\mathbb{C},0}$. Then the following properties are equivalent:*

i) *$\hat{\varphi} \in \mathscr{O}_{\mathbb{C},0}$ is Ω-continuable,*

ii) *$\mathfrak{p}_\Omega^*\hat{\varphi} \in \mathscr{O}_{X_\Omega,\underline{0}_\Omega}$ can be analytically continued along any path on X_Ω,*

iii) *$\mathfrak{p}_\Omega^*\hat{\varphi} \in \mathscr{O}_{X_\Omega,\underline{0}_\Omega}$ can be extended to $\Gamma(X_\Omega, \mathscr{O}_{X_\Omega})$.*

So, the morphism $\mathfrak{p}_\Omega^*$ allows us to identify an Ω-continuable function with a function holomorphic on the whole of the Riemann surface X_Ω:

$$\mathfrak{p}_\Omega^*\big|_{\hat{\mathscr{R}}_\Omega}\colon \hat{\mathscr{R}}_\Omega \xrightarrow{\sim} \Gamma(X_\Omega, \mathscr{O}_{X_\Omega}).$$

The reader is referred to [KS] for the details.

Notation 3.4. For $\delta > 0$ small enough so that $\Omega_\delta = \emptyset$ and for $L > 0$, we set

$$\Pi_\Omega^{\delta,L} := \big\{\, \gamma \in \Pi \mid L(\gamma) \le L \ \text{ and } \ |L(\gamma_{|t}) - \lambda|^2 + |\gamma(t) - \omega|^2 \ge \delta^2 \ \text{ for all } (\lambda,\omega) \in \mathcal{S}_\Omega \text{ and } t \,\big\},$$

$$K_\Omega^{\delta,L} := \big\{\, \underline{\zeta} \in X_\Omega \mid \exists t_* \ge 0 \text{ and } \exists \gamma\colon\ [0,t_*] \to \mathbb{C} \text{ path of } \Pi_\Omega^{\delta,L} \text{ such that } \underline{\zeta} = \underline{\gamma}(t_*) \,\big\}.$$

The condition $\Omega_\delta = \emptyset$ is meant to ensure that $\Pi_\Omega^{\delta,L}$ is a nonempty subset of Π_Ω and hence $K_\Omega^{\delta,L}$ is a nonempty compact subset of X_Ω. In fact, with the notation of Lemma 2.13, $\Omega_\delta = \emptyset$ as soon as $\delta < L_1$ and then, for every $\gamma \in \Pi$,

$$L(\gamma) + \delta \le L_1 \quad \Rightarrow \quad \gamma \in \Pi_\Omega^{\delta,L} \ \text{ for all } L \ge L(\gamma).$$

(In particular, if $L + \delta \le L_1$, then $\Pi_\Omega^{\delta,L}$ consists exactly of the paths of Π which have length $\le L$.) Notice that, if $\Omega_\delta = \emptyset$, then $\Omega_\delta^{*n} = \emptyset$ for all $n \ge 1$ and $\Omega_\delta^{*\infty} = \emptyset$.

On the other hand, for a given d.f.s. Ω, the family $(K_\Omega^{\delta,L})_{\delta,L>0}$ yields an exhaustion of X_Ω by compact subsets.

We are now ready to state the main result of this section, which is the analytical core of our study of the convolution of endelssly continuable functions:

Theorem 3.5. *Let Ω be a d.f.s. and let $\delta, L > 0$ be reals such that $\Omega_{2\delta} = \emptyset$. Then there exist $c, \delta' > 0$ such that $\delta' \le \delta$ and, for every integer $n \ge 1$ and for every $\hat{f}_1, \dots, \hat{f}_n \in \hat{\mathscr{R}}_\Omega$, the function $1 * \hat{f}_1 * \cdots * \hat{f}_n$ belongs to $\hat{\mathscr{R}}_{\Omega^{*n}}$ and satisfies the following estimates:*

