On a Bilinear Estimate of Schrödinger Waves

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Abstract

In this paper we want to consider a bilinear space-time estimate for homogeneous Schrödinger equations. We give an elementary proof for the estimates in Bourgain space, which is in a form of scaling invariance.

1 Introduction

Consider the homogeneous Schrödinger equations

$$\begin{cases} iu_t - \Delta u = 0, \ (x,t) \in R^n \times R, \\ u(0) = f; \end{cases} \qquad \begin{cases} iv_t - \Delta v = 0, \ (x,t) \in R^n \times R, \\ v(0) = g. \end{cases} \tag{1.1}$$

Via Fourier transform, the solution u and v can be written as

$$u(t) = e^{-it\Delta} f \quad \text{and} \quad v(t) = e^{-it\Delta} g. \tag{1.2}$$

Thus to study the estimates of the product of Schrödinger waves, uv, is to study the estimates of the product

$$e^{-it\Delta} f e^{-it\Delta} g. \tag{1.3}$$

There are many literature investigating on the topic of bilinear estimates for Schödinger waves. In '98, Ozawa and Tsutsumi [OT] proved an L^2 estimate for $u\bar{v}$ with $\frac{1}{2}$ derivative for $n = 1$,

$$\left\| (-\Delta)^{\frac{1}{4}} (e^{it\Delta} f)(e^{-it\Delta} g) \right\|_{L^2_{t,x}} = \frac{1}{\sqrt{2}} \|f\|_{L^2} \|g\|_{L^2}. \tag{1.4}$$

In '98, Bourgain [Bo] showed a refinements of Strichartz' inequality for $n = 2$. If $\widehat{f}$ is supported on $|\xi| \sim N$, $\widehat{g}$ is supported on $|\xi| \sim M$, and $M << N$, then

$$\left\| (e^{it\Delta} f)(e^{\pm it\Delta} g) \right\|_{L^2_{t,x}} \lesssim \left(\frac{M}{N} \right)^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}. \tag{1.5}$$

In '01, Kenig etc. [CDKS] obtained a bilinear estimate in Bourgain space for nonlinear Schrödinger equation in two dimension. Let $b = \frac{1}{2}+$. If $-\frac{1}{4} - (1-b) < s$ and $\sigma < \min\big(s + \frac{1}{2}, 2s + 2(1-b)\big)$, then

$$\|uv\|_{X^{\sigma,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}. \tag{1.6}$$

In '03, Tao [T1] obtained a sharp bilinear restriction estimate for paraboloids. Let $q > \dfrac{n+3}{n+1}$, $n \geq 2$, $N > 0$, and f and g have Fourier transform supported in the region $|\xi| \leq N$. Suppose that $dist(supp\,\widehat{f}, supp\,\widehat{g}) \geq cN$. Then we have

$$\left\|e^{-it\Delta} f e^{-it\Delta} g\right\|_{L^q_{t,x}(R^{n+1})} \lesssim N^{n-\frac{n+2}{q}} \|f\|_{L^2} \|g\|_{L^2}. \tag{1.7}$$

In '05, Burq, Jérard, and Tzvetkov [BGT] derived bilinear eigenfunction estimates on sphere and on Zoll surfaces. In '09, Keraani and Vargas [KV] showed a bilinear estimate of uv in $L^{\frac{n+2}{n}}$ norm, where $n \geq 2$. If $b \in (0, \frac{2}{n+2})$, then

$$\left\|e^{-it\Delta} f e^{-it\Delta} g\right\|_{L^{\frac{n+2}{n}}(R^{n+1})} \lesssim C \|f\|_{\dot{H}^b} \|g\|_{\dot{H}^{1-b}}. \tag{1.8}$$

In '09, Kishimoto [K] derived an improved bilinear estimate for quadratic Schrödinger equation in one and two dimensions. The estimate is in a variant of Bourgain space with weighted norm. In '10, Chae, Cho, and Lee [CCL] proved an interactive estimate of uv in a mixed norm. Let $n \geq 2$. If $\frac{2}{q} = n(1-\frac{1}{r})$, $1 < r \leq 2$, $\quad q > 1$, $|s| < 1 - \frac{1}{r}$, then

$$\left\|e^{-it\Delta} f e^{-it\Delta} g\right\|_{L^q_t L^r_x} \leq C \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^{-s}}. \tag{1.9}$$

Let D, S_+, and S_- be the operators with the symbols

$$\widehat{D} \stackrel{\text{def}}{=} |\xi|, \quad \widehat{S}_+ \stackrel{\text{def}}{=} \big||\tau| + |\xi|^2\big|, \quad \text{and} \quad \widehat{S}_- \stackrel{\text{def}}{=} \big||\tau| - |\xi|^2\big|, \tag{1.10}$$

respectively.

