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Modified Amoeba Formulation
of the Non-Hermitian Skin Effect
in One-Dimensional Multiband Systems

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ABSTRACT

Topology has become a central concept in modern condensed matter physics, providing profound insights into the qualitative behavior of quantum systems. Recently, non-Hermitian physics has opened a new frontier of topological phenomena, among which the non-Hermitian skin effect (NHSE) stands out as a striking example. The NHSE is characterized by an extreme sensitivity to boundary conditions, where an extensive number of bulk eigenstates accumulate at the system boundaries even in the thermodynamic limit. This behavior invalidates conventional Bloch band theory and often leads to severe numerical instabilities in real-space diagonalization.

While non-Bloch band theory successfully captures the NHSE in one-dimensional systems, its extension to higher dimensions remains a formidable challenge. The Amoeba formulation provides a promising approach by introducing a spectral potential obtained through the optimization of the Ronkin function, and is mathematically grounded in the strong Szegő limit theorem and its topological generalizations. However, its rigorous validity has been established primarily for single-band systems. In multiband systems, competing contributions from different bands obstruct the optimization procedure, leaving both the validity criteria and the topological origin of the Amoeba formulation unclear. In particular, symmetry-protected degeneracies can invalidate the conventional optimization scheme and lead to incorrect evaluations even in one-dimensional systems.

In this thesis, we address these issues by establishing a comprehensive framework that extends the Amoeba formulation to one-dimensional multiband systems using Hermitian doubling and the Wiener–Hopf factorization (WHF). We demonstrate that the partial indices determined by the WHF provide a definitive criterion for whether the spectral potential is independent of boundary conditions. Physically, these partial indices correspond to the presence or absence of topological edge modes in the associated doubled Hermitian system. Even when the spectral potentials under different boundary conditions do not coincide, we show that their deviation can be quantitatively evaluated in terms of the localization lengths of these edge modes. As a result, the conventional condition of a vanishing spectral winding number in single-band systems is generalized to the requirement of vanishing residual partial indices in multiband systems.

We further extend this framework to non-Hermitian systems respecting time-reversal symmetry of transpose type ($\text{TRS}^\dagger$). In such systems, $\text{TRS}^\dagger$ protects Kramers pairs with opposite localization lengths, rendering the conventional optimization of the Ronkin function inapplicable. By employing the WHF, we introduce a symmetry-based decomposition of the Ronkin function that isolates the contributions from each symmetry-protected sector, allowing for a correct evaluation of the spectral potential.

Taken together, this work establishes a firm theoretical foundation for the Amoeba formulation in one-dimensional multiband systems and provides a systematic guideline for its generalization to symmetry-protected cases. Moreover, it lays the groundwork for extending non-Bloch band theory to higher-dimensional and geometry-dependent non-Hermitian systems, where the interplay of topology, symmetry, and boundary geometry plays a crucial role.

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List of publications

1. **Shin Kaneshiro** and Robert Peters,
Symplectic-Amoeba formulation of the non-Bloch band theory for one-dimensional two-band systems,
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Wiener-Hopf factorization and non-Hermitian topology for Amoeba formulation in one-dimensional multiband systems,
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List of Abbreviation

NHSE	Non-Hermitian Skin Effect
OBC	Open Boundary Condition
PBC	Periodic Boundary Condition
SIBC	Semi-Infinite Boundary Condition
BBC	Bulk-Boundary Correspondence
BZ	Brillouin Zone
GBZ	Generalized Brillouin Zone
DOS	Density Of States
WHF	Wiener–Hopf factorization
SWHF	Symmetric Wiener–Hopf factorization

Chapter 1

Introduction

In this chapter, we establish the theoretical framework of non-Hermitian topology and characterize the fundamental properties of the non-Hermitian skin effect (NHSE). The NHSE represents a fundamental departure from Hermitian physics, where an extensive number of bulk modes localize at the boundaries under open boundary conditions.