(3.1)
$$\sup_{K_{\Omega^{*n}}^{\delta,L}} \Big|\mathfrak{p}_{\Omega^{*n}}^*\big(1 * \hat{f}_1 * \cdots * \hat{f}_n\big)\Big| \le \frac{c^n}{n!} \sup_{\substack{L^{(1)},\dots,L^{(n)}>0 \\ L^{(1)}+\cdots+L^{(n)}=L}} \ \sup_{K_\Omega^{\delta',L^{(1)}}} |\mathfrak{p}_\Omega^* \hat{f}_1| \cdots \sup_{K_\Omega^{\delta',L^{(n)}}} |\mathfrak{p}_\Omega^* \hat{f}_n|.$$

The main step of the proof of Theorem 3.5 is sketched in Section 4 (possible values for c and δ' are indicated in (4.7); the point is that they depend on Ω, δ and L, but not on n nor on $\hat{f}_1, \dots, \hat{f}_n$). The full proof of Theorem 3.5 is in [KS].

Theorem 3.5 implies Theorem 2.11. Let $r \ge 1$ and let $\Omega_1, \dots, \Omega_r$ be d.f.s. Let $\Omega := \Omega_1 * \cdots * \Omega_r$, so that $\Omega_i \subset \Omega$ and $\hat{\mathscr{R}}_{\Omega_i} \subset \hat{\mathscr{R}}_\Omega$ for $i = 1, \dots, r$.

Let $F \in \mathbb{C}\{w_1, \dots, w_r\}$. Denote its coefficients by $(c_k)_{k \in \mathbb{N}^r}$ as in (1.3) and pick $C, \Lambda > 0$ such that $|c_k| \le C\Lambda^{|k|}$ for all k, with the notation $|k| := k_1 + \cdots + k_r$.

Let $\tilde{\varphi}_1, \dots \tilde{\varphi}_r$ be formal series without constant term such that $\hat{\varphi}_i := \mathcal{B}\tilde{\varphi}_i \in \mathscr{R}_{\Omega_i}$ for each i. We get the uniform convergence of $\sum c_k\, \mathfrak{p}_{\Omega^{*\infty}}^*(1 * \hat{\varphi}_1^{*k_1} * \cdots * \hat{\varphi}^{*k_r})$ on every compact subset of $X_{\Omega^{*\infty}}$ by means of estimates of the form

$$\sup_{K_{\Omega^{*\infty}}^{\delta,L}} |c_k\, \mathfrak{p}_{\Omega^{*\infty}}^*(1 * \hat{\varphi}_1^{*k_1} * \cdots * \hat{\varphi}^{*k_r})| \le \sup_{K_{\Omega^{*n}}^{\delta,L}} |c_k\, \mathfrak{p}_{\Omega^{*n}}^*(1 * \hat{\varphi}_1^{*k_1} * \cdots * \hat{\varphi}^{*k_r})| \le C\Lambda^n \cdot \frac{c^n}{n!} \cdot M^n,$$

where $\delta, L > 0$ with $\Omega_{2\delta} = \emptyset$, $n := |k| \ge 1$, $M := \sup_{1 \le i \le r} \sup_{K_\Omega^{\delta',L}} |\mathfrak{p}_\Omega^* \hat{\varphi}_i|$, and the positive reals c and δ' stem from Theorem 3.5. $\square$

§ 4. Sketch of the proof of Theorem 3.5 – construction of adapted deformations of the identity

§ 4.1. Preliminaries

Let Ω be a d.f.s. Recall from Definition 2.2 and 2.3 that Ω-continuability is defined by means of the allowed open subset $\mathcal{M}_\Omega$ of $\mathbb{R}_{\geq 0} \times \mathbb{C}$ associated with Ω, and of paths $\gamma \in \Pi$ which satisfy

$$\tilde{\gamma}(t) := \big(L(\gamma_{|t}), \gamma(t)\big) \in \mathcal{M}_\Omega \quad \text{for all } t.$$

The latter condition is $\gamma \in \Pi_\Omega$. Conversely, notice that

(4.1) if $t \in [0, t_*] \mapsto \tilde{\gamma}(t) = \big(\lambda(t), \gamma(t)\big) \in \mathcal{M}_\Omega$ is a piecewise C^1 path such that $\tilde{\gamma}(0) = (0,0)$ and $\lambda'(t) = |\gamma'(t)|$ for a.e. t, then $\gamma \in \Pi_\Omega$.