Theorem 1. *Let $n \geq 2$. If for $j = 1, 2$,*

$$\begin{gathered} \beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2, \\ \beta_- \geq 0, \quad \beta_- - \alpha_j + \frac{n-1}{2} \geq 0, \\ (\beta_-, \alpha_j) \neq (0, \frac{n-1}{2}), \ \textit{and} \quad \beta_0 > -\frac{n-1}{2}, \end{gathered} \tag{1.11}$$

then

$$\left\| D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \left(e^{-it\Delta} f e^{-it\Delta} g \right) \right\|_{L^2(R^{n+1})} \leq C \|f\|_{\dot{H}^{\alpha_1}(R^n)} \|g\|_{\dot{H}^{\alpha_2}(R^n)}. \tag{1.12}$$

Notice that Strichartz Estimate for Homogeneous Schrödinger equation for $n = 2$ reads

$$\|u\|_{L^4} \lesssim \|f\|_{L^2},$$

which coincides with the bilinear estimate

$$\|u\|_{L^4}^2 = \|uu\|_{L^2} \lesssim \|f\|_{L^2} \|f\|_{L^2},$$

when

$$\beta_0 = \beta_+ = \beta_- = \alpha_1 = \alpha_2 = 0.$$

The estimate is given in the form of scaling invariance. The conditions stated in the theorem come from the scaling invariance and interactions between frequencies.

The proof of Theorem 1 is based on the ideas of the work of Foschi and Klainerman [FK], and the work of Klainerman and Machedon [KM], however some modifications for adapting the case of Schrödinger are required. The purpose of this work is to derive a new estimate with an elementary proof.

The paper is organized as follows: In Section 2, we prove Theorem 1. In Section 3, we state and prove some properties which are the technical parts left in the proof of Theorem 1.

2 Bilinear Estimates for Schrödinger waves

We denote the Fourier transform of the function $u(t,x)$ by $\widehat{u}(t,\xi)$ with respect to the space variable and by $\widetilde{u}(\tau,\xi)$ with respect to the space-time variables. For simplicity, we call

$$\widehat{A} \ \equiv\ \widehat{D}^{2\beta_0}\ \widehat{S}_+^{2\beta_+}\ \widehat{S}_-^{2\beta_-}.$$

We now prove Theorem 1.

Proof. First we compute the Fourier transform of the product $e^{-it\Delta}fe^{-it\Delta}g$ with respect to space variables,

$$\widehat{e^{-it\Delta}f} * \widehat{e^{-it\Delta}g}(\xi) = \int e^{it|\xi-\eta|^2}\widehat{f}(\xi-\eta)e^{it|\eta|^2}\widehat{g}(\eta)d\eta = \int e^{it(|\xi-\eta|^2+|\eta|^2)}\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta.$$

Thus its Fourier transform with respect to space-time variables is

$$\int \delta(\tau-|\xi-\eta|^2-|\eta|^2)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta,$$

where $\delta(\tau-|\xi-\eta|^2-|\eta|^2)d\eta$ is viewed as a measure supported on surfaces $\{\eta : \tau = |\xi-\eta|^2+|\eta|^2\}$.

We split the integral into three parts in the following way. First we define a function

$$h(\gamma) \ \stackrel{\text{def}}{=}\ \frac{\sqrt{2\gamma-1}}{\gamma} \tag{2.1}$$

which will appear in the proof later, see figure 1. Since the equation $h(\gamma)=1/3$ has two roots $9\pm6\sqrt{2}$, we denote the two roots by $\gamma_1 = 9-6\sqrt{2}$ and $\gamma_2 = 9+6\sqrt{2}$. Then we decompose the η-space into $S_a\cup S_b\cup S_c$, see figure 2, where

$$\begin{aligned} &S_a \stackrel{\text{def}}{=} \{\eta : \frac{1}{2}|\xi|^2 \le |\xi-\eta|^2+|\eta|^2 \le \gamma_1|\xi|^2\},\\ &S_b \stackrel{\text{def}}{=} \{\eta : \gamma_1|\xi|^2 \le |\xi-\eta|^2+|\eta|^2 \le \gamma_2|\xi|^2\},\ \text{and}\\ &S_c \stackrel{\text{def}}{=} \{\eta : \gamma_2|\xi|^2 \le |\xi-\eta|^2+|\eta|^2\}. \end{aligned} \tag{2.2}$$

Thus we have

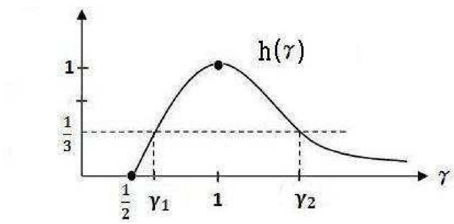

Figure 1:

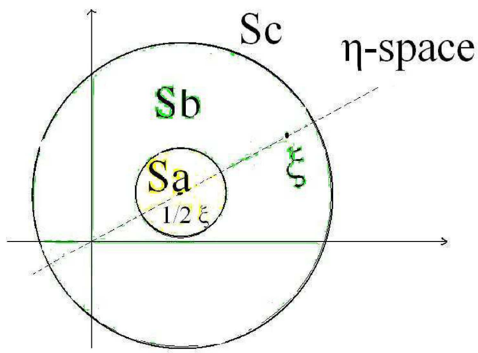

Figure 2:

$$
\begin{aligned}
&\left\| D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \left(e^{-it\Delta} f e^{-it\Delta} g \right) \right\|_{L^2} \\
= &\iint \widehat{A} \left| \int \delta(\tau - |\xi-\eta|^2 - |\eta|^2) \widehat{f}(\xi-\eta) \widehat{g}(\eta) d\eta \right|^2 d\tau d\xi \qquad (2.3) \\
= &\iint \widehat{A} \left| \int_{S_a} + \int_{S_b} + \int_{S_c} \delta(\tau - |\xi-\eta|^2 - |\eta|^2) \widehat{f}(\xi-\eta) \widehat{g}(\eta) d\eta \right|^2 d\tau d\xi.
\end{aligned}
$$

