

O'Shea's Defect Relation for Slowly Moving Targets

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Dedicated to
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In 1924 Rolf Nevanlinna [10] proved his famous defect relation

$$(1) \quad \sum_{a \in G} \delta(f, a) \leq 2$$

for a transcendental meromorphic function f on $\mathbb{C}$ and a finite subset G of $\mathbb{P}_1$. Here $0 \leq \delta(f, a) \leq 1$. If $f^{-1}(a)$ is finite, then $\delta(f, a) = 1$. Thus Picard's theorem [7] follows. In 1933 Henri Cartan extended this defect relation to linearly non-degenerated, holomorphic maps $f : \mathbb{C} \rightarrow \mathbb{P}(V)$. Here V is a complex vector space of dimension $n+1$. Put $V_* = V - \{0\}$. Then $\mathbb{P}(V) = V_*/\mathbb{C}_*$ is the complex projective space defined by V . Let $\mathbb{P} : V_* \rightarrow \mathbb{P}(V)$ be the quotient map. The dual complex vector space V^* consists of all $\mathbb{C}$ -linear functions $\alpha : V \rightarrow \mathbb{C}$. Write also $\langle x, \alpha \rangle = \alpha(x)$ if $x \in V$. If $\alpha \in V^*$ then $a = \mathbb{P}(\alpha)$ and $\mathbb{P}(\alpha^{-1}(0))_* = E[a]$ is a hyperplane in $\mathbb{P}(V)$. The map $\alpha \rightarrow E[a]$ parameterizes the set of all hyperplanes in $\mathbb{P}(V)$ bijectively. Now f is said to be linearly non-degenerated if $f(\mathbb{C})$ is not contained in any hyperplane. If so, the defect $\delta(f, a)$ is defined for all $a \in \mathbb{P}(V^*)$ with $0 \leq \delta(f, a) \leq 1$. The subset G of $\mathbb{P}(V)$ with $n+1 \leq \#G < \infty$ is said to be in general position if $\#G \cap E[b] \leq n$ for all $b \in \mathbb{P}(V)$. Under these assumptions Cartan [1] proved

$$(2) \quad \sum_{a \in G} \delta(f, a) \leq n+1.$$

In 1973 Philipp Griffiths and James King[2] proved a defect relation for dominant, holomorphic maps $f : M \rightarrow N$. Here dominant means that $f(M)$ contains a non-empty open subset of N . Thus $\dim M = m \geq \text{rank } f = n = \dim N$. They assume that M is a connected, affine algebraic manifold spread over $\mathbb{C}^m$ by a proper, surjective holomorphic map $\pi : M \rightarrow \mathbb{C}^m$. Later Stoll [13] extended the theory to parabolic manifolds M . Griffiths and King assume, that N is a connected, compact complex manifold with a positive holomorphic line bundle L on N . Thus N is projective algebraic. Let K_N be the canonical bundle on N and let K_N^* be its dual bundle. Define

$$[K_N^* : L] = \inf \left\{ \frac{p}{q} \mid L^p \otimes K_N^q \text{ positive, } 0 \leq p \in \mathbb{Z}, \text{ and } 0 < q \in \mathbb{Z} \right\}.$$

The set $\Gamma(N, L)$ of all global holomorphic sections of L is a finite dimensional complex vector space. We assume that $\dim \Gamma(N, L) \geq 2$. If $a \in \mathbb{P}(\Gamma(N, L))$, then $a = \mathbb{P}(\alpha)$ with $0 \neq \alpha \in \Gamma(N, L)$. The divisor of the section α depends on a only and is denoted by μ_a . The assignment $a \rightarrow \mu_a$ is injective. Hence we identify $a = \mu_a$. Let $G \neq \emptyset$ a finite subset of $\mathbb{P}(\Gamma(N, L))$ with strictly normal crossings. If f has sufficient growth Griffiths and King show that

$$(3) \quad \sum_{a \in G} \delta(f, a) \leq [K_N^* : L].$$

Here again $0 \leq \delta(f, a) \leq 1$. For instance if $N = \mathbb{P}(V)$ and if $L = \mathcal{O}(p) = H^p$ is the p^{th} power of the hyperplane section bundle, then