This chapter is structured into three primary components. First, we elucidate the topological origin of the NHSE through the lens of Hermitian doubling, which maps non-Hermitian systems onto doubled Hermitian counterparts, uncovering their topological nature. Second, we examine the analytical properties of the NHSE using the non-Bloch band theory. This framework introduces the generalized Brillouin zone, providing a consistent description of the energy spectra and localization profiles that conventional Bloch theory fails to capture. Finally, we discuss the extension of non-Bloch band theory to higher dimensions. Specifically, we present an overview of the Amoeba theory, a multifaceted mathematical framework for defining the generalized Brillouin zone in higher dimensions.

By highlighting the current challenges and open questions in higher-dimensional non-Hermitian systems, this chapter provides the necessary context and motivation for the investigations presented in the subsequent chapters of this thesis.

1.1 Non-Hermiticity and Topological phases

The Hermiticity of the Hamiltonian constitutes one of the fundamental postulates of quantum mechanics. Dynamics governed by Hermitian Hamiltonians ensure the unitarity of time evolution, which preserves the norm of state vectors and results in conservative dynamics. The reality of the energy spectrum mathematically guarantees this conservation law. Within this framework, quantum mechanics has successfully elucidated the microscopic mechanisms underlying a vast array of natural phenomena.

However, recent advancements in experimental techniques have enabled the realization of quantum systems where energy and particle numbers are not strictly conserved. In such non-

conservative dynamics, a non-Hermitian Hamiltonian can emerge as an effective description by “renormalizing” the environmental degrees of freedom into the system Hamiltonian [1–8]. Such formulations are not only limited to open quantum systems [9–13] but are also increasingly employed in correlated electron systems to describe quasiparticles with finite lifetimes [14–24].

By treating the non-Hermitian Hamiltonian as a general matrix, this framework extends naturally beyond quantum systems to a wide variety of classical physical platforms [25, 26]. In these contexts, non-Hermiticity naturally arises from dissipation, such as damping in acoustic [27] or mechanical vibrations [28], or can be intentionally engineered through gain and loss in structured platforms such as photonic lattices [29–35] and electric circuits [36–40]. These systems provide highly controllable environments for exploring exotic phenomena beyond the reach of Hermitian physics.

A defining feature of a non-Hermitian Hamiltonian is the presence of complex energy eigenvalues corresponding to gain and loss, which introduce entirely new degrees of freedom into the system. Non-Hermiticity profoundly affects system properties, often leading to singular spectral structures such as exceptional points [41–53], accompanied by unique eigenstate distributions that have no counterparts in Hermitian systems.

Parallel to these developments, topology has established itself as a cornerstone of modern condensed matter physics. In systems possessing translational symmetry, the energy spectrum forms a band structure in momentum space, which serves as a fundamental analytical tool. When the bulk spectrum is gapped, nontrivial topological structures in momentum space can give rise to robust gapless modes localized at the system boundaries. These edge modes are spatially localized near the boundaries, and crucially, their number is predicted by the topological invariants defined in the bulk. This principle, known as the bulk-boundary correspondence (BBC) [54–58], signifies the emergence of topological phases distinct from spontaneous symmetry breaking and has been the subject of extensive research in Hermitian systems.

Motivated by these developments, recent studies have vigorously extended topological concepts to the non-Hermitian regime. The complex nature of the energy spectrum gives rise to novel spectral structures, leading to a richer topological landscape. Notably, the non-Hermitian skin effect (NHSE) has attracted significant attention as a hallmark feature of non-Hermitian topology. The NHSE is a phenomenon in which an extensive number of bulk eigenstates localize at the system boundaries, defying conventional Bloch band theory and challenging the traditional understanding of BBC. In the remainder of this chapter, we provide a brief review of the NHSE and the non-Bloch band theory, which serves as the theoretical framework for addressing these unique non-Hermitian phenomena.

1.2 Basis properties of non-Hermitian topology

In this section, we summarize the basic properties of non-Hermitian topology. First, we review the framework of topological classification in non-Hermitian systems, focusing on the roles of symmetries and the distinct types of energy gaps. We also briefly discuss the concept of spectral flattening, which is a key procedure in the classification process.