Hence it is sufficient to bound each of the above integrals. For simplicity, we denote $\Phi(\eta) \equiv \tau - |\xi-\eta|^2 - |\eta|^2$. Using Hölder inequality, we can bound the first integral in (2.3) as follows.

$$
\begin{aligned}
&\iint \widehat{A} \left| \int_{S_a} \delta(\tau - |\xi-\eta|^2 - |\eta|^2) \widehat{f}(\xi-\eta) \widehat{g}(\eta) d\eta \right|^2 d\tau d\xi \\
\le &\iint \widehat{A} \int \frac{\delta(\Phi(\eta))}{|\xi-\eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \int \delta(\Phi(\eta)) \left| |\xi-\eta|^{\alpha_1} \widehat{f}(\xi-\eta) |\eta|^{\alpha_2} \widehat{g}(\eta) \right|^2 d\eta d\tau d\xi \\
\lesssim &\iint \left\{ \int \delta(\Phi(\eta)) d\tau \right\} \left| |\xi-\eta|^{\alpha_1} \widehat{f}(\xi-\eta) |\eta|^{\alpha_2} \widehat{g}(\eta) \right|^2 d\eta d\xi \le C \|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}},
\end{aligned}
$$

provided that

$$
\widehat{A} \int_{S_a} \frac{\delta(\Phi(\eta))}{|\xi-\eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \le C, \quad \text{for all } \tau, \xi. \qquad (2.4)
$$

For the second integral we can get the desired bound in the same vain,

$$\iint \widehat{A}\left|\int_{S_b} \delta(\Phi(\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\right|^2 d\tau d\xi \leq C\|f\|_{\dot{H}^{\alpha_1}}\|g\|_{\dot{H}^{\alpha_2}}, \tag{2.5}$$

provided that

$$\widehat{A}\int_{S_b} \frac{\delta(\Phi(\eta))}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta \leq C, \quad \text{for all } \tau,\, \xi. \tag{2.6}$$

Notice that we have $|\xi-\eta| \sim |\eta|$ and $|\xi-\varphi| \sim |\varphi|$ on the set S_c. Using the fact that $|z|^2 = z\bar{z}$, the Fubini theorem, and change of variables, we can bound the third integral in (2.3) as follows.

$$\begin{aligned}
&\iint \widehat{A}\left|\int_{S_c} \delta(\Phi(\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\right|^2 d\tau d\xi \\
= &\iint \widehat{A}\int \delta(\Phi(\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta \int \delta(\Phi(\varphi))\overline{\widehat{f}}(\xi-\varphi)\overline{\widehat{g}}(\varphi)d\varphi d\tau d\xi \\
= &\iiint \widehat{A}\delta(\Phi(\eta)-\Phi(\xi-\varphi))\widehat{f}(\xi-\eta)\widehat{g}(\eta)\overline{\widehat{f}}(\varphi)\overline{\widehat{g}}(\xi-\varphi)d\varphi d\eta d\xi \\
\lesssim &\left(\iiint \frac{\widehat{A}\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\left||\varphi|^{\alpha_1}\overline{\widehat{f}}(\varphi)|\eta|^{\alpha_2}\widehat{g}(\eta)\right|^2 d\varphi d\eta d\xi\right)^{1/2} \cdot \\
&\left(\iiint \frac{\widehat{A}\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\xi-\varphi||\xi-\eta|)^{\alpha_1+\alpha_2}}\left||\xi-\varphi|^{\alpha_1}\overline{\widehat{f}}(\xi-\varphi)|\xi-\eta|^{\alpha_2}\widehat{g}(\xi-\eta)\right|^2 d\varphi d\eta d\xi\right)^{1/2}.
\end{aligned}$$

Hence we have the following bound

$$\iint\int_{T_c} \widehat{A}\frac{\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}d\xi\left||\varphi|^{\alpha_1}\overline{\widehat{f}}(\varphi)|\eta|^{\alpha_2}\widehat{g}(\eta)\right|^2 d\varphi d\eta \leq C\|f\|^2_{\dot{H}^{\alpha_1}}\|g\|^2_{\dot{H}^{\alpha_2}},$$

provided that

$$\int_{T_c} \widehat{A}\frac{\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\, d\xi \leq C, \text{ for all } \varphi \text{ and } \eta, \tag{2.7}$$

where $T_c \stackrel{\text{def}}{=} \{\xi : |\xi-\varphi|^2+|\varphi|^2,\ |\xi-\eta|^2+|\eta|^2 \geq \gamma_2|\xi|^2\}$ and $\tau = |\xi-\varphi|^2+|\varphi|^2 = |\xi-\eta|^2+|\eta|^2$.

