

Residuated mapping and CPS-translation

– Extended abstract –

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Abstract

We provide a call-by-name CPS-translation from polymorphic λ -calculus λ_2 into existential λ -calculus $\lambda^\exists$. Then we prove that the CPS-translation is a residuated mapping from the preordered set of λ_2 -terms to that of $\lambda^\exists$ -terms. From the inductive proof, its residual (inverse translation) can be extracted, which constitutes the so-called Galois connection. It is also obtained that given the CPS-translation the existence of its inverse is unique.

1 Preliminaries

By a preordered set $\langle A, \sqsubseteq \rangle$, we mean a set A on which there is defined a preorder, i.e., a reflexive and transitive relation $\sqsubseteq$. If $\langle A_1, \sqsubseteq_1 \rangle$ and $\langle A_2, \sqsubseteq_2 \rangle$ are preordered sets, then we say that a mapping $f : A_1 \rightarrow A_2$ is monotone, if $x \sqsubseteq_1 y$ implies $f(x) \sqsubseteq_2 f(y)$ for any $x, y \in A_1$. A direct image under f is denoted by $f[X]$ for every $X \subseteq A_1$, and an inverse image is denoted by $f^{\leftarrow}[Y]$ for every $Y \subseteq A_2$. A subset $B \subseteq A$ is a down-set of a preordered set $\langle A, \sqsubseteq \rangle$, if $y \sqsubseteq x$ together with $y \in A$ and $x \in B$ implies $y \in B$. By a principal down-set, we mean a down-set of the form $\{y \in A \mid y \sqsubseteq x\}$, which is denoted by $\downarrow x$.

Definition 1 (Residuated mapping) *A mapping $f : A \rightarrow B$ that satisfies the following condition is said to be residuated: The inverse image under f of every principal down-set of B is a principal down-set of A .*

2 Source calculus: λ_2

We introduce our source calculus of 2nd order λ -calculus (Girard-Reynolds), denoted by λ_2 . For simplicity, we adopt its domain-free style.

Definition 2 (Types)

$$A ::= X \mid A \Rightarrow A \mid \forall X. A$$

Definition 3 ((Pseudo) λ_2 -terms)

$$\lambda_2 \ni M ::= x \mid \lambda x. M \mid MM \mid \lambda X. M \mid MA$$

Definition 4 (Reduction rules) $(\beta) (\lambda x.M_1)M_2 \rightarrow M_1[x := M_2]$

$(\eta) \lambda x.Mx \rightarrow M$, if $x \notin FV(M)$

$(\beta_t) (\lambda X.M)A \rightarrow M[X := A]$

$(\eta_t) \lambda X.MX \rightarrow M$, if $X \notin FV(M)$

$FV(M)$ denotes a set of free variables in M .

We write $\rightarrow_{\lambda 2}$ for the compatible relation obtained from the reflexive and transitive closure of the one step reduction relation, and $\rightarrow_{\lambda 2}^+$ for that from the transitive closure. In particular, $\rightarrow_R$ denotes the subrelation of $\rightarrow$ restricted to the reduction rules $R \subseteq \{\beta, \eta, \beta_t, \eta_t\}$. We may write simply (β) for either (β) or (β_t) , and (η) for either (η) or (η_t) , if clear from the context. We employ the notation $\equiv$ to indicate the syntactic identity under renaming of bound variables.

3 Target calculus: $\lambda^\exists$

We next define our target calculus denoted by $\lambda^\exists$, which is logically a subsystem of minimal logic consisting of constant $\perp$, negation, conjunction and 2nd order existential quantification¹.