1.2.1 Real-space and Momentum-space Hamiltonians

We begin by considering a generic tight-binding model defined on a lattice. The real-space Hamiltonian is generally expressed as

$$\hat{H} = \sum_{x,a,y,b} H_{xa,yb} \hat{c}_{x,a}^\dagger \hat{c}_{y,b}. \quad (1.2.1)$$

where $x, y \in \{1, \dots, N\}$ denote the unit-cell indices, and $a, b \in \{1, \dots, M\}$ label the internal degrees of freedom (e.g., orbitals or sublattices) within each cell. Here, $\hat{c}_{x,a}^\dagger$ and $\hat{c}_{x,a}$ are the creation and annihilation operators, respectively, of a fermion at unit cell x and internal state a , satisfying the canonical anticommutation relations $\{\hat{c}_{x,a}, \hat{c}_{y,b}^\dagger\} = \delta_{xy} \delta_{ab}$.

When the system possesses translational invariance, the matrix elements $H_{xa,yb}$ depend only on the relative distance $x - y$. In this case, the real-space Hamiltonian matrix takes the form of a block-Toeplitz matrix,

$$H_{xa,yb} = h_{x-y,ab}. \quad (1.2.2)$$

Under periodic boundary conditions (PBC), this Toeplitz structure allows the Hamiltonian to be block-diagonalized via a Fourier transform. The system is then described in momentum space as

$$\hat{H} = \int_0^{2\pi} \frac{dk}{2\pi} \sum_{a,b} [h(k)]_{ab} \hat{c}_{k,a}^\dagger \hat{c}_{k,b}, \quad (1.2.3)$$

where the Bloch Hamiltonian $h(k)$ for a crystal momentum $k \in [0, 2\pi)$ is defined as:

$$h(k) = \sum_j h_j e^{ikj}. \quad (1.2.4)$$

Here, h_j is the $M \times M$ hopping matrix between unit cells separated by distance j .

In standard Hermitian systems, the condition $h(k) = h(k)^\dagger$ holds for all k , which implies $h_j = h_{-j}^\dagger$ and ensures that the energy eigenvalues remain real. In contrast, in the non-Hermitian systems discussed in this thesis, this equality is generally violated ($h(k) \neq h(k)^\dagger$), leading to complex-valued energy spectra.

It should be noted that the pointwise definition of the Hermitian conjugate and the use of real momentum follow the conventional notation widely used in the literature. While this

k -space description is sufficient for this introductory chapter, it will be extended in Chapter 2 to the framework of non-Bloch band theory. In that context, we will treat the complex variable and its algebraic properties more fundamentally to account for phenomena such as the NHSE.

1.2.2 Symmetries

In Hermitian systems, the topological classification is established based on the 10-fold Altland-Zirnbauer (AZ) classes [59]. These classes are defined by the presence or absence of three fundamental internal symmetries: time-reversal symmetry (TRS), particle-hole symmetry (PHS), and chiral symmetry (CS). Their defining relations, formulated for the Hamiltonian in the real-space representation, are given by

$$U_T H^* U_T^{-1} = H, \quad U_T U_T^* = \pm 1, \quad (1.2.5)$$

$$U_C H^T U_C^{-1} = -H, \quad U_C U_C^* = \pm 1, \quad (1.2.6)$$

$$\Gamma H^\dagger \Gamma^{-1} = -H, \quad \Gamma^2 = 1, \quad (1.2.7)$$

where U_T, U_C and Γ are unitary operators. Notably, CS is defined as the combination of TRS and PHS, where $\Gamma = U_T U_C^*$. In Hermitian systems, the constraint $H = H^\dagger$ implies that complex conjugation and transposition are intimately linked via $H^* = H^T$. Consequently, the distinction between conjugate-type and transpose-type operations is redundant, yielding the standard 10-fold classification.

However, in non-Hermitian systems where $H \neq H^\dagger$, this equivalence breaks down [60]. The operations of complex conjugation (H^*) and transposition (H^T) become independent, leading to a ramification of fundamental symmetries. This requires a generalized framework that distinguishes between symmetries acting on the complex conjugate H^* and those acting on the transpose H^T , which introduces the dual symmetries defined as

$$U_T H^T U_T^{-1} = H, \quad U_T U_T^* = \pm 1, \quad (1.2.8)$$

$$U_C H^* U_C^{-1} = -H, \quad U_C U_C^* = \pm 1. \quad (1.2.9)$$

Here, the first equation represents the transpose-type time-reversal symmetry ($\text{TRS}^\dagger$), and the second represents the conjugate-type particle-hole symmetry ($\text{PHS}^\dagger$).