Therefore the proof of the Theorem is complete once the claims, (2.4), (2.6), and (2.7) are proved. $\square$

Remark 1. *What left to be done are the following estimates. Claims: There is a constant C which is independent of τ, ξ, φ, and η such that the following inequalities hold.*

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_a} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta \leq C, \quad \textit{for all } \tau,\, \xi. \tag{2.8}$$

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_b} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta \leq C, \quad \textit{for all } \tau,\, \xi. \tag{2.9}$$

$$\int_{T_c} \widehat{D}^{2\beta_0} \widehat{S}_+^{2\beta_+} \widehat{S}_-^{2\beta_-} \frac{\delta(|\xi-\varphi|^2+|\varphi|^2-|\xi-\eta|^2-|\eta|^2)}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\, d\xi \leq C, \text{ for all } \varphi \text{ and } \eta, \tag{2.10}$$

where $\tau = |\xi-\varphi|^2+|\varphi|^2 = |\xi-\eta|^2+|\eta|^2$. The proofs of the above claims will be given in the next section.

3 Proofs of Claims

Now we are ready to prove the claims. First we prove the claims which come from the proof of bilinear estimates for uv.

Lemma 1 (Claim (2.8))**.** *Let $S_a = \{\eta : \frac{1}{2}|\xi|^2 \leq |\eta|^2+|\xi-\eta|^2 \leq \gamma_1|\xi|^2\}$. If $\beta_0+2\beta_++2\beta_-+\frac{n-2}{2} = \alpha_1+\alpha_2$ and $n \geq 2$, then*

$$\widehat{D}^{2\beta_0} \widehat{S}_+^{2\beta_+} \widehat{S}_-^{2\beta_-} \int_{S_a} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta \leq C, \tag{3.1}$$

for all τ and ξ.

Proof. We set $\zeta \overset{\text{def}}{=} R\Big(\eta-\frac{\xi}{2}\Big)$, where R is the rotation such that $R\xi = |\xi|e_1$, then we have the indentities

$$|\xi-\eta| = \Big|\zeta-\frac{1}{2}|\xi|e_1\Big| \quad \text{and} \quad |\eta| = \Big|\zeta+\frac{1}{2}|\xi|e_1\Big|. \tag{3.2}$$

Then we use spherical coordinates

$$\zeta = (X_1,\cdots,X_n) \overset{\text{def}}{=} \rho(\cos\phi, \sin\phi\,\omega') \overset{\text{def}}{=} \rho\omega, \tag{3.3}$$

where $\omega \in S^n$ and $\omega' \in S^{n-1}$, so that we can rewrite the integral in (3.1) as

$$\int_{S_a} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta = \int \frac{1}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}} \frac{\rho_0^{n-1}}{|-4\rho_0|} d\omega, \tag{3.4}$$

where $\rho_0^2 = \frac{1}{4}(2\tau-|\xi|^2)$. Using the identity $d\omega = (\sin\phi)^{n-2}d\phi d\omega'$, the identities for $|\xi-\eta|$ and $|\eta|$ in (3.2), and the change of variables $p = \cos\phi$, we can simplify the integral further.

$$\iint \frac{1}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}} \rho_0^{n-2} \sin^{n-2}\phi\, d\phi d\omega \sim \frac{\rho_0^{n-2}}{\tau^{\alpha_1+\alpha_2}} \int_{-1}^{1} \frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}} dp, \tag{3.5}$$

where $\lambda = \frac{2|\xi|\rho_0}{\tau} = \frac{|\xi|\sqrt{2\tau-|\xi|^2}}{\tau}$. Notice that $0 \leq \lambda \leq \frac{1}{3}$ under the restriction $\tau = |\xi-\eta|^2+|\eta|^2$ for $\eta \in S_a$.

We set $\tau \stackrel{\text{def}}{=} \gamma|\xi|^2$ which implies that $\lambda = \dfrac{\sqrt{2\gamma-1}}{\gamma} = h(\gamma)$. Then we have $\dfrac{1}{2} \leq \gamma \leq \gamma_1$ which implies that $\tau \sim |\xi|^2$, and $\rho_0 = \frac{1}{2}\sqrt{2\gamma-1}|\xi| \leq |\xi|$. Thus we can estimate the quantity in (3.1) as follows.

$$\begin{aligned}
&\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_a}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta\\
\leq\ & |\xi|^{2\beta_0+4\beta_++4\beta_--2\alpha_1-2\alpha_2}\Big||\gamma|-1\Big|^{2\beta_-}\left(\frac{\sqrt{2\gamma-1}|\xi|}{2}\right)^{n-2}\int_{-1}^{1}\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp.
\end{aligned}$$

The above quantity is bounded if we require that $n \geq 2$ and $\beta_0 + 2\beta_+ + 2\beta_- + \dfrac{n-2}{2} = \alpha_1 + \alpha_2$. □

Lemma 2 (Claim (2.9)). *Let $S_b = \{\eta : \gamma_1|\xi|^2 \leq |\eta|^2 + |\xi-\eta|^2 \leq \gamma_2|\xi|^2\}$. If*

$$\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2,$$

$$n \geq 2, \quad \beta_- \geq \alpha_j - \frac{n-1}{2}, \quad \beta_- \geq 0, \quad \textit{and} \quad (\beta_-, \alpha_j) \neq (0, \frac{n-1}{2}),$$

for $j = 1, 2$, then

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_b}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta \leq C, \tag{3.6}$$

for all τ and ξ.