Definition 5 (Types)

$$A ::= \perp \mid X \mid \neg A \mid A \wedge A \mid \exists X.A$$

Definition 6 ((Pseudo) $\lambda^\exists$ -terms)

$$\begin{aligned} \Lambda^\exists \ni M ::= & x \mid \lambda x.M \mid MM \mid \langle M, M \rangle \mid \text{let } \langle x, x \rangle = M \text{ in } M \\ & \mid \langle A, M \rangle \mid \text{let } \langle X, x \rangle = M \text{ in } M \end{aligned}$$

Definition 7 (Reduction rules) $(\beta) (\lambda x.M_1)M_2 \rightarrow M_1[x := M_2]$

$(\eta) \lambda x.Mx \rightarrow M$, if $x \notin FV(M)$

$(\text{let}_\wedge) \text{let } \langle x_1, x_2 \rangle = \langle M_1, M_2 \rangle \text{ in } M \rightarrow M[x_1 := M_1, x_2 := M_2]$

$(\text{let}_{\wedge, \eta}) \text{let } \langle x_1, x_2 \rangle = M_1 \text{ in } M[z := \langle x_1, x_2 \rangle] \rightarrow M[z := M_1],$

if $x_1, x_2 \notin FV(M)$

$(\text{let}_\exists) \text{let } \langle X, x \rangle = \langle A, M_1 \rangle \text{ in } M \rightarrow M[X := A, x := M_1]$

$(\text{let}_{\exists, \eta}) \text{let } \langle X, x \rangle = M_1 \text{ in } M[z := \langle X, x \rangle] \rightarrow M[z := M_1],$

if $X, x \notin FV(M_2)$

We also write simply (let) for either $(\text{let}_\wedge)$ or $(\text{let}_\exists)$, and (let_η) for $(\text{let}_{\wedge, \eta})$ or $(\text{let}_{\exists, \eta})$. Similarly we write $\rightarrow_{\lambda^\exists}$ and $\rightarrow_{\lambda^\exists}^+$ as done for $\lambda 2$.

¹For further introduction of the CPS target calculus $\lambda^\exists$ with let -expressions, see also [5].

4 CPS-translation * from Λ^2 into $\Lambda^\exists$

We define a translation, so-called modified CPS-translation * from pseudo λ^2 -terms into pseudo $\lambda^\exists$ -terms. In each case, a fresh and free variable a is introduced, which is called a continuation variable.

Definition 8 1. $x^* = xa$

2. $(\lambda x.M)^* = \text{let } \langle x, a \rangle = a \text{ in } M^*$

3. $(M_1 M_2)^* = \begin{cases} M_1^*[a := \langle x, a \rangle] & \text{for } M_2 \equiv x \\ M_1^*[a := \langle \lambda a.M_2^*, a \rangle] & \text{otherwise} \end{cases}$

4. $(\lambda X.M)^* = \text{let } \langle X, a \rangle = a \text{ in } M^*$

5. $(MA)^* = M^*[a := \langle A^*, a \rangle]$

6. $X^* = X$; $(A_1 \Rightarrow A_2)^* = \neg A_1^* \wedge A_2^*$; $(\forall X.A)^* = \exists X.A^*$

Remark that M^* contains exactly one free occurrence of a continuation variable a , and M^* has neither β -redex nor η -redex. Let $\lambda X.M$ have type $\forall X.A$. Then, under the translation, the parametric polymorphic function $\lambda X.M$ with respect to X becomes an abstract data type $(\lambda X.M)^*$ for X , which is waiting for an implementation a with type $\exists X.A^*$ together with an interface (a signature) with type A^* , i.e., $(\lambda X.M)^*$ is

abstype X with $a:A^*$ is a in M^*

in a familiar notation.

Lemma 1 (Monotone *) If we have $M_1 \rightarrow_{\lambda^2} M_2$, then $M_1^* \rightarrow_{\lambda^\exists}^+ M_2^*$ holds.

In particular, if $M_1 \rightarrow_\beta M_2$, then $M_1^* \rightarrow_{\beta \text{let}}^+ M_2^*$. And if $M_1 \rightarrow_\eta M_2$, then $M_1^* \rightarrow_{\text{let}, \eta} M_2^*$.

