

On the Two-Phase Obstacle Problem

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1 Introduction

Although the regularity in one-phase free boundary problems has by now been extensively studied, the methods used there prove in many cases to be unsuitable for the corresponding two-phase problems.

Here we announce a result concerning the two-phase obstacle problem

$$\Delta u = \frac{\lambda_+}{2} \chi_{\{u>0\}} - \frac{\lambda_-}{2} \chi_{\{u<0\}} . \quad (1)$$

The nonlinearities of this equation suggest that the solution should be locally a $H^{2,\infty}$ -function. We obtain this regularity in the form of a growth estimate (Proposition 3.1). The proof uses new ideas as well as a monotonicity formula introduced by the author in [7]. A consequence is that the Hausdorff dimension of the free boundary $\partial\{u > 0\} \cup \partial\{u < 0\}$ is less than or equal to $n - 1$ (Corollary 4.1).

Note that our approach can also be used to derive Lipschitz continuity of minimizers of the functional $v \mapsto \int_{\Omega} (|\nabla v|^2 + \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v<0\}})$ (Remark 4.1); Lipschitz continuity of minimizers of this functional has been proven

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in [1] using a result on optimal Poincaré constants with respect to spherical domains ([2]).

2 The equation

Let $n \geq 2$ and let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary, assume that $u_D \in H^{1,2}(\Omega)$ and let $A := \{v \in H^{1,2}(\Omega) : v - u_D \in H_0^{1,2}(\Omega)\}$. Then the functional $E(v) := \int_{\Omega} (|\nabla v|^2 + \lambda_+ \max(v, 0) - \lambda_- \min(v, 0))$, being real-valued, non-negative, convex and weakly lower semicontinuous, attains its infimum on the affine subspace A of $H^{1,2}(\Omega)$ at the point $u \in A$.

Throughout the whole paper u shall denote this minimizer, however the reader may replace the boundary condition in the definition of A at his own convenience, since from now on everything we do will be completely local.

Let us compute the first variation of the energy E at the point u . Using $v := u + \epsilon\phi$ as test function for the minimality of u , where $\epsilon > 0$ and $\phi \in H_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, we obtain that

$$\int_{\Omega} (2\nabla u \cdot \nabla \phi + \phi \lambda_+ \chi_{\{u \geq -\epsilon\phi\}} - \phi \lambda_- \chi_{\{u \leq -\epsilon\phi\}}) \geq -\epsilon \int_{\Omega} |\nabla \phi|^2,$$

and, as $\epsilon \rightarrow 0$, that

$$\begin{aligned} & \int_{\Omega \cap \{u=0\}} (-\lambda_+ \max(\phi, 0) + \lambda_- \min(\phi, 0)) \leq \\ & \int_{\Omega} (2\nabla u \cdot \nabla \phi + \phi \lambda_+ \chi_{\{u>0\}} - \phi \lambda_- \chi_{\{u<0\}}) \\ & \leq \int_{\Omega \cap \{u=0\}} (\lambda_+ \max(-\phi, 0) - \lambda_- \min(-\phi, 0)) \end{aligned} \quad (2)$$

for every $\phi \in H_0^{1,2}(\Omega)$. By the characterization of non-negative distributions this implies that $v \mapsto \int (\nabla u \cdot \nabla \phi + \frac{\lambda_+}{2} \phi)$ is locally in Ω represented by a finite regular measure. Hence, (2) yields by Radon-Nikodym's theorem that $\Delta u \in L_{\text{loc}}^1(\Omega)$ and it follows that $\Delta u = \frac{\lambda_+}{2} \chi_{\{u>0\}} - \frac{\lambda_-}{2} \chi_{\{u<0\}}$ a.e. in Ω .