$$(4) \quad \sum_{a \in G} \delta(f, a) \leq \frac{n+1}{p}.$$

In 1929, Rolf Nevanlinna [4] asked if (1) remains valid if G is a finite set of meromorphic functions growing slower than f . In 1986 Norbert Steinmetz [12] proved this conjecture. Min Ru and myself [8], [9] obtained the corresponding result for Cartan's defect relation (2). The proofs for (1) and (2) require the construction of auxiliary Steinmetz maps $h_p : \mathbb{C} \rightarrow \mathbb{P}_{q(p)}$ where $q(p) \rightarrow \infty$ for $p \rightarrow \infty$. Even if we would find analogous Steinmetz maps $h_p : \mathbb{C} \rightarrow N_{q(p)}$ with $\dim N_{q(p)} = q(p) \rightarrow \infty$ for $p \rightarrow \infty$, the map h_p could not possibly be dominant as soon as $q(p) > m$. If (3) can be saved for non-dominant maps is one of the most difficult unsolved problems in value distribution theory.

Thus Ann O'Shea used an older, but more restrictive method of Bernard Shiffman [10], [11], which Stoll employed to study slowly growing associated target maps by the Ta -operator [14]. In 1983, he gave a lengthy report about this theory at a RIMS conference. Because (4) is explicit, O'Shea considers only this situation. The main difficulty is to construct a parameterized Carlson-Griffiths form and to measure the deterioration of strictly normal crossings. This task is made more difficult than the corresponding problem of general position for hyperplanes.

First some basic concepts have to be explained before the result can be stated.

Divisors. Let N be a connected complex manifold of dimension n . A function $\nu : N \rightarrow \mathbb{Z}$ is said to be an (effective) divisor if and only if every point $p \in N$ has an open, connected neighborhood U with a holomorphic function $g \not\equiv 0$ on U such that for each $z \in U$ the number $\nu(z)$ is the zero multiplicity of g at z . Here g is called a defining function of ν on U . The set

$$\text{supp } \nu = \{z \in N \mid \nu(z) > 0\}$$

is called the support of ν . If $\nu \not\equiv 0$, then $\text{supp } \nu$ is a pure $(n-1)$ -dimensional analytic subset of N . If $\nu \equiv 0$, then $\text{supp } \nu = \emptyset$. If $f \not\equiv 0$ is a holomorphic function or a holomorphic section of a holomorphic line bundle on N , the zero divisor μ_f of f is defined.

Let $\nu_1, \dots, \nu_q$ be divisors on N . Put $S_j = \text{supp } \nu_j$ and $S = S_1 \cup \dots \cup S_q$. Take any $a \in S$. Define $I(a) = \{j \in \mathbb{N}[1, q] \mid a \in S_j\}$. Then $I(a) \neq \emptyset$. Thus $j_\lambda \in \mathbb{N}[1, q]$ exist uniquely for $\lambda = 1, \dots, k$ such that $I(a) = \{j_1, \dots, j_k\}$ and $j_1 < \dots < j_k$. Then there is an open, connected neighborhood U of a and for each $\lambda \in \mathbb{N}[1, k]$ a defining function g_λ of ν_{j_λ} . Then $\nu_1, \dots, \nu_q$ are said to have strictly normal crossings at a if and only if

$$dg_1(a) \wedge \dots \wedge dg_k(a) \neq 0.$$

The condition is independent of the choice of the defining functions g_j . Trivially $k \leq n$. Here $\nu_1, \dots, \nu_q$ are said to have strictly normal crossings, if they have strictly normal crossings at every $a \in S$. If $f_1 \not\equiv 0, \dots, f_q \not\equiv 0$ are holomorphic functions on N , respectively holomorphic sections in a holomorphic line bundle on N they are said to have strictly normal crossings (at a) if this is the case for their divisors.

Symmetric Tensor Product. Let V be a complex vector space of dimension $n+1 > 1$ with a hermitian metric attached. Then V is called a hermitian vector space. Take $p \in \mathbb{N}$. The p -fold symmetric tensor product $\odot_p V$ is a hermitian vector space of dimension

$\binom{n+p}{p}$. If $x \in V$, Then $x^p = x \odot \cdots \odot x \in \odot_p V$ with $\|x^p\| = \|x\|^p$. Hence if $x \neq 0$, then $x^p \neq 0$. If $x \in \mathbb{P}(V)$, then $x = \mathbb{P}(x)$ with $0 \neq x^p \in \odot_p V$ and $x^p = \mathbb{P}(x^p) \in \mathbb{P}(\odot_p V)$ is well defined. The Veronese map $\varphi_p : \mathbb{P}(V) \rightarrow \mathbb{P}(\odot_p V)$ defined by $\varphi_p(x) = x^p$ embeds $\mathbb{P}(V)$ into $\mathbb{P}(\odot_p V)$.