Proof. By induction on the derivation. □

In order to give an inverse translation, first we provide the mutual inductive definitions, respectively for denotations $Univ$ and continuations $\mathcal{C}$, as follows. Both $Univ$ and $\mathcal{C}$ are down-sets in the above sense.

$$\begin{array}{c}
 \begin{array}{c} a \in \mathcal{C} \\[10pt] \frac{C \in \mathcal{C} \quad P \in Univ}{\langle \lambda a.P, C \rangle \in \mathcal{C}} \\[10pt] \frac{C \in \mathcal{C}}{xC \in Univ} \\[10pt] \frac{C \in \mathcal{C} \quad P \in Univ}{\text{let } \langle x, a \rangle = C \text{ in } P \in Univ} \end{array}
 \qquad
 \begin{array}{c} \frac{C \in \mathcal{C}}{\langle x, C \rangle \in \mathcal{C}} \\[10pt] \frac{C \in \mathcal{C}}{\langle A^*, C \rangle \in \mathcal{C}} \\[10pt] \frac{C \in \mathcal{C} \quad P \in Univ}{(\lambda a.P)C \in Univ} \\[10pt] \frac{C \in \mathcal{C} \quad P \in Univ}{\text{let } \langle X, a \rangle = C \text{ in } P \in Univ} \end{array}
 \end{array}$$

We write $\langle R_1, R_2, \dots, R_n \rangle$ for $\langle R_1, \langle R_2, \dots, R_n \rangle \rangle$ with $n > 1$, and $\langle R_1 \rangle$ for R_1 with $n = 1$. $C \in \mathcal{C}$ is in the form of $\langle R_1, \dots, R_n, a \rangle$ where R_i ($1 \leq i \leq n$) is x , $\lambda a.P$, or A^* with $n \geq 0$. We explicitly mention that $C \in \mathcal{C}$ has exactly one occurrence of free variable a such that $C \equiv \langle R_1, \dots, R_n, a \rangle$ with $n \geq 0$. $P \in \text{Univ}$ also has exactly one occurrence of free variable a in such C as a proper subterm of P .

The inductively defined sets Univ , $\mathcal{C} \subseteq \Lambda^\exists$ are down-sets with respect to $\rightarrow_{\lambda\exists}$.

Lemma 2 1. If $P_1 \in \text{Univ}$ and $P_1 \rightarrow_{\lambda\exists} P_2$, then $P_2 \in \text{Univ}$.

2. If $C_1 \in \mathcal{C}$ and $C_1 \rightarrow_{\lambda\exists} C_2$, then $C_2 \in \mathcal{C}$.

Proof. Let $P, P_1 \in \text{Univ}$ and $C, C_1 \in \mathcal{C}$. Then $P[a := C_1], P[x := \lambda a.P_1], P[X := A^*] \in \text{Univ}$, and $C[a := C_1], C[x := \lambda a.P_1], C[X := A^*] \in \mathcal{C}$. $\square$

Proposition 1 1. Univ is strongly normalizing with respect to $\rightarrow_{\beta\eta}$, i.e., for any $P \in \text{Univ}$, there is no infinite reduction sequence of $\rightarrow_{\beta\eta}$ starting with P .

2. Univ is Church-Rosser with respect to $\rightarrow_{\beta\eta}$, i.e., for any $P, P_1, P_2 \in \text{Univ}$, if we have $P \rightarrow_{\beta\eta} P_1$ and $P \rightarrow_{\beta\eta} P_2$, then there exists some $P_3 \in \text{Univ}$ such that $P_1 \rightarrow_{\beta\eta} P_3$ and $P_2 \rightarrow_{\beta\eta} P_3$.

Proof.

1. Since every λ -abstraction $\lambda a.P \in \text{Univ}$ is linear, for any $P_1 \rightarrow_{\beta\eta} P_2$, the contractum P_2 has less length than that of P_1 .
2. Univ is weak Church-Rosser with respect to $\rightarrow_{\beta\eta}$, and hence the property of Church-Rosser holds from Newman's Lemma. $\square$

Any (pseudo) term $P \in \text{Univ}$ is Church-Rosser and strongly normalizing with respect to $\beta\eta$ -reductions, and the unique $\beta\eta$ -normal form is denoted by $\Downarrow_{\beta\eta} P$. The same property naturally holds for $\mathcal{C}$ as well. A normalization function $\Downarrow_{\beta\eta}$ can be inductively defined as follows:

Definition 9 ($\Downarrow_{\beta\eta}$) 1. For $P \in \text{Univ}$:

- (a) $\Downarrow_{\beta\eta}(xC) = x(\Downarrow_{\beta\eta} C)$
- (b) $\Downarrow_{\beta\eta}((\lambda a.P)C) = \Downarrow_{\beta\eta}(P[a := C])$
- (c) $\Downarrow_{\beta\eta}(\text{let } \langle \chi, a \rangle = C \text{ in } P) = \text{let } \langle \chi, a \rangle = \Downarrow_{\beta\eta} C \text{ in } \Downarrow_{\beta\eta} P$

2. For $C \equiv \langle R_1, \dots, R_n, a \rangle \in \mathcal{C}$ with $n \geq 0$, where $R_i \equiv x, \lambda a.P$, or A^* :

$$\Downarrow_{\beta\eta} \langle R_1, \dots, R_n, a \rangle = \langle \Downarrow_{\beta\eta} R_1, \dots, \Downarrow_{\beta\eta} R_n, a \rangle$$

- (a) $R \equiv x$:
 $\Downarrow_{\beta\eta} x = x$
- (b) $R \equiv \lambda a.P$:
 - i. $\Downarrow_{\beta\eta}(\lambda a.xa) = x$, if $P \equiv xa$;
 - ii. $\Downarrow_{\beta\eta}(\lambda a.P) = \lambda a.(\Downarrow_{\beta\eta} P)$, otherwise;
- (c) $R \equiv A^*$:
 $\Downarrow_{\beta\eta} A^* = A^*$

5 Residuated CPS-translation

Proposition 2 *The following conditions are equivalent.*

1. $f : A \rightarrow B$ is a residuated mapping.
2. $f : A \rightarrow B$ is monotone and there exists a monotone mapping $g : B \rightarrow A$ such that $A \ni a \sqsubseteq g(f(a))$ and $f(g(b)) \sqsubseteq b \in B$.

Proof. A residuated mapping is monotone in general. On the other hand, from the condition 1, for any $b \in B$ there exists $a \in A$ such that $f^\leftarrow[\downarrow b] = \downarrow a$ which cannot be empty, whence one has a choice function $g : B \rightarrow A$ by $g(b) = a$. Hence $g(b) \in \downarrow g(b) = f^\leftarrow[\downarrow b]$ holds true, so that we have $f(g(b)) \sqsubseteq b$. We also have $a \in f^\leftarrow[\downarrow f(a)] = \downarrow g(f(a))$ by the definition, and hence we have $a \sqsubseteq g(f(a))$.

From the condition 2, we have that $f(a) \sqsubseteq b$ if and only if $a \sqsubseteq g(b)$. Hence, we have $f^\leftarrow[\downarrow b] = \downarrow g(b)$ for every $b \in B$. $\square$

We write $M \sqsubseteq N$ for $N \rightarrow M$, i.e., the contextual and reflexive-transitive closure of one-step reduction $\rightarrow$.

Lemma 3 *For any $P \in \text{Univ}$, there uniquely exists $M \in \Lambda 2$ such that $\Downarrow_{\beta\eta} P \equiv M^*$.*

Proof. By induction on $P \in \text{Univ}$.

1. Case of $P \equiv xC \equiv x\langle R_1, \dots, R_n, a \rangle$ with $n \geq 0$

(a) If $R_i \equiv x_i$, then we take $N_i \equiv x_i$, whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.

(b) Case of $R_i \equiv \lambda a.P_i$

If $P_i \equiv x_i a$, then we take $N_i \equiv x_i$, and whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.

Otherwise, from the induction hypothesis for P_i , there uniquely exists N_i such that $\Downarrow_{\beta\eta} P_i \equiv N_i^*$. Now we have $\Downarrow_{\beta\eta} R_i = \lambda a.(\Downarrow_{\beta\eta} P_i) \equiv \lambda a.N_i^*$.

(c) If $R_i \equiv A_i^*$, then we take $N_i \equiv A_i$.