At this point we observe that any other function $v \in H^{1,2}(\Omega)$ with boundary data u_D on $\partial\Omega$ that satisfies the weak equation

$$\int_{\Omega} (2\nabla v \cdot \nabla \phi + \phi \lambda_+ \chi_{\{v>0\}} - \phi \lambda_- \chi_{\{v<0\}}) = 0 \quad \text{for every } \phi \in H_0^{1,2}(\Omega)$$

must coincide with u : subtracting the weak equation for u and inserting $\phi := v - u$ as test function we obtain that

$$\int_{\Omega} 2|\nabla(v-u)|^2 \leq$$

$\int_{\Omega} (2\nabla(v-u) \cdot \nabla(v-u) + \lambda_+(\chi_{\{v>0\}} - \chi_{\{u>0\}})(v-u) - \lambda_-(\chi_{\{v<0\}} - \chi_{\{u<0\}})(v-u))$
 $= 0$. Thus the weak solution is *unique* and it is therefore no restriction to confine our study to the minimizer u .

In what follows, the term “solution” shall always denote a $H^{2,1}$ -function solving the strong equation $\Delta v = \frac{\lambda_+}{2} \chi_{\{v>0\}} - \frac{\lambda_-}{2} \chi_{\{v<0\}}$ a.e. in a given open set.

A powerful tool is now a monotonicity formula introduced in [7] by the author for a class of semilinear free boundary problems. For the sake of completeness let us state the two-phase obstacle problem case here:

Theorem 2.1 (the monotonicity formula) *Suppose that $B_{\delta}(x_0) \subset \Omega$. Then for all $0 < \rho < \sigma < \delta$ the function*

$$\Phi_{x_0}(r) := r^{-n-2} \int_{B_r(x_0)} (|\nabla u|^2 + \lambda_+ \max(u, 0) + \lambda_- \max(-u, 0)) \\ - 2 r^{-n-3} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1} ,$$

defined in $(0, \delta)$, satisfies the monotonicity formula

$$\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) = \int_{\rho}^{\sigma} r^{-n-2} \int_{\partial B_r(x_0)} 2 \left(\nabla u \cdot \nu - 2 \frac{u}{r} \right)^2 d\mathcal{H}^{n-1} dr \geq 0 .$$

3 Pointwise regularity and non-degeneracy

By L^p -theory the solution $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$ for every $\alpha \in (0, 1)$. The set $R := \Omega \cap \{u = 0\} \cap \{\nabla u \neq 0\}$ is therefore open relative to $\Omega \cap (\partial\{u > 0\} \cup \partial\{u < 0\})$ and the implicit function theorem implies that R is a $C^{1,\alpha}$ -surface for every $\alpha \in (0, 1)$. The set of interest is therefore the set $S := \Omega \cap \{\nabla u = 0\} \cap (\partial\{u > 0\} \cup \partial\{u < 0\})$.

Lemma 3.1 *Let $\alpha - 1 \in \mathbb{N}$, let $w \in H^{1,2}(B_1(0))$ be a harmonic function in $B_1(0)$ and assume that $D^j w(0) = 0$ for $0 \leq j \leq \alpha - 1$.*

$$\text{Then } \int_{B_1(0)} |\nabla w|^2 - \alpha \int_{\partial B_1(0)} w^2 d\mathcal{H}^{n-1} \geq 0,$$

and equality implies that w is homogeneous of degree α in $B_1(0)$.

The proof is based on the well-known fact that the mean frequency of a harmonic function is a non-decreasing function of the radius.

The following proposition gives an estimate on the growth of the solution near S :

Proposition 3.1 *There exists for each $\delta > 0$ a constant $C < \infty$ such that*

$$\int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1} \leq C r^{n-1+4}$$

for every $r \in (0, \delta)$ and every $x_0 \in S$ satisfying $B_{2\delta}(x_0) \in \Omega$.

Furthermore the estimate

$$\begin{aligned} & r^{1-n-4} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1} \\ & \leq \frac{1}{2} r_0^{-n-2} \int_{B_{r_0}(x_0)} (|\nabla u|^2 + \lambda_+ \max(u, 0) + \lambda_- \max(-u, 0)) \end{aligned}$$

holds for every $0 < r < r_0$ and $x_0 \in S$ satisfying $B_{r_0}(x_0) \subset \Omega$.