The dual vector space V^* of V carries the dual hermitian metric. If $\eta \in V^*$, then $\eta : V \rightarrow \mathbb{C}$ is a $\mathbb{C}$ -linear function. We define the inner product $\langle, \rangle$ between V and V^* by $\langle x, \eta \rangle = \eta(x)$ if $x \in V$ and $\eta \in V^*$. Then

$$|\langle x, \eta \rangle| \leq \|x\| \|\eta\|.$$

Thus if $x = \mathbb{P}(x) \in \mathbb{P}(V)$ and $y = \mathbb{P}(\eta) \in \mathbb{P}(V^*)$, then

$$0 \leq \square x, y \square = \frac{|\langle x, \eta \rangle|}{\|x\| \|\eta\|} \leq 1$$

is well defined. The hermitian metric on V induces a hermitian metric λ along the fibers of the hyperplane section bundle $H = \mathcal{O}(1)$ whose Chern form $c(h, \lambda) = \Omega > 0$ is called the Fubini Study form on $\mathbb{P}(V)$. If $p \in \mathbb{N}$, then $c(H^p, \lambda^p) = p\Omega$ is the Chern form of $H^p = \mathcal{O}(p)$. As hermitian vector spaces. We have the identity $(\odot_p V)^* = \odot_p(V^*)$ and as complex vector spaces

$$\odot_p V^* = \Gamma(\mathbb{P}(V), H^p).$$

Thus each $\eta \in \odot_p V^*$ can be regarded as a holomorphic section in H^p and if $\eta \neq 0$, this section defines a divisor $\mu_\eta = \mu_y$ which in fact depends only on the projective value $y = \mathbb{P}(\eta)$. The assignment $y \rightarrow \mu_y$ is injective and we can identify $y = \mu_y$. As such each element $y \in \mathbb{P}(\odot_p V^*)$ is said to be a hypersurface of degree p on $\mathbb{P}(V)$, which is not to be mixed up with its support

$$\text{supp } y = \text{supp } \mu_y = \{x \in \mathbb{P}(V) | \square x^p, y \square = 0\}.$$

How does $\eta \in \odot_p V^*$ become a holomorphic section in H^p ? In order of explanation, we will also denote η as $\tilde{\eta}$ in its section capacity and we have to calculate $\tilde{\eta}(x)$ for any given $x \in \mathbb{P}(V)$. Observe $(H^p)_x = ((\mathcal{O}(-1)_x)^p)^*$. Take any $x \in V_*$ with $x = \mathbb{P}(x)$. Then x is a base of $\mathcal{O}(-1)_x$. Thus x^p is a base of $(\mathcal{O}(-1)_x)^p$ and $\tilde{\eta}(x) : (\mathcal{O}(-1)_x)^p \rightarrow \mathbb{C}$ a $\mathbb{C}$ -linear function. Hence if $z \in (\mathcal{O}(-1)_x)^p$, then $z \in \mathbb{C}$ exist uniquely such that $z = z x^p$ and the assignment $z \rightarrow z$ is $\mathbb{C}$ -linear. Then $\tilde{\eta}(x)(z) = z \langle x^p, \eta \rangle$.

A holomorphic function $f : V \rightarrow \mathbb{C}$ is said to be a homogeneous polynomial of degree p if and only if $f(zx) = z^p f(x)$ for all $(z, x) \in \mathbb{C} \times V$. The vector space of all homogeneous

polynomials of degree p is $\mathbb{C}$ -linear isomorphic to $\odot_p V^*$. Thus $\eta \in \odot_p V^*$ in its capacity as homogeneous polynomial on V is denoted by $\hat{\eta}$. If $x \in V$, then $\hat{\eta}(x) = \langle x^p, \eta \rangle$. The distinction is important:

$$\begin{aligned} d\hat{\eta}(x, v) &= p \langle x^{p-1} \odot v, \eta \rangle & \text{if } x \in V_1 \text{ and } v \in V \\ d\eta(z, w) &= \langle w, \eta \rangle & \text{if } z \in \odot_p V \text{ and } w \in \odot_p V \end{aligned}$$

and $d\hat{\eta}$ does not make sense.