Hence, we take $M \equiv xN_1 \dots N_n$, and then there uniquely exists $M \in \Lambda 2$ such that

$$\Downarrow_{\beta\eta} P$$

$$= x\langle \Downarrow_{\beta\eta} R_1, \dots, \Downarrow_{\beta\eta} R_n, a \rangle$$

$$\equiv x\langle N_1^{*'}, \dots, N_n^{*'}, a \rangle$$

$$= M^*,$$

where $N_i^{*'} = \lambda a.N_i^*$ if $R_i \equiv \lambda a.P_i$ with no outmost η -redex; otherwise $N_i^{*'} = N_i^*$.

2. Case of $P \equiv (\lambda a.P')C$

Since a is a linear variable, by the induction hypothesis for $P'[a := C]$, there uniquely exists $M \in \Lambda 2$ such that $\Downarrow_{\beta\eta}(P'[a := C]) \equiv M^*$. Therefore, we have a unique $M \in \Lambda 2$ such that $\Downarrow_{\beta\eta} P \equiv M^*$.

3. Case of $P \equiv \text{let } \langle x, a \rangle = C \text{ in } P_1$ with $C = \langle R_1, \dots, R_n, a \rangle$ and $n \geq 0$

- (a) From the induction hypothesis for P_1 , there uniquely exists $M_1 \in \Lambda 2$ such that $\Downarrow_{\beta\eta} P_1 \equiv M_1^*$.
- (b) If $R_i \equiv x_i$, then we take $N_i \equiv x_i$, whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.
- (c) Case of $R_i \equiv \lambda a. P_i$
 If $P_i \equiv x_i a$, then we take $N_i \equiv x_i$, and whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.
 Otherwise, from the induction hypothesis for P_i , there uniquely exists N_i such that $\Downarrow_{\beta\eta} P_i \equiv N_i^*$. Now we have $\Downarrow_{\beta\eta} R_i = \lambda a. (\Downarrow_{\beta\eta} P_i) \equiv \lambda a. N_i^*$.
- (d) If $R_i \equiv A_i^*$, then we take $N_i \equiv A_i$.

Hence, we take $M \equiv x N_1 \dots N_n$, and then there uniquely exists $M \in \Lambda 2$ such that

$$\begin{aligned}
 & \Downarrow_{\beta\eta} P \\
 &= \text{let } \langle x, a \rangle = \langle \Downarrow_{\beta\eta} R_1, \dots, \Downarrow_{\beta\eta} R_n, a \rangle \text{ in } (\Downarrow_{\beta\eta} P_1) \\
 &\equiv \text{let } \langle x, a \rangle = \langle N_1^{*'}, \dots, N_n^{*'}, a \rangle \text{ in } M_1^* \\
 &= M^*, \\
 &\text{where } N_i^{*'} = \lambda a. N_i^* \text{ if } R_i \equiv \lambda a. P_i \text{ with no outmost } \eta\text{-redex; otherwise } N_i^{*'} = N_i^*.
 \end{aligned}$$

4. Case of $P \equiv \text{let } \langle X, a \rangle = C \text{ in } P'$ is handled similarly. $\square$

From the inductive proof of Lemma 3 above, an extracted function giving a witness is written down here.

1. $x^\# = x$; $(\lambda a. P)^\# = P^\#$; $(A^*)^\# = A$
2. $(x \langle R_1, \dots, R_n, a \rangle)^\# = x R_1^\# \dots R_n^\#$
3. $((\lambda a. P) C)^\# = (P[a := C])^\#$
4. $(\text{let } \langle x, a \rangle = \langle R_1, \dots, R_n, a \rangle \text{ in } P)^\# = (\lambda x. P^\#) R_1^\# \dots R_n^\#$
5. $(\text{let } \langle X, a \rangle = \langle R_1, \dots, R_n, a \rangle \text{ in } P)^\# = (\lambda X. P^\#) R_1^\# \dots R_n^\#$

where the clause 1 is for R_i appeared in $\langle R_1, \dots, R_n, a \rangle \in C$, and the clause 2 through 5 are for $P \in \text{Univ}$.

Corollary 1 (Composition of $*$ and $\#$) 1. For any $P \in \text{Univ}$, we have $P \twoheadrightarrow_{\beta\eta} (P^\#)^*$.
 2. For any $M \in \Lambda 2$, we have $(M^*)^\# \equiv M$.

Proof.