Recall from Section 3 that, if $\gamma\colon [0, t_*] \to \mathbb{C}$ is a path of Π_Ω, then there is a unique path $\underline{\gamma}\colon [0, t_*] \to X_\Omega$ such that $\underline{\gamma}(0) = \underline{0}_\Omega$ and $\mathfrak{p}_\Omega \circ \underline{\gamma} = \gamma$.

We fix $\rho > 0$ such that $\Omega_{2\rho} = \emptyset$ and set

$$U := \{\, \zeta \in \mathbb{C} \mid |\zeta| < 2\rho \,\}.$$

For every $\zeta \in U$, the path $\gamma_\zeta\colon t \in [0,1] \mapsto t\zeta$ is in Π_Ω and the formula

$$\zeta \in U \mapsto \mathscr{L}(\zeta) := \underline{\gamma}_\zeta(1) \in X_\Omega$$

defines a holomorphic section $\mathscr{L}$ of $\mathfrak{p}_\Omega$ on U. Let $\underline{U} := \mathscr{L}(U)$: this is an open subset of X_Ω containing $\underline{0}_\Omega$ and we have mutually inverse biholomorphisms

$$\mathscr{L}\colon U \xrightarrow{\sim} \underline{U}, \qquad \mathfrak{p}_\Omega\big|_{\underline{U}}\colon \underline{U} \xrightarrow{\sim} U.$$

Notation 4.1. For any $n \geq 1$, we denote by Δ_n the n-dimensional simplex

$$\Delta_n := \{\, (s_1, \dots, s_n) \in \mathbb{R}^n \mid s_1, \dots, s_n \geq 0 \text{ and } s_1 + \cdots + s_n \leq 1 \,\}$$

with the standard orientation, and by $[\Delta_n] \in \mathscr{E}_n(\mathbb{R}^n)$ the corresponding integration current. For every $\zeta \in U$, we consider the map

$$\vec{\mathscr{D}}(\zeta)\colon \vec{s} = (s_1, \dots, s_n) \mapsto \vec{\mathscr{D}}(\zeta, \vec{s}) := \big(\mathscr{L}(s_1\zeta), \dots, \mathscr{L}(s_n\zeta)\big) \in \underline{U}^n \subset X_\Omega^n,$$

defined in a neighbourhood of Δ_n in $\mathbb{R}^n$, and denote by $\vec{\mathscr{D}}(\zeta)_\#[\Delta_n] \in \mathscr{E}_n(X_\Omega^n)$ the push-forward of $[\Delta_n]$ by $\vec{\mathscr{D}}(\zeta)$.

The reader is referred to [S3] for the notations and notions related to integration currents. Let $n \geq 1$, $\hat{f}_1, \dots, \hat{f}_n \in \hat{\mathscr{R}}_\Omega$ and $\hat{g} := 1 * \hat{f}_1 * \cdots * \hat{f}_n$. Our starting point is

Lemma 4.2 ([S3]). *Let*

$$\beta := (\mathfrak{p}_\Omega^* \hat{f}_1)(\underline{\zeta}_1) \cdots (\mathfrak{p}_\Omega^* \hat{f}_n)(\underline{\zeta}_n)\, \mathrm{d}\underline{\zeta}_1 \wedge \cdots \wedge \mathrm{d}\underline{\zeta}_n,$$

where we denote by $\mathrm{d}\underline{\zeta}_1 \wedge \cdots \wedge \mathrm{d}\underline{\zeta}_n$ *the pullback of the* n*-form* $\mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n$ *in* X_Ω^n *by* $\mathfrak{p}^{\otimes n}\colon X_\Omega^n \to \mathbb{C}^n$. *Then*

$$\zeta \in U \quad \Rightarrow \quad \hat{g}(\zeta) = \vec{\mathscr{D}}(\zeta)_\#[\Delta_n](\beta).$$