Parabolic manifolds (M, τ) is said to be a parabolic manifold of dimension m if and only if

- 1: M is a connected, complex manifold of dimension m .
- 2: $\tau \geq 0$ is a non-negative, unbounded function of class C^∞ on M .
- 3: If $0 \leq r \in \mathbb{R}$ and $S \subseteq M$, abbreviate

$$\begin{aligned} S[r] &= \{x \in S | \tau(x) \leq r^2\} & S(r) &= \{x \in S | \tau(x) < r^2\} \\ S < r > &= \{x \in S | \tau(x) = r^2\} & S_* &= \{x \in S | \tau(x) > 0\} \end{aligned}$$

$$v = dd^c \tau \quad \omega = dd^c \log \tau \quad \sigma = d^c \log \tau \wedge \omega^{m-1}.$$

- 4: $M[r]$ is compact for all $r > 0$.
- 5: $\omega^m \equiv 0 \not\equiv v^m$ and $\omega \geq 0$.

Then $v \geq 0$. Define $M^+ = \{x \in M | v(x) > 0\}$. A positive constant $\varsigma > 0$ exists such that for all $r > 0$ we have

$$\int_{M < r >} \sigma = \varsigma \quad \int_{M[r]} v^m = \varsigma r^{2m}.$$

Let $\psi > 0$ be a positive form of degree $2m$ and of class C^∞ on M . A non-negative function u on M is defined by $v^m = u^2 \psi$. For $0 < s < r$ the Ricci function of τ is defined by

$$Ric_\tau(r, s) = \int_{M < r >} \log u \, \sigma - \int_{M < s >} \log u \, \sigma + \int_s^r \int_{M[t]} ric \psi \wedge v^{m-1} t^{1-2m} dt$$

and does not depend on the choice of ψ .

Let ν be a divisor on M with $S = \text{supp } \nu$. For $0 < s < r$, the valence function N_ν of ν is defined by

$$N_\nu(r, s) = \int_s^r \int_{S[t]} \nu v^{m-1} t^{1-2m} dt.$$

The First Main Theorem Let $f : M \rightarrow \mathbb{P}(V)$ be a meromorphic map with indeterminacy I_f . Let $U \neq \emptyset$ be an open connected subset of M . A holomorphic map $v : u \rightarrow V$ is said

to be a reduced representation of f , if $U \cap I_f = \mathfrak{v}^{-1}(0)$ and $f|(U - I_f) = \mathbb{P} \circ \mathfrak{v}$. Every point of M has an open, connected neighborhood admitting a reduced representation of f .

For $0 < s < r$ the characteristic function T_f of f is defined by

$$T_f(r, s) = \int_s^r \int_{M[t]} f^* \Omega \wedge v^{m-1} t^{1-2m} dt \geq 0.$$

Then $T_f \equiv 0$ if and only if f is constant. If f is not constant, $T_f(r, s) \rightarrow \infty$ for $r \rightarrow \infty$. Actually T_f is the characteristic in respect to H . Thus for H^p we have the characteristic pT_f .

Let $g : M \rightarrow \mathbb{P}(\odot_p V^*)$ be a meromorphic map. Then (f, g) are said to be free if $\square f^p, g \square \neq 0$. If so, the compensation function $m_{f,g}$ is defined for $r > 0$ by

$$m_{f,g}(r) = \int_{M < r >} \log \frac{1}{\square f^p, g \square} \sigma \geq 0.$$

Here (f, g) is free, if $\langle \mathfrak{v}^p, \mathfrak{w} \rangle \neq 0$ for one and therefore every choice of reduced representations $\mathfrak{v} : U \rightarrow V$ of f and $\mathfrak{w} : U \rightarrow \odot_p V^*$ of g . If so, there exists one and only one divisor $\mu_{f,g}$ such that $\mu_{f,g}|U$ is the zero divisor of the holomorphic function $\langle \mathfrak{v}^p, \mathfrak{w} \rangle \neq 0$ on U for each possible choice of $U, \mathfrak{v}, \mathfrak{w}$. Abbreviate $N_{\mu_{f,g}} = N_{f,g}$. For $0 < s < r$ we have the First Main Theorem

$$pT_f(r, s) + T_g(r, s) = N_{f,g}(r, s) + m_{f,g}(r) - m_{f,g}(s).$$

Assume that f or g or both are not constant. Then the defect of (f, g) is defined by

$$0 \leq \delta(f, g) = \lim_{r \rightarrow \infty} \frac{m_{f,g}(r)}{pT_f(r, s) + T_g(r, s)} = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{N_{f,g}(r, s)}{pT_f(r, s) + T_g(r, s)} \leq 1.$$

If g moves slower than f , that is, if $T_g(r, s)/T_f(r, s) \rightarrow 0$ for $r \rightarrow \infty$, then we can omit the term T_g in the definition of the defect.