Proof. As in the proof of Lemma 1, we set $\zeta \stackrel{\text{def}}{=} R\Big(\eta - \dfrac{\xi}{2}\Big)$, where the rotation $R\xi = |\xi|e_1$, and

$$\zeta = (X_1, \cdots, X_n) \stackrel{\text{def}}{=} \rho(\cos\phi, \sin\phi\,\omega') \stackrel{\text{def}}{=} \rho\omega. \tag{3.7}$$

Using the above, the identity $d\omega = (\sin\phi)^{n-2}d\phi d\omega'$, and the identities for $|\xi-\eta|$ and $|\eta|$, we can rewrite the integral in (3.6) as

$$\int_{S_b}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta \sim \frac{\rho_0^{n-2}}{\tau^{\alpha_1+\alpha_2}}\int_{-1}^{1}\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp, \tag{3.8}$$

where $\rho_0^2 = \dfrac{1}{4}(2\tau - |\xi|^2)$, the change of variables $p = \cos\phi$, and $\lambda = \dfrac{2|\xi|\rho_0}{\tau} = \dfrac{|\xi|\sqrt{2\tau-|\xi|^2}}{\tau}$.

Again we set $\tau \stackrel{\text{def}}{=} \gamma|\xi|^2$ and then we have $\gamma_1 \leq \gamma \leq \gamma_2$ and $\dfrac{1}{3} \leq \lambda \leq 1$ under the restriction $\tau = |\xi-\eta|^2 + |\eta|^2$ for $\eta \in S_b$. These imply that

$$\tau \sim |\xi|^2, \quad (\gamma-1)^2 \sim (1-\lambda), \quad \text{and} \quad \rho_0 = \frac{1}{2}\sqrt{2\gamma-1}|\xi| \sim |\xi|. \tag{3.9}$$

Now we can combine the above observations to simplify the quantity in (3.1) as follows.

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_b}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta$$

$$\sim|\xi|^{2\beta_0+4\beta_++4\beta_-+n-2-2\alpha_1-2\alpha_2}\Big|\gamma-1\Big|^{2\beta_-}\int_{-1}^{1}\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp.$$

To bound the above integral we split it into two parts, one is over [-1, 0] while the other is over [0, 1]. Since the estimates for the two parts are the same, thus we only prove the second part. First we have

$$\int_0^1\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp\sim\int_0^1\frac{(1-p)^{(n-3)/2}}{(1-\lambda p)^{\alpha_2}}dp \tag{3.10}$$

Using the change of variables $p=-(1-\lambda)q+\lambda$, the integral is changed into

$$(1-\lambda)^{\frac{n-3}{2}+1-\alpha_2}\int_{-1}^{\frac{\lambda}{1-\lambda}}\frac{(1+q)^{(n-3)/2}}{(1+\lambda+\lambda q)^{\alpha_2}}\,dq.$$

Again we split the above integral into two parts. For the first part, we have

$$\int_{-1}^{1}\frac{(1+q)^{(n-3)/2}}{(1+\lambda+\lambda q)^{\alpha_2}}\,dq\le C,$$

provided that $n\ge 2$. For the second part, we have

$$\int_{1}^{\frac{\lambda}{1-\lambda}}\frac{(1+q)^{(n-3)/2}}{(1+\lambda+\lambda q)^{\alpha_2}}\,dq\sim\begin{cases}(1-\lambda)^{-\frac{n-1}{2}+\alpha_2} & \text{for}\quad \alpha_2<\frac{n-1}{2},\\ |\log(1-\lambda)| & \text{for}\quad \alpha_2=\frac{n-1}{2},\\ C & \text{for}\quad \alpha_2>\frac{n-1}{2},\end{cases}$$

Thus we get

$$(1-\lambda)^{\beta_-}\int_0^1\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp$$

$$\sim C(1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}+\begin{cases}(1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}(1-\lambda)^{-\frac{n-1}{2}+\alpha_2} & \text{for}\quad \alpha_2<\frac{n-1}{2},\\ (1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}|\log(1-\lambda)| & \text{for}\quad \alpha_2=\frac{n-1}{2},\\ (1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}C & \text{for}\quad \alpha_2>\frac{n-1}{2},\end{cases}$$

Hence the above integral is bounded if (β_-,α_2) is in the set $S_1\cap(S_2\cup S_3\cup S_4)$, where

$$S_1\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2):\beta_-\ge\alpha_2-\frac{n-1}{2}\},\quad S_2\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2):\beta_-\ge 0,\alpha_2<\frac{n-1}{2}\},$$

$S_3\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2)\colon\beta_->\alpha_2-\frac{n-1}{2},\alpha_2=\frac{n-1}{2}\}$, and $S_4\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2)\colon\beta_-\ge\alpha_2-\frac{n-1}{2},\alpha_2>\frac{n-1}{2}\}$.