1. From Lemma 3, we have $\Downarrow_{\beta\eta} P \equiv (P^\#)^*$ and $P \twoheadrightarrow_{\beta\eta} \Downarrow_{\beta\eta} P$. Therefore, $P \twoheadrightarrow_{\beta\eta} (P^\#)^*$ holds for any $P \in \text{Univ}$.
2. From the definition of $*$, M^* has neither β - nor η -redex. Hence, $\Downarrow_{\beta\eta} (M^*) \equiv M^*$ holds, and then $(M^*)^\# \equiv M$ for any $M \in \Lambda 2$. $\square$

Lemma 4 (Monotone $\#$) *The above mapping $\# : \text{Univ} \rightarrow \Lambda 2$ is monotone.*

Proof. By the definition of $\sharp$. In particular, let $P_1, P_2 \in \text{Univ}$, then the following holds.

1. If $P_1 \rightarrow_{\beta\eta} P_2$, then $P_1^\sharp \equiv P_2^\sharp$.
2. If $P_1 \rightarrow_{1\text{et}} P_2$, then $P_1^\sharp \rightarrow_\beta P_2^\sharp$.
3. If $P_1 \rightarrow_{1\text{et}\eta} P_2$, then $P_1^\sharp \rightarrow_\eta P_2^\sharp$. □

6 Residuated CPS-translation

As expected from the previous results, the CPS-translation forms a residuated mapping from $\Lambda 2$ to Univ .

Theorem 1 (Residuated CPS-trans.) *The CPS-translation $*$ is a residuated mapping from $\Lambda 2$ to Univ .*

Proof. From Proposition 2, Lemmata 1 and 4, and Corollary 1, the translation $*$ is a residuated mapping. In other words, for any $P \in \text{Univ}$, we have

$$\{M \in \Lambda 2 \mid M^* \sqsubseteq P\} = \downarrow P^\sharp.$$

In fact, from Lemma 1 and Corollary 1, we have $\downarrow P^\sharp \subseteq \{M \in \Lambda 2 \mid M^* \sqsubseteq P\}$. On the other hand, from Lemma 4 and Corollary 1, the inverse direction $\{M \in \Lambda 2 \mid M^* \sqsubseteq P\} \subseteq \downarrow P^\sharp$ holds true. □

We summarize results induced from the discussion above.

Corollary 2 1. $\Lambda 2$ is strongly normalizing if and only if Univ is strongly normalizing.

2. $\Lambda 2$ is weakly normalizing if and only if Univ is weakly normalizing.

3. $\Lambda 2$ is Church-Rosser if and only if Univ is Church-Rosser.

We remark that $\Lambda^\exists$ itself is not Church-Rosser.

4. Let $\downarrow P$ be $\{Q \mid P \rightarrow_{\lambda^\exists} Q\}$ for $P \in \text{Univ}$. Then the inverse image under $*$ of $\downarrow P$ is a principal down-set generated by $P^\sharp \in \Lambda 2$.

5. Given the CPS-translation $*$. Then an existence of its residual (inverse translation) is unique.

6. Define $P_1 \sim_{\beta\eta} P_2$ by $\downarrow_{\beta\eta} P_1 \equiv \downarrow_{\beta\eta} P_2$ for $P_1, P_2 \in \text{Univ}$. There exists a bijection $\star$ between $\Lambda 2$ and $\text{Univ} / \sim_{\beta\eta}$. In particular, there exists a one-to-one correspondence between $\Lambda 2$ -normal forms and Univ -normal forms.

7. Let $\downarrow_{\lambda^\exists} [\Lambda 2]^*$ be the down-set generated by $[\Lambda 2]^*$, i.e., $\{P \mid M^* \rightarrow_{\lambda^\exists} P \text{ for some } M \in \Lambda 2\}$. Let $\uparrow_{\beta\eta} [\Lambda 2]^*$ be the up-set generated by $[\Lambda 2]^*$, i.e., $\{P \in \text{Univ} \mid P \rightarrow_{\beta\eta} M^* \text{ for some } M \in \Lambda 2\}$.