Based on these ramifications, the classification scheme can be structured into complex and real sectors. The standard 10-fold AZ classes are divided into 2 complex classes (class A and AIII) and 8 real classes (the other classes), depending on the presence of TRS and PHS. In the non-Hermitian framework, this structure is mirrored in the $\dagger$ -sector. Corresponding to the 8 real AZ classes, there emerge 8 real $\text{AZ}^\dagger$ classes, defined by $\text{TRS}^\dagger$ and $\text{PHS}^\dagger$.

The non-Hermiticity necessitates the consideration of additional symmetry constraints beyond the conventional AZ-type ones: sublattice symmetry (SLS) and pseudo-Hermiticity

Symmetry Class	TRS	PHS	TRS [†]	PHS [†]	CS
Complex AZ	A	0	0	0	0
	AIII	0	0	0	1
Real AZ	AI	+1	0	0	0
	BDI	+1	+1	0	1
	D	0	+1	0	0
	DIII	-1	+1	0	1
	AII	-1	0	0	0
	CII	-1	-1	0	1
	C	0	-1	0	0
	CI	+1	-1	0	1
	AI [†]	0	0	+1	0
Real AZ [†]	BDI [†]	0	0	+1	1
	D [†]	0	0	0	+1
	DIII [†]	0	0	-1	+1
	AII [†]	0	0	-1	0
	CII [†]	0	0	-1	-1
	C [†]	0	0	0	-1
	CI [†]	0	0	+1	-1

Table 1.1: AZ and AZ[†] symmetry classes for non-Hermitian systems.

(PH), defined as

$$SHS^{-1} = -H, \quad S^2 = 1, \quad (1.2.10)$$

$$\eta H^\dagger \eta^{-1} = H, \quad \eta^2 = 1, \quad (1.2.11)$$

where S and η are unitary operators. It is crucial to distinguish SLS from the conventional CS. While CS relates $H^\dagger$ to $-H$, the SLS in the non-Hermitian context specifically relates H to $-H$, which is equivalent to CS only in Hermitian systems. In non-Hermitian physics, SLS and PH act as independent symmetry generators that link the AZ and AZ[†] sectors. See Ref. [60] for details.

In principle, various combinations of symmetry constraints, including both AZ-type and AZ[†] type symmetries, could appear in the classification. However, these symmetries are not all independent. Specifically, under the transformation $H \mapsto iH$, TRS and PHS[†] are mapped onto each other in a one-to-one manner.

Consequently, the redundant symmetry sectors can be consolidated. The possible symmetry constraints are then conveniently organized into representative pairs: the standard AZ-type (TRS and PHS) and the AZ[†]-type (TRS[†] and PHS[†]). By incorporating SLS and PH as additional unitary symmetries, these combinations form a complete set of constraints.