Therefore the quantity in (3.6) is bounded if we require that

$$\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2,$$

$$n \geq 2, \quad \beta_- \geq \alpha_2 - \frac{n-1}{2}, \quad \beta_- \geq 0, \quad \text{and} \quad (\beta_-, \alpha_2) \neq (0, \frac{n-1}{2}).$$

The conditions for α_1 is the same as that of α_2. This completes the proof. $\square$

Remark 2. *Notice that for the integral over S_c we have $0 \leq \lambda \leq \frac{1}{3}$, $\gamma_2 \leq \gamma$, $\tau = \gamma|\xi|^2$, and $\rho_0 = \sqrt{2\gamma - 1}|\xi|/2 \sim \sqrt{\gamma}|\xi|$. If we follow the same path to estimates it, then we obtain*

$$\begin{aligned} &\widehat{D}^{2\beta_0} \widehat{S}_+^{2\beta_+} \widehat{S}_-^{2\beta_-} \int_{S_b} \frac{\delta(\tau - |\xi-\eta|^2 - |\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta \\ \sim\ & |\xi|^{2\beta_0 + 4\beta_+ + 4\beta_- + n - 2 - 2\alpha_1 - 2\alpha_2} \gamma^{2\beta_+ + 2\beta_- + \frac{n-2}{2} - \alpha_1 - \alpha_2} \int_{-1}^{1} \frac{(1-u^2)^{\frac{n-3}{2}}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}\, dp \leq C, \end{aligned} \tag{3.11}$$

provided that $\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2$, $n \geq 2$, and $2\beta_+ + 2\beta_- + \frac{n-2}{2} - \alpha_1 - \alpha_2 \leq 0$. The last condition implies that $\beta_0 \geq 0$ which we shall see that this is not good enough.

Lemma 3 (Claim (2.10))**.** *Let $T_c(\eta, \varphi) \stackrel{\text{def}}{=} \{\xi : |\xi-\eta|^2 + |\eta|^2,\ |\xi-\varphi|^2 + |\varphi|^2 \geq \gamma_2|\xi|^2\}$. If $\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2$ and $\beta_0 > -\frac{n-1}{2}$, then*

$$\int_{T_c} \widehat{D}^{2\beta_0} \widehat{S}_+^{2\beta_+} \widehat{S}_-^{2\beta_-} \frac{\delta(|\xi-\varphi|^2 + |\varphi|^2 - |\xi-\eta|^2 - |\eta|^2)}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}} d\xi \leq C, \tag{3.12}$$

where $\tau = |\xi-\varphi|^2 + |\varphi|^2 = |\xi-\eta|^2 + |\eta|^2$.

Proof. Let $\Phi(\xi) \stackrel{\text{def}}{=} |\xi-\varphi|^2 + |\varphi|^2 - |\xi-\eta|^2 - |\eta|^2$ and $P(\varphi,\eta) \stackrel{\text{def}}{=} \{\xi : \Phi(\xi) = 0\}$. Since $|\xi-\varphi|^2 + |\varphi|^2 \geq \gamma_2|\xi|^2$ and $\gamma_2 > 16$, thus we have $|\varphi| + |\xi-\varphi| \geq 4|\xi|$ and analogously we have $|\eta| + |\xi-\eta| \geq 4|\xi|$.

Using triangle inequality, we get

$$\frac{3}{5}|\eta| \leq |\xi-\eta| \leq \frac{5}{3}|\eta| \quad \text{and} \quad \frac{3}{5}|\varphi| \leq |\xi-\varphi| \leq \frac{5}{3}|\varphi|,$$

and then

$$|\xi| \leq \frac{2}{3}\min\{|\eta|,\ |\xi-\eta|,\ |\varphi|,\ |\xi-\varphi|\}.$$

On the plane P we have $|\xi-\varphi|^2 + |\varphi|^2 = |\xi-\eta|^2 + |\eta|^2$ which implies that

$$|\xi-\varphi| \sim |\varphi| \sim |\xi-\eta| \sim |\eta|.$$

Set $(\xi-\eta)\cdot(-\eta) \stackrel{\text{def}}{=} |\xi-\eta||\eta|\cos\theta$, through some calculations we can show that

$$\cos\theta \geq \cos\theta_0 = \frac{\sqrt{5}}{3} > \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}.$$

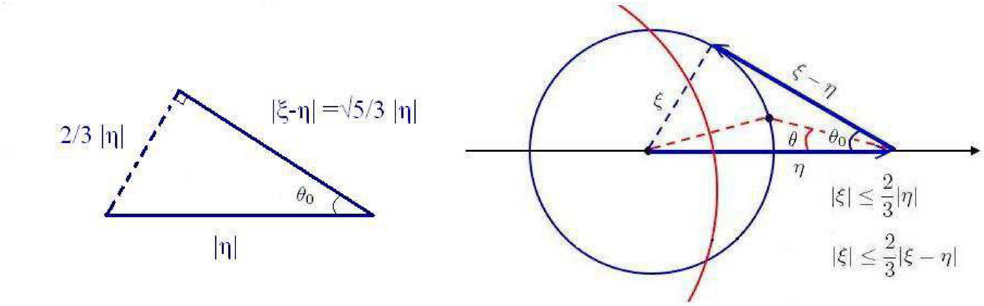

Figure 3:

Hence the angle between $-\eta$ and $\xi-\eta$ is restricted on $0 \leq \theta \leq \theta_0 \leq \dfrac{\pi}{4}$.