Then we have $\downarrow_{\lambda^\exists} [\Lambda 2]^* \subseteq \text{Univ} = \uparrow_{\beta\eta} [\Lambda 2]^*$. We remark that $\subseteq$ is strict. For instance, $xa \in \downarrow_{\lambda^\exists} [\Lambda 2]^*$ and $(\lambda a.xa)a \in \text{Univ}$, but $(\lambda a.xa)a \notin \downarrow_{\lambda^\exists} [\Lambda 2]^*$.

Proof.

1. If $M_1 \rightarrow_{\lambda 2} M_2$, then we have $M_1^* \rightarrow_{\lambda \exists}^+ M_2^*$ by induction on the derivation. Therefore, strong normalization of *Univ* implies that of $\lambda 2$.

On the other hand, $\rightarrow_{\beta\eta}$ in *Univ* is strongly normalizing. If *Univ* has an infinite reduction path of $\rightarrow_{\lambda \exists}$, then the path should contain an infinite reduction path consisting of $\rightarrow_{1\text{et}, 1\text{et}_\eta}$. Now, from Lemma 4, $\lambda 2$ has an infinite reduction path of $\rightarrow_{\beta\eta}$. Hence, strong normalization of $\lambda 2$ implies that of *Univ*.

2. From the monotone translations between $\Lambda 2$ and *Unvi*, and the one-to-one correspondence between $\lambda 2$ -normal forms and *Univ*-normal forms.
3. $\Lambda 2$ and *Univ* form the so-called Galois connection under $*$ and $\sharp$.
4. The CPS-translation $*$ forms a residuated mapping.
5. Suppose we had two inverse translations $\sharp_1$ and $\sharp_2$, then $P^{\sharp_1} \equiv P^{\sharp_2}$ for any $P \in \text{Univ}$. Because we have $P \rightarrow_{\beta\eta} P^{\sharp_1*}$ for any $P \in \text{Univ}$ from Corollary 1 (1). Hence, we have $P^{\sharp_2} \equiv (P^{\sharp_1*})^{\sharp_2} \equiv P^{\sharp_1}$ from Lemma 4 (1).

6. Since $\sim_{\beta\eta}$ is an equivalence relation over *Univ*, we take

$$[P]_{\sim_{\beta\eta}} = \{P' \in \text{Univ} \mid P \sim_{\beta\eta} P'\} \text{ for } P \in \text{Univ}.$$

Then we define $\star(M) = [M^*]_{\sim_{\beta\eta}}$. In other words,

$$\star(M) = \uparrow_{\beta\eta}(M^*) = \{P \in \text{Univ} \mid P \rightarrow_{\beta\eta} M^*\}.$$

Then $\star : \Lambda 2 \rightarrow \text{Univ} / \sim_{\beta\eta}$ is a bijection. In fact, for any $[P] \in \text{Univ} / \sim_{\beta\eta}$, there exists $M \in \Lambda 2$ such that $\star(M) = [P]$. Because we take $M \equiv P^\sharp$, whence $P \rightarrow_{\beta\eta} (P^\sharp)^*$ and $\star(P^\sharp) = [P]$. On the other hand, suppose $M_1 \not\equiv M_2$. Then $\star(M_1) \neq \star(M_2)$, since M_1^* and M_2^* are distinct $\beta\eta$ -normal forms.

7. For any $M \in \Lambda 2$, we have $M^* \in \text{Unvi}$, and *Univ* is a down-set with respect to $\rightarrow_{\lambda \exists}$. Then we have $\downarrow_{\lambda \exists} [\Lambda 2]^* \subseteq \text{Univ}$.

For any $P \in \text{Univ}$, we have $P^\sharp \in \Lambda 2$ and $P \rightarrow_{\beta\eta} P^{\sharp*}$ from Lemma 1. Hence, $P \in \uparrow_{\beta\eta} [\Lambda 2]^*$ holds true. The inverse direction is clear, and therefore we have $\text{Univ} = \uparrow_{\beta\eta} [\Lambda 2]^*$. $\square$

It is remarked that instead of pseudo-terms, when we take account of well-typed terms, the binary relations $\rightarrow_{\lambda 2}$ and $\rightarrow_{\lambda \exists}$ form partial orders on λ -terms.

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