Take $\mathfrak{a} \in \odot_p V$ and $\mathfrak{b} \in (\odot_p V)^*$. Put $b = \mathbb{P}(\mathfrak{b})$. Assume that (b, g) is free. Then there exist a unique meromorphic function $g_{\mathfrak{a}\mathfrak{b}}$ on M , called a coordinate function of g such that for each reduced representation $\mathfrak{w} : U \rightarrow \odot_p V^*$ of g we have

$$g_{\mathfrak{a}\mathfrak{b}}|U = \frac{\langle \mathfrak{a}, \mathfrak{w} \rangle}{\langle \mathfrak{b}, \mathfrak{w} \rangle}.$$

If so, there is a constant C_s for each $s > 0$, such that

$$T_{g_{\mathfrak{a}\mathfrak{b}}}(r, s) \leq T_g(r, s) + C_s$$

for all $r > s$.

The Second Main Theorem More definitions are needed.

Let $g_j : M \rightarrow \mathbb{P}(\odot_p V^*)$ be holomorphic maps for $j = 1, \dots, q$. Then $g_1, \dots, g_q$ are said to have strictly normal crossings if there exists at least one point $x_0 \in M$ such that $g_j(x_0) \neq g_t(x_0)$ for $1 \leq j < t \leq q$ and if $g_1(x_0), \dots, g_q(x_0)$ have strictly normal crossings.

Let P_q be the set of all bijective maps $\alpha : \mathbb{N}[1, q] \rightarrow \mathbb{N}[1, q]$. A function

$$\Gamma : \mathbb{P}(V) \times \mathbb{P}(\odot_p V^*)^q \rightarrow \mathbb{R}[0, 1]$$

shall be defined. Take $x \in \mathbb{P}(V)$ and $y = (y_1, \dots, y_q) \in \mathbb{P}(\odot_p V^*)^q$. Take $\mathfrak{x} \in V_*$ with $\mathbb{P}(\mathfrak{x}) = x$ and $0 \neq \eta_j \in \odot_p V^*$ with $\mathbb{P}(\eta_j) = y_j$ for $j = 1, \dots, q$. Put $\eta = (\eta_1, \dots, \eta_q)$. For $(\alpha, s) \in P_q \times \mathbb{N}[1, q]$ define $W_{\alpha s}(\mathfrak{x}, \eta) \in \wedge_s V^*$ by

$$W_{\alpha s}(\mathfrak{x}, \eta) = \left(\prod_{\lambda = s+1}^q \langle \mathfrak{x}^\lambda, \eta_{\alpha(\lambda)} \rangle \right) d\hat{\eta}_{\alpha(1)}(\mathfrak{x}) \wedge \dots \wedge d\hat{\eta}_{\alpha(s)}(\mathfrak{x})$$

$$\Gamma_{\alpha s}(x, y) = \frac{\|W_{\alpha s}(\mathfrak{x}, \eta)\|}{p^s \|\mathfrak{x}\|^{pq-s} \|\eta_1\| \dots \|\eta_s\|}$$

$$0 \leq \Gamma(x, y) = \frac{1}{(q!)n} \sum_{s=1}^q \sum_{\alpha \in P_q} \Gamma_{\alpha s}(x, y) \leq 1.$$

Then

$$I_q = \bigcap_{s=1}^q \bigcap_{\alpha \in P_q} (W_{\alpha s})^{-1}(0) \text{ is analytic in } V \times (\odot_p V^*)^q.$$