Without loss of generality, we assume $|\varphi| > |\eta|$. Follow the idea used in [FK], we decompose $S^{n-2} = \bigcup_{j=1}^{N} \Omega_j$, where Ω_j are disjoint and the angle between any two unit vectors lie in the same Ω_j is less than θ_0, and N is a finite integer. Denote

$$\Gamma_j \stackrel{\text{def}}{=} \left\{ \xi \in \mathbb{R}^n/\{0\} : \frac{\xi}{|\xi|} \in \Omega_j \right\},\ \chi_j \stackrel{\text{def}}{=} \text{characteristic function of}\, \Gamma_j,\ \ f_j \stackrel{\text{def}}{=} \chi_j\, f,\ \text{ and } g_j \stackrel{\text{def}}{=} \chi_j\, g.$$

Thus we have $f = \sum_{j=1}^{N} f_j$ and $g = \sum_{j=1}^{N} g_j$. Then we can split the integral into finitely many pieces,

$$\begin{aligned}
&\left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi-\eta|^2 - |\eta|^2) \widehat{f}(\xi-\eta) \widehat{g}(\eta)\ d\eta \right\|_{L^2(\tau \geq 16|\xi|^2)} \\
\leq\ & \sum_{j,k} \left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi-\eta|^2 - |\eta|^2) \widehat{f}_j(\xi-\eta) \widehat{g}_k(\eta)\ d\eta \right\|_{L^2(\tau \geq 16|\xi|^2,\ \theta_0 < \pi/4)}.
\end{aligned}$$

There exists a cone Γ with an aperture $2\theta_0$ such that $\eta \in \Gamma_k \subset \Gamma$ and $\xi - \eta \in \Gamma_j \subset -\Gamma$.

$$\begin{aligned}
&\left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi-\eta|^2 - |\eta|^2) \widehat{f}_j(\xi-\eta) \widehat{g}_k(\eta)\ d\eta \right\|^2_{L^2(\tau \geq 16|\xi|^2)} \\
=\ & \iiint D^{2\beta_0} S_+^{2\beta_+} S_-^{2\beta_-} \delta(\Phi(\xi)) \widehat{f}_j(\xi-\eta) \widehat{g}_k(\eta) \bar{\widehat{f}}_j(\varphi) \bar{\widehat{g}}_k(\xi-\varphi)\ d\varphi\ d\eta\ d\xi,
\end{aligned}$$

where $\eta \in \Gamma_k \subset \Gamma$, $\xi - \eta \in \Gamma_j \subset -\Gamma$, $\varphi \in \Gamma_j \subset -\Gamma$, and $\xi - \varphi \in \Gamma_k \subset \Gamma$. Through elementary argument, we have the following identity, see [H],

$$\int D^{2\beta_0} S_+^{2\beta_+} S_-^{2\beta_-} \frac{\delta(\Phi(\xi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}} d\xi = \int \frac{D^{2\beta_0} S_+^{2\beta_+} S_-^{2\beta_-}}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}} \frac{d\mu}{|\nabla\Phi(\xi)|}, \tag{3.13}$$

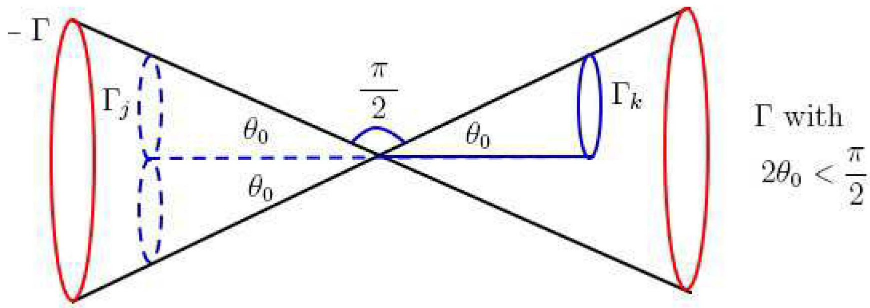

Figure 4:

where $d\mu$ is the surface measure on the surface $\{\xi : \Phi(\xi) = 0\}$. The facts $\xi - \varphi \in \Gamma$ and $\xi - \eta \in -\Gamma$ imply that $|\nabla\Phi(\xi)| \sim |\varphi|$ since

$$|\nabla\Phi(\xi)| = |-2(\varphi - \eta)| = 2\sqrt{|\varphi|^2 + |\eta|^2 - 2\varphi \cdot \eta} \sim |\varphi|.$$