§ 4.2. γ-adapted deformations of the identity

Let $L > 0$ and $\delta \in (0, \rho)$. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a path of $\Pi_{\Omega^{*n}}^{\delta,L} \subset \Pi_{\Omega^{*n}}$. We want to study the analytic continuation of $\hat{g}$ along γ, which amounts to studying $(\mathfrak{p}_{\Omega^{*n}}^* \hat{g})\big(\underline{\gamma}(t)\big)$, where $\underline{\gamma}$ is the lift of γ in $X_{\Omega^{*n}}$ which starts at $\underline{0}_{\Omega^{*n}}$. Without loss of generality, we may assume that there exists $a \in (0,1)$ such that

$$0 < |\gamma(a)| < \rho, \qquad \gamma(t) = \tfrac{t}{a}\gamma(a) \text{ for } t \in [0,a], \qquad \gamma\big|_{[a,1]} \text{ is } C^1.$$

Definition 4.3. For $\zeta \in \mathbb{C}$ and $1 \le i \le n$, we set

$$\mathcal{N}(\zeta) := \big\{\, (\underline{\zeta}_1, \ldots, \underline{\zeta}_n) \in X_\Omega^n \mid \mathfrak{p}_\Omega(\underline{\zeta}_1) + \cdots + \mathfrak{p}_\Omega(\underline{\zeta}_n) = \zeta \,\big\},$$
$$\mathcal{N}_i := \big\{\, (\underline{\zeta}_1, \ldots, \underline{\zeta}_n) \in X_\Omega^n \mid \underline{\zeta}_i = \underline{0}_\Omega \,\big\}.$$

We call *γ-adapted deformation of the identity on* $\underline{V}$ any family $(\Psi_t)_{t\in[a,1]}$ of maps

$$\Psi_t\colon \underline{V} \to X_\Omega^n,$$

where $\underline{V}$ is a neighbourhood of $\vec{\mathscr{D}}\big(\gamma(a)\big)(\Delta_n)$ in X_Ω^n, such that $\Psi_a = \mathrm{Id}$, the map $\big(t, \vec{\underline{\zeta}}\big) \in [a,1] \times \underline{V} \mapsto \Psi_t\big(\vec{\underline{\zeta}}\big) \in X_\Omega^n$ is locally Lipschitz, and for any $t \in [a,1]$ and $i = 1, \ldots, n$,

$$\vec{\underline{\zeta}} \in \mathcal{N}\big(\gamma(a)\big) \quad \Rightarrow \quad \Psi_t\big(\vec{\underline{\zeta}}\big) \in \mathcal{N}\big(\gamma(t)\big),$$
$$\vec{\underline{\zeta}} \in \mathcal{N}_i \quad \Rightarrow \quad \Psi_t\big(\vec{\underline{\zeta}}\big) \in \mathcal{N}_i.$$

The above notion is a slight generalization of the "γ-adapted origin-fixing isotopies" which appear in [S3, Def. 5.1]. Adapting the proof of [S3, Prop. 5.2], we get

Proposition 4.4 ([S3]). *If* $(\Psi_t)_{t\in[a,1]}$ *is a* γ*-adapted deformation of the identity, then*

$$(\mathfrak{p}_{\Omega^{*n}}^* \hat{g})\big(\underline{\gamma}(t)\big) = \big(\Psi_t \circ \vec{\mathscr{D}}\big(\gamma(a)\big)\big)_\#[\Delta_n](\beta) \qquad \textit{for } t \in [a,1]. \tag{4.2}$$