If $0 \neq \mathfrak{x} \in V$ and $0 \neq \eta_j \in \odot_p V^*$ for $j = 1, \dots, q$ and if $\eta = (\eta_1, \dots, \eta_q)$, then $(\mathfrak{x}, \eta) \in I_q$ if

and only if $\mathfrak{x} \in \bigcup_{j=1}^q \hat{\eta}_j^{-1}(0)$ and the zero divisors of $\hat{\eta}_1, \dots, \hat{\eta}_q$ do not have strictly normal crossings at $\mathfrak{x}$. Similar

$$\ddot{I}_q = \{(x, y) \in \mathbb{P}(V) \times \mathbb{P}(\odot_p V^*)^q \mid \Gamma(x, y) = 0\}$$

is an analytic subset of $\mathbb{P}(V) \times \mathbb{P}(\odot_p V^*)^q$. If $x \in \mathbb{P}(V)$ and $y_j \in \mathbb{P}(\odot_p V^*)$ for $j = 1, \dots, q$ are

given, if $y = (y_1, \dots, y_q)$, then $(x, y) \in \ddot{I}_q$ if and only if $x \in \bigcup_{j=1}^q \text{supp } y_j$ and if the divisors $y_1, \dots, y_q$ do not have strictly normal crossings at x . Let $f : M \rightarrow \mathbb{P}(V)$ and $g_j : M \rightarrow \odot_p V^*$ be holomorphic maps. Put $h = (f, g_1, \dots, g_q)$ and $g = (g_1, \dots, g_q)$. Thus if $g_1, \dots, g_q$ have strictly normal crossings, then $\Gamma \circ h \not\equiv 0$ and

$$\Gamma(r) = \int_{M \langle r \rangle} \log \frac{1}{\Gamma \circ h} \sigma \geq 0$$

is defined for $r > 0$.

Let M and N be connected, complex manifolds of dimensions m and n respectively with $k = m - n > 0$. Let $f : M \rightarrow N$ be a holomorphic map. Let B be a holomorphic form of bidegree $(k, 0)$ on N . Then B is said to be effective for f if for every open connected subset U of N with $\tilde{U} = f^{-1}(U) \neq \emptyset$ and for every holomorphic form χ of bidegree $(n, 0)$ on U without zeroes $B \wedge f^*(\chi) \not\equiv 0$ on each component of $\tilde{U}$. If so, then there is one and only one divisor ρ on M called the ramification divisor for B, f on M such that $\rho|_{\tilde{U}}$ is the zero divisor of $B \wedge f^*(\chi)$ for each choice of U and χ .

Define

$$i_k = \left(\frac{i}{2\pi}\right)^k (-1)^{\frac{k(k-1)}{2}} k!.$$

Then B is said to be majorized by a function $Y : \mathbb{R}_+ \rightarrow \mathbb{R}[1, +\infty)$ if

$$i_k B \wedge \bar{B} \leq \left(\frac{Y(r)}{m}\right)^n v^k.$$

on $M[r]$ for every $r \geq 0$. If so, Y can be taken optimal.

If $r_0 \in \mathbb{R}$ and if u and v are real valued functions on $\mathbb{R}[r_0, +\infty)$. Then $u \leq v$ means that there is some subset E of finite measure in $\mathbb{R}$ such that $u \leq v$ on $\mathbb{R}[r_0, +\infty) - E$.

Second Main Theorem (O'Shea [6]). We assume

- (A1) Let V be a hermitian vector space of dimension $n + 1 > 1$.
- (A2) Let (M, τ) be a parabolic manifold of dimension m with $k = m - n > 0$.
- (A3) Let p and q be positive integers with $pq > n + 1$.
- (A4) Let $f : M \rightarrow \mathbb{P}(V)$ and $g_j : M \rightarrow \mathbb{P}(\odot_p V^*)$ for $j = 1, \dots, q$ be holomorphic maps.
- (A5) Let (f, g_j) be free for $j = 1, \dots, q$.
- (A6) Let $g_1, \dots, g_q$ have strictly normal crossings.
- (A7) Let B be a holomorphic form of bidegree $(s, 0)$ on M . Assume that B is effective for f with ramification divisor ρ and that B is majorized by Y such that

$$f^*(\Omega^n) \wedge B \wedge \bar{B} \not\equiv 0.$$

- (A8) $dg_{jab} \wedge B \equiv 0$ for every coordinate function of every $g_1, \dots, g_q$.