Let ξ' be the projection of ξ onto the plane P, see figure 5,

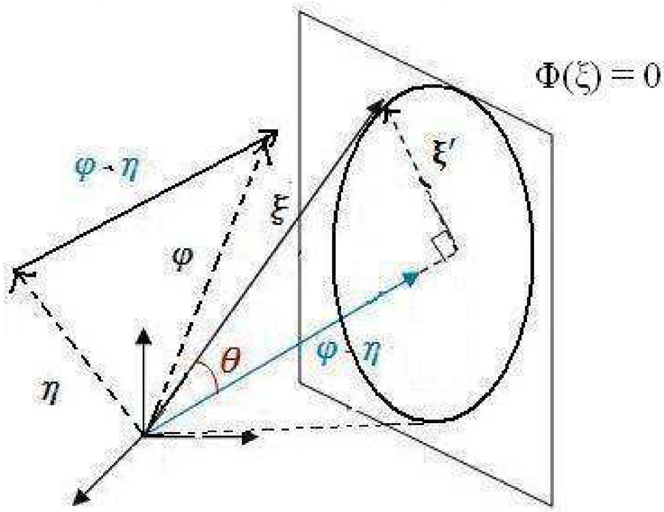

Figure 5:

$$\xi' \stackrel{\text{def}}{=} \xi - \xi \cdot \frac{\varphi - \eta}{|\varphi - \eta|} \frac{\varphi - \eta}{|\varphi - \eta|},$$

and $P_{\frac{\varphi+\eta}{2}}$ the projection of $\dfrac{\varphi + \eta}{2}$ onto the plane P,

$$P_{\frac{\varphi+\eta}{2}} \stackrel{\text{def}}{=} \frac{\varphi + \eta}{2} - \frac{\varphi + \eta}{2} \cdot \frac{\varphi - \eta}{|\varphi - \eta|} \frac{\varphi - \eta}{|\varphi - \eta|}.$$

Denote the rotation taking $\varphi - \eta$ to $|\varphi - \eta| e_1$ by R and the change of coordinates

$$\nu \stackrel{\text{def}}{=} R\left(\xi - \frac{\varphi + \eta}{2}\right) \stackrel{\text{def}}{=} (X_1, X_2, \cdots, X_n).$$

Thus we get $|\xi'| \leq |\xi|$ and

$$\begin{aligned} R\left(\xi' - P_{\frac{\varphi+\eta}{2}}\right) &= R\left(\xi - \frac{\varphi+\eta}{2}\right) - R\left(\xi - \frac{\varphi+\eta}{2}\right) \cdot R\left(\frac{\varphi-\eta}{|\varphi-\eta|}\right) R\left(\frac{\varphi-\eta}{|\varphi-\eta|}\right) \\ &= \nu - \nu \cdot e_1\, e_1 = (X_1, X_2, \cdots, X_n) - (X_1, 0, \cdots, 0) = (0, X_2, \cdots, X_n), \end{aligned}$$

and then $d\xi' = dX_2 \cdots dX_n = d\mu$. Hence we obtain $\delta(\Phi(\xi))d\xi = \dfrac{d\mu}{|\nabla\Phi(\xi)|} \sim \dfrac{d\xi'}{|\varphi|}$.

Therefore we can now bound the integral (3.13) as follows.

$$\begin{aligned} &\int_{P(\eta,\varphi),\,|\xi|\leq\frac{2}{3}|\eta|} |\xi|^{2\beta_0} \Big| |\xi-\varphi|^2 + |\varphi|^2 + |\xi|^2 \Big|^{2\beta_+} \Big| |\xi-\varphi|^2 + |\varphi|^2 - |\xi|^2 \Big|^{2\beta_-} \frac{\delta(\Phi(\xi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\, d\xi \\ &\sim |\varphi|^{4\beta_+ + 4\beta_- - 2\alpha_1 - 2\alpha_2} \int_{|\xi|<|\varphi|} |\xi|^{2\beta_0} \frac{d\xi'}{|\varphi|}. \end{aligned}$$

If $\beta_0 \geq 0$, then we get the bound

$$\int_{|\xi|<|\varphi|} |\xi|^{2\beta_0}\, d\xi' \leq \int_{|\xi|\leq|\varphi|} |\varphi|^{2\beta_0}\, d\xi' \sim |\varphi|^{2\beta_0+n-1}.$$

If $\beta_0 < 0$, then we get the bound

$$\int_{|\xi'|\leq|\varphi|} |\xi|^{2\beta_0}\, d\xi' \leq \int_{|\xi'|\leq|\varphi|} |\xi'|^{2\beta_0}\, d\xi' \sim |\varphi|^{2\beta_0+n-1},$$

provided that $2\beta_0 + n - 1 > 0$. Finally combining the above results we have

$$(3.13) \lesssim |\varphi|^{4\beta_+ + 4\beta_- - 2\alpha_1 - 2\alpha_2 + 2\beta_0 + n - 2} \leq C,$$

provided that $2\beta_+ + 2\beta_- + \beta_0 + \frac{n-2}{2} = \alpha_1 + \alpha_2$ and $2\beta_0 + n - 1 > 0$. This completes the proof. $\square$

Acknowledgement: The author was partially supported by NSC and NCTS(South) in Taiwan. The author also wants to express his gratitude for the hospitality of the Department of Mathematics, Kyoto University, and RIMS during the visit from July 03 2010 to July 10 2010.

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