Notice that, with the notations

$$(4.3)\qquad \big(\underline{\zeta}_1^t,\dots,\underline{\zeta}_n^t\big) := \Psi_t\circ\vec{\mathscr{D}}\big(\gamma(a)\big)\colon\ \Delta_n\to X_\Omega^n,\qquad \zeta_i^t := \mathfrak{p}_\Omega\circ\underline{\zeta}_i^t\quad\text{for } 1\le i\le n,$$

formula (4.2) can be rewritten as

$$(4.4)\qquad (\mathfrak{p}_{\Omega^{*n}}^*\hat{g})\big(\underline{\gamma}(t)\big) = \int_{\Delta_n}(\mathfrak{p}_\Omega^*\hat{f}_1)\big(\underline{\zeta}_1^t\big)\cdots(\mathfrak{p}_\Omega^*\hat{f}_n)\big(\underline{\zeta}_n^t\big)\det\left[\frac{\partial\zeta_i^t}{\partial s_j}\right]_{1\le i,j\le n}\mathrm{d}s_1\cdots\mathrm{d}s_n$$

(for each t, the partial derivatives $\frac{\partial\zeta_i^t}{\partial s_j}$ exist almost everywhere on Δ_n by Rademacher's theorem, for the functions $\vec{s}\mapsto\zeta_i^t(\vec{s})$ are Lipschitz).

§ 4.3. Sketch of the proof of Theorem 3.5

We define a function $\eta\ge 0$ by the formula

$$v=(\lambda,\xi)\in\mathcal{M}_\Omega\mapsto\eta(v):=\operatorname{dist}\big((\lambda,\xi),\{(0,0)\}\cup\overline{\mathcal{S}}_\Omega\big),$$

where $\operatorname{dist}(\cdot,\cdot)$ is a notation for the Euclidean distance in $\mathbb{R}\times\mathbb{C}\simeq\mathbb{R}^3$.

The following three lemmas allow to prove Theorem 3.5. The reader is referred to [KS] for their proofs.

Lemma 4.5. *The function D defined by the formula*

$$D\big(t,(v_1,\dots,v_n)\big):=\eta(v_1)+\cdots+\eta(v_n)+\operatorname{dist}\big(\big(L(\gamma_{|t}),\gamma(t)\big),v_1+\cdots+v_n\big)$$

is everywhere positive on $[a,1]\times\mathcal{M}_\Omega^n$ and the formula

$$(4.5)\qquad \vec{X}(t,\vec{v})=\left|\begin{array}{l} X_1:=\dfrac{\eta(v_1)}{D(t,\vec{v})}\big(|\gamma'(t)|,\gamma'(t)\big)\\[2ex] \qquad\vdots\\[2ex] X_n:=\dfrac{\eta(v_1)}{D(t,\vec{v})}\big(|\gamma'(t)|,\gamma'(t)\big)\end{array}\right.$$

defines a non-autonomous vector field $\vec{X}(t,\vec{v})\in T_{\vec{v}}\big(\mathcal{M}_\Omega^n\big)\simeq(\mathbb{R}\times\mathbb{C})^n$ on $[a,1]\times\mathcal{M}_\Omega^n$, which admits a flow map $\Phi_t\colon\ \mathcal{M}_\Omega^n\to\mathcal{M}_\Omega^n$ between time a and time t for every $t\in[a,1]$.

Lemma 4.6. *One can define a γ-adapted deformation of the identity $(\Psi_t)_{t\in[a,1]}$ on $\underline{U}^n$ as follows: for every $\vec{\underline{\zeta}}=\big(\mathscr{L}(\zeta_1),\dots\mathscr{L}(\zeta_n)\big)\in\underline{U}^n$ and $i\in\{1,\dots,n\}$, we set $v_j:=(|\zeta_j|,\zeta_j)$ for each $j\in\{1,\dots,n\}$ and define a path $\tilde{\gamma}_{i,\vec{\underline{\zeta}}}\colon\ [0,1]\to\mathcal{M}_\Omega$ by*