Now, take any $\varepsilon > 0$; then

$$\begin{aligned} N_\rho(r, s) + \sum_{j=1}^q m_{f, g_j}(r) &\leq (n+1)T_f(r, s) + Ric_\tau(r, s) + \Gamma(r) + \varepsilon \log r \\ &\quad + \frac{n}{2} \varsigma(1 + \varepsilon)(\log^+ T_f(r, s) + \log Y(r) + \log^+ \sum_{j=1}^q T_{g_j}(r, s)). \end{aligned}$$

For the proof Julann O'Shea constructs a "Griffiths" form ξ on $\mathbb{P}(V) \times \mathbb{P}(\odot_p V^*)^q = X$ such that there is a constant $\gamma > 0$ and a form Θ on X such that

$$(Ric\xi)^n \geq \gamma(\Omega^n + \Gamma^2\xi) + \Theta.$$

If $h = (f, g_1, \dots, g_q) : M \rightarrow X$, then Θ is constructed such that $h^*(\Theta) \wedge B \wedge \bar{B} \equiv 0$.

Theorem of O'Shea ([6]) Assume that (A1) - (A8) hold. then there are integers $a_j \geq 0$ for $j = 1, \dots, q$ and $b > 0$, and for each $s > 0$ a constant C_s , such that $r > s$ implies

$$\Gamma(r) \leq \sum_{j=1}^q b a_j T_{g_j}(r, s) + C_s.$$

The proof shall be sketched. Abbreviate

$$\begin{aligned} Z &= V \times (\odot_p V^*)^q & Z_* &= V_* \times ((\odot_p V^*)_*)^q \\ Z^1 &= (\odot_p V^*)^q & Z_*^1 &= ((\odot_p V^*)_*)^q \\ X &= \mathbb{P}(V) \times \mathbb{P}(\odot_p V^*)^q & X^1 &= \mathbb{P}(\odot_p V^*)^q. \end{aligned}$$

Let $\psi : Z \rightarrow Z^1, \psi_0 : Z_* \rightarrow Z_*^1, \pi : X \rightarrow X^1$ be the projections and define $\mathbb{P} : Z_* \rightarrow X$ and $\mathbb{P} : Z_*^1 \rightarrow X^1$ by

$$\begin{aligned} \mathbb{P}(\mathfrak{x}, \eta_1, \dots, \eta_q) &= (\mathbb{P}(\mathfrak{x}), \mathbb{P}(\eta_1), \dots, \mathbb{P}(\eta_q)) \\ \mathbb{P}(\eta_1, \dots, \eta_q) &= (\mathbb{P}(\eta_1), \dots, \mathbb{P}(\eta_q)) \end{aligned}$$

if $\mathfrak{x} \in V_*$ and $\eta_j \in (\odot_p V^*)_*$ for $j = 1, \dots, q$. Then $\mathbb{P} \circ \psi_0 = \pi \circ \mathbb{P}$. The analytic subsets I_q of Z and $\tilde{I}_q$ of X where defined above. Then $I_q \neq Z$ and $\tilde{I}_q \neq X$ and $\tilde{I}_q = \mathbb{P}(I_q \cap Z_*)$. Also the projections $I_q^1 = \psi(I_q)$ and $\tilde{I}_q^1 = \pi(\tilde{I}_q)$ are analytic in Z^1 respectively X^1 with $\mathbb{P}(I_q^1 \cap Z_*^1) = \tilde{I}_q^1$. If $\eta = (\eta_1, \dots, \eta_q) \in Z_*^1$ then $\eta \in I_q^1$ if and only if the divisor of $\hat{\eta}_1, \dots, \hat{\eta}_q$ do not have strictly normal crossings. Also if $y = (y_1, \dots, y_q) \in X^1$, then $y \in \tilde{I}_q^1$ if and only if $y_1, \dots, y_q$ do not have strictly normal crossings.

There is a polynomial $Q \neq 0$ of multidegree $(a_1, \dots, a_q) \in \mathbb{Z}^q$ with $a_j \geq 0$ for $j = 1, \dots, q$ on Z^1 such that

$$Q(z_1 \eta_1, \dots, z_q \eta_q) = z_1^{a_1} \dots z_q^{a_q} Q(\eta_1, \dots, \eta_q)$$