Remark 3.1 *Note that in the one-phase case $\lambda_- = 0$, $u_D \geq 0$ the first estimate of Proposition 3.1 can be proved via a Harnack inequality argument: introducing for $r > 0$ the scaled function $u_r(x) := \frac{u(x_0+rx)}{r^2}$ and supposing that $u(x_0) = 0$ and $B_{r_0}(x_0) \subset \subset \Omega$ we obtain that $\Delta u_r = \frac{1}{2} \chi_{\{u_r > 0\}}$ in $B_1(0)$ for $r \in (0, r_0)$. Now the fact that $u \in H^{2,p}(B_{r_0}(x_0))$ allows us to apply Harnack's inequality Theorem 8.18 of [3] to deduce that $\sup_{B_1(0)} u_r \leq C(n)$ and, in the original scaling, that $\sup_{B_r(x_0)} u \leq C(n) r^2$.*

Lemma 3.2 (non-degeneracy) *For every $x_0 \in \overline{\{u > 0\}} \cup \overline{\{u < 0\}}$ and every $B_{2r}(x_0) \subset \Omega$ the estimate*

$$\sup_{\partial B_r(x_0)} |u| \geq \frac{1}{4n} \min(\lambda_+, \lambda_-) r^2 \quad \text{holds.}$$

Proof: We observe that it is sufficient to prove the statement for every $x_0 \in \{u > 0\}$ such that $B_{2r}(x_0) \subset \Omega$. Assuming that $\sup_{\partial B_r(x_0)} u \leq \frac{1}{4n} \lambda_+ r^2$, the comparison principle yields that $u(x) \leq v(x) := \frac{1}{4n} \lambda_+ |x - x_0|^2$ in $B_r(x_0)$. This, however, contradicts the assumption $u(x_0) > 0$.

4 A Hausdorff dimension estimate

From now on we assume that $\min(\lambda_+, \lambda_-) > 0$. The results of the previous section lead to the following consequences.

Lemma 4.1 *Let $x_0 \in S$ and let $u_k(x) := \frac{u(x_0 + \rho_k x)}{\rho_k^2}$ be a blow-up sequence, i.e. assume that $\rho_k \rightarrow 0$ as $k \rightarrow \infty$. Then $(u_k)_{k \in \mathbb{N}}$ is for each open $D \subset \subset \mathbb{R}^n$ and each $p \in (1, \infty)$ bounded in $H^{2,p}(D)$, and each limit u_0 with respect to a subsequence $k \rightarrow \infty$ is a nontrivial homogeneous solution of degree 2 in $\mathbb{R}^n$ and satisfies the following:*

for each compact set $K \subset \mathbb{R}^n$ and each open set $U \supset K \cap S_0$ there exists $k_0 < \infty$ such that $S_k \cap K \subset U$ for $k \geq k_0$; here $S_0 := \{\nabla u_0 = 0\} \cap (\partial\{u_0 > 0\} \cup \partial\{u_0 < 0\})$ and $S_k := \{\nabla u_k = 0\} \cap (\partial\{u_k > 0\} \cup \partial\{u_k < 0\})$.

Applying standard geometric measure theoretic tools we obtain the following theorem:

Theorem 4.1 *The Hausdorff dimension of the set S is less than or equal to $n - 1$.*

Corollary 4.1 *The Hausdorff dimension of $\partial\{u > 0\} \cup \partial\{u < 0\}$ is less than or equal to $n - 1$.*

Remark 4.1 *The procedure of Proposition 3.1 yields a new proof for the regularity of a minimizer $\tilde{u}$ of the functional $v \mapsto \int_{\Omega} (|\nabla v|^2 + \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v<0\}})$.*

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