$$t\in[0,a]\ \Rightarrow\ \tilde{\gamma}_{i,\vec{\underline{\zeta}}}(t):=\tfrac{t}{a}v_i,\qquad t\in[a,1]\ \Rightarrow\ \tilde{\gamma}_{i,\vec{\underline{\zeta}}}(t):=\pi_i\circ\Phi_t(v_1,\dots,v_n),$$

where $\pi_i\colon\ \mathcal{M}_\Omega^n\to\mathcal{M}_\Omega$ is the projection onto the ith factor; then, by virtue of (4.1), the $\mathbb{C}$-projection of $\tilde{\gamma}_{i,\vec{\underline{\zeta}}}$ is a path $\gamma_{i,\vec{\underline{\zeta}}}\in\Pi_\Omega$, and we set

$$\Psi_t\big(\vec{\underline{\zeta}}\big):=\big(\underline{\gamma}_{1,\vec{\underline{\zeta}}}(t),\dots,\underline{\gamma}_{n,\vec{\underline{\zeta}}}(t)\big)\in X_\Omega^n\quad\textit{for } t\in[a,1].$$

Lemma 4.7. *Consider the functions* $\vec{s} \in \Delta_n \mapsto v_i^a(\vec{s}) := \big(s_i|\gamma(a)|, s_i\gamma(a)\big) \in \mathcal{M}_\Omega$ *for* $1 \le i \le n$ *and, for each* $t \in [a,1]$,

$$(v_1^t, \dots, v_n^t) := \Phi_t \circ (v_1^a, \dots, v_n^a)\colon\ \Delta_n \to \mathcal{M}_\Omega^n.$$

Suppose $|\gamma(a)| > \delta$ *and let*

$$\mathcal{M}_\Omega^{\delta,L} := \big\{\, (\lambda,\zeta) \in \mathbb{R}_{\ge 0} \times \mathbb{C} \mid \lambda \le L \ \text{ and } \ \operatorname{dist}\big((\lambda,\zeta),u\big) \ge \delta \ \text{ for all } u \in \{(0,0)\} \cup \overline{\mathcal{S}}_\Omega \,\big\}.$$

Then

$$(v_1^t, \dots, v_n^t)(\Delta_n) \subset \bigcup_{L^{(1)}+\cdots+L^{(n)}=L(\gamma_{|t})} \mathcal{M}_\Omega^{\delta(t),L^{(1)}} \times \cdots \times \mathcal{M}_\Omega^{\delta(t),L^{(n)}}, \tag{4.6}$$

where $\delta(t) := \frac{1}{2}\rho\,\mathrm{e}^{-2\sqrt{2}\delta^{-1}(L(\gamma_{|t})-|\gamma(a)|)}$. *Moreover, for each* i, *the* $\mathbb{C}$*-projection of* v_i^t *is a Lipschitz function* $\zeta_i^t\colon\ \Delta_n \to \mathbb{C}$ *and the almost everywhere defined partial derivatives* $\frac{\partial \zeta_i^t}{\partial s_j}$ *satisfy*

$$\left|\det\left[\frac{\partial \zeta_i^t}{\partial s_j}\right]_{1\le i,j\le n}\right| \le \big(c(t)\big)^n,$$

where $c(t) := |\gamma(a)|\,\mathrm{e}^{3\sqrt{2}\delta^{-1}(L(\gamma_{|t})-|\gamma(a)|)}$.

We set

$$\delta' := \frac{1}{2}\rho\,\mathrm{e}^{-2\sqrt{2}\delta^{-1}L}, \qquad c := |\gamma(a)|\,\mathrm{e}^{3\sqrt{2}\delta^{-1}L}, \tag{4.7}$$

so that $\delta' \le \delta(t)$ and $c \ge c(t)$ for all $t \in [a,1]$. Theorem 3.5 follows from the previous estimates and the identity (4.4) (the functions ζ_i^t in (4.3) and in Lemma 4.7 are indeed the same).

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