

THE Nil-Nil THEOREM IN ALGEBRAIC K -THEORY

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The reduced Nil-groups are certain reduced K -theory groups defined by Friedhelm Waldhausen for pure amalgams and tensor algebras [1]. They measure the defect in his Mayer–Vietoris sequence in the algebraic K -theory of rings. In the main paper [2], we recently showed that the apparently more complicated amalgam Nil can be computed in terms of the apparently simpler tensor Nil.

Theorem 1. *Let R be a (unital, associative) ring. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be R -bimodules. Suppose I is a small, filtered category and $\mathcal{B}_2 = \operatorname{colim}_{\alpha \in I} \mathcal{B}_2^\alpha$ is a direct limit of R -bimodules such that the left R -module structure of each $\mathcal{B}_2^\alpha$ is finitely generated and projective. Then, for every $n \in \mathbb{Z}$, there is an induced isomorphism*

$$\tilde{K}_n(j) : \tilde{\operatorname{Nil}}_n(R; \mathcal{B}_1, \mathcal{B}_2) \longrightarrow \tilde{\operatorname{Nil}}_n(R; \mathcal{B}_1 \otimes_R \mathcal{B}_2) .$$

An important special case are those amalgams of group rings which are induced by an epimorphism onto the infinite dihedral group $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes_{-1} \mathbb{Z}/2$.

Corollary 2. *Suppose G is a group with an epimorphism $p : G \rightarrow D_\infty$. Denote the p -induced injective amalgamated product decomposition $G = G_1 *_F G_2$. Consider the index-two subgroup $\bar{G} := p^{-1}(\mathbb{Z})$ of G . Denote the p -induced injective HNN-extension $\bar{G} = F \rtimes_\alpha \mathbb{Z}$. Then, for all rings R and for all $n \in \mathbb{Z}$, there is an isomorphism of abelian groups:*

$$\tilde{\operatorname{Nil}}_n(R[F]; R[G_1 - F], R[G_2 - F]) \cong NK_{n+1}(R[F], \alpha) .$$

The right-hand side of the isomorphism is the twisted Bass Nil-group [3] of F.T. Farrell and W.C. Hsiang [4]. These are more readily computable since they involve the Wang sequence in K -theory of the twisted polynomial ring $R[F]_\alpha[x]$.

The following application [2] of the above corollary is a sharpening of the fibered isomorphism conjecture of F. T. Farrell and L. E. Jones in algebraic K -theory. Given a group G , denote vc as the class of virtually cyclic subgroups and fbc as the subclass of finite-by-cyclic subgroups. The elements of the complement $\operatorname{vc} - \operatorname{fbc}$ are exactly those subgroups of G which are finite-by- D_∞ .

Theorem 3. *Let $\varphi : \Gamma \rightarrow G$ be an epimorphism of groups. Then, for all rings R and for all $n \in \mathbb{Z}$, the following induced map is an isomorphism:*

$$H_n^\Gamma(E_{\varphi^* \operatorname{fbc}} \Gamma; \mathbf{K}_R) \longrightarrow H_n^\Gamma(E_{\varphi^* \operatorname{vc}} \Gamma; \mathbf{K}_R) .$$

Both sides are equivariant homology groups, whose coefficients are given by the spectrum-valued functor $\mathbf{K}_R : \operatorname{Or} \Gamma \rightarrow \operatorname{SPECTRA}$ of the Bredon orbit category [5].

Recently, the Farrell–Jones conjecture has been proven by A. Bartels, W. Lück, and H. Reich for a large class of infinite groups with torsion [6]. Consider the non-fibered case of $\varphi = \operatorname{id}_\Gamma$.

Corollary 4. *Let Γ be a word-hyperbolic group. Then, for all rings R and for all $n \in \mathbb{Z}$, the algebraic K -theory assembly map is an isomorphism:*

$$H_n^\Gamma(E_{\text{fbc}}\Gamma; \mathbf{K}_R) \longrightarrow K_n(R[\Gamma]) .$$

This isomorphism yields specific fruit. In the following calculation [2], the Bass NK -groups vanish if R is a regular Noetherian ring, such as if $R = \mathbb{Z}$. Here $K_n(R[x]) = K_n(R) \oplus NK_n(R)$ by definition.

Theorem 5. *Consider the modular group $\Gamma = PSL(2, \mathbb{Z})$. Then, for any ring R and integer n , we have*

$$K_n(R[\Gamma]) = (K_n(R[\mathbb{Z}/2]) \oplus K_n(R[\mathbb{Z}/3])) / K_n(R) \oplus \bigoplus_{\mathbb{N}_0} NK_n(R) .$$

Finally, [2] provides the first example of a non-vanishing amalgam Nil-group.

Example 6. Consider the group $G_0 := \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}$. Then

$$\widetilde{\text{Nil}}_0(\mathbb{Z}[G_0]; \mathbb{Z}[G_0], \mathbb{Z}[G_0]) = NK_1(\mathbb{Z}[G_0])$$

is a non-zero abelian group (see [3]), which is a summand of the Whitehead group $\text{Wh}(G_0 \times D_\infty)$. Therefore we obtain the following topological consequence. Consider the finite CW-complexes

$$\begin{aligned} W &:= \mathbb{RP}^2 \times \mathbb{RP}^2 \times S^1 \\ X &:= W \times S^2 \\ Y &:= W \times (\mathbb{RP}^3 - \text{int } D^3) . \end{aligned}$$

Given any non-zero element of the above amalgam Nil-group, one can construct [7] the first known example of a homotopy equivalence $h : K \rightarrow Y \cup_X Y$, where K is a certain finite CW-complex, such that h is **not** splittable along X . Here, we say h is *splittable along X* if there exist a simple homotopy equivalence $s : K' \rightarrow K$ of finite CW-complexes and a homotopy equivalence $h' : K' \rightarrow Y \cup_X Y$ such that $h \circ s \simeq h'$ and the cellular restriction $(h')^{-1}(X) \rightarrow X$ is a homotopy equivalence.

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