for all $z_j \in \mathbb{C}$ and $\eta_j \in \odot_p V^*$ for $j = 1, \dots, q$ and such that $Q|_{I_q^1} = 0$. Moreover if we take reduced representations $\mathfrak{w}_j : U \rightarrow \odot_p V^*$ for $j = 1, \dots, q$, put $\mathfrak{w} = (\mathfrak{w}_1, \dots, \mathfrak{w}_q) : U \rightarrow Z^1$, then $Q \circ \mathfrak{w} \neq 0$ on u . Let $\mathfrak{R}$ be the ring of all holomorphic polynomials on Z . Let $\mathfrak{a}$ be the ideal generated by the coordinate functions in respect to a fixed base of all the W_{α_s} within $\mathfrak{R}$. Then $\mathfrak{a}$ is finitely generated and $\text{loc } \mathfrak{a} = I_q$. Also $(Q \circ \psi)|_{I_q} = 0$. By the Hilbertsche Nullstellensatz $b \in \mathbb{N}$ exists such that $(Q \circ \psi)^b \in \mathfrak{a}$. A function $\hat{\Gamma} : X^1 \rightarrow \mathbb{R}[0,1]$ is defined by

$$\hat{\Gamma}(y) = \frac{|Q(\eta)|}{\|\eta_1\|^{a_1} \dots \|\eta_q\|^{a_q}}$$

for all $y = (y_1, \dots, y_q) \in X^1$ with $\eta = (\eta_1, \dots, \eta_q) \in Z_*^1$ such that $\mathbb{P}(\eta_j) = y_j$ for $j = 1, \dots, q$. W.l.o.g. we can assume $0 \leq \hat{\Gamma} \leq 1$ by multiplying Q by a constant. Since $(Q \circ \psi)^b \in \mathfrak{a}$, we obtain a constant $c > 0$ such that

$$\hat{\Gamma}(y)^b \leq c \Gamma(x, y)$$

for all $x \in \mathbb{P}(V)$ and $y \in X^1$. According to Stoll [14] Q can be regarded as projective operation homogenous of degree $(a_1, \dots, a_q)$ (see page 58) which is free for $g = (g_1, \dots, g_q) : M \rightarrow X^1$ (page 60). We abbreviate the symbol $g_1 \dot{Q} g_2 \dot{Q} \dots \dot{Q} g_q$ to $\dot{Q}g$. Then Theorem 3.4[14] page 141 yields the First Main Theorem

$$\sum_{j=1}^q a_j T_{g_j}(r, s) = N_{\dot{Q}g}(r, s) + m_{\dot{Q}g}(r) - m_{\dot{Q}g}(s)$$

where $N_{\dot{Q}g} \geq 0$ and

$$m_{\dot{Q}g}(r) = \int_{M < r} \log \frac{1}{\hat{\Gamma}(y)} \sigma.$$

Thus

$$\Gamma(r) = \int_{M < r} \log \frac{1}{\Gamma_0 h} \sigma \leq b m_{\dot{Q}g}(r) + \varsigma \log c.$$

Take $s > 0$. Define $C_s = b m_{\dot{Q}g}(s) + \varsigma \log c$. Take any $r > s$. Then

$$\Gamma(r) \leq \sum_{j=1}^q b a_j T_{g_j}(r, s) + C_s.$$

If, in addition we make the standard assumptions

$$(A9) \quad T_{g_j}(r, s)/T_f(r, s) \rightarrow 0 \quad \text{for } r \rightarrow \infty \text{ for } j = 1, \dots, q$$

$$(A10) \quad Ric_\tau(r, s)/T_f(r, s) \rightarrow 0 \quad \text{for } r \rightarrow \infty$$

$$(A11) \quad \log Y(r)/T_f(r, s) \rightarrow 0 \quad \text{for } r \rightarrow \infty.$$

O'Shea's Defect Relation [6]

$$\sum_{j=1}^q \delta(f, g_j) \leq \frac{n+1}{r}$$

follows. Observe that we have to divide by pT_f .

If M is a connected, complex manifold of dimension m and if $\pi = (\pi_1, \dots, \pi_m) : M \rightarrow \mathbb{C}^m$ is a proper surjective holomorphic map, defined $\tau = \|\pi\|^2$. Then (M, τ) is a parabolic manifold of dimension m called a parabolic covering manifold of $\mathbb{C}^m$. The zero divisor β of $d\pi_1 \wedge \dots \wedge d\pi_m$ is called the branching divisor. Then $Ric_\tau(r, s) = N_\beta(r, s) \geq 0$. In this case we can replace (A10) by

(A10') $j \in \mathbb{N} [1, q]$ exists such that g_j separates the fibers of π .

If so, then

$$\limsup_{r \rightarrow \infty} \frac{N_\beta(r, s)}{T_{g_j}(r, s)} \leq 2\varsigma - 2$$

by a theorem of Junjiro Noguchi [5]. Thus (A9) implies (A10).

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