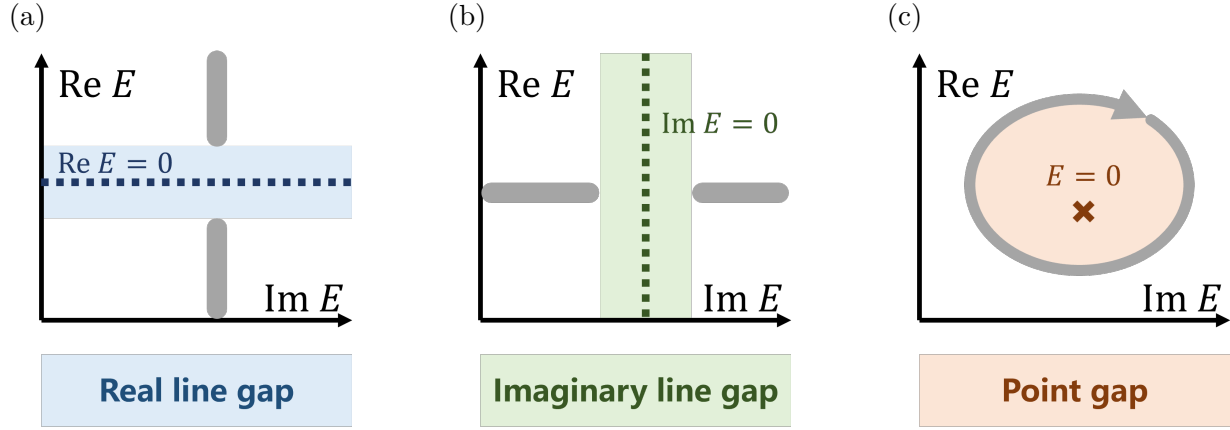


Figure 1.1: Schematic illustration of gap structures in non-Hermitian systems. Gray curves represent the energy spectrum under PBC. (a) Real line gap, where the spectrum avoids the imaginary axis. (b) Imaginary line gap, where the spectrum avoids the real axis. (c) Point gap, where the spectrum avoids a reference point in the complex energy plane and may exhibit a nontrivial winding around it.

This unified perspective underlies the 38-fold Bernard-LeClair (BL) classification of non-Hermitian Hamiltonians [60].

Due to the extensive nature of the full classification table, we present only the symmetry classes that are directly relevant to this thesis. For the comprehensive table covering all 38 classes, we refer the reader to Ref. [60].

1.2.3 Gap structures

One of the most distinctive features of non-Hermitian systems is that their energy spectra generally extend into the complex plane. As a result, the notion of a spectral gap is no longer restricted to gaps on the real axis. In this context, it is useful to distinguish between line gaps and point gaps.

A line gap is defined when the complex spectrum avoids a reference line in the complex energy plane. Typical examples include a real or an imaginary line gap, where the real or imaginary part of the energy remains nonzero, respectively. Systems with a real line gap can be continuously deformed into Hermitian systems without closing the gap, and their topology therefore reduces to conventional Hermitian topology. Similarly, systems with an imaginary line gap can be deformed into anti-Hermitian systems. Since an anti-Hermitian Hamiltonian can be mapped to a Hermitian one via the transformation $H \mapsto -iH$, both cases are topologically equivalent.

In contrast, a point gap is defined when the complex spectrum avoids a reference point, typically chosen as $E = 0$. In this case, the Hamiltonian is invertible, while the spectrum may

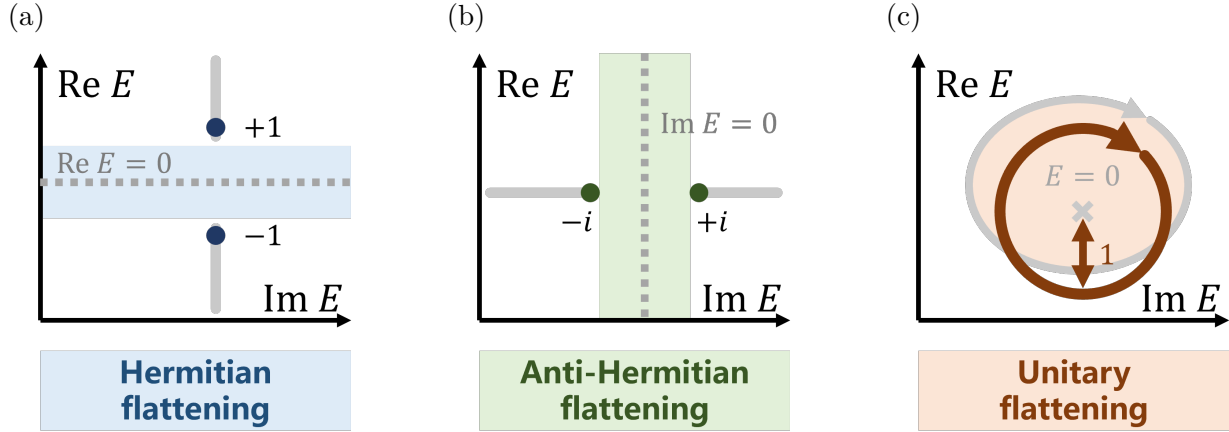


Figure 1.2: Schematic illustration of spectral flattening procedures in non-Hermitian systems. Gray curves represent the energy spectrum under periodic boundary conditions. (a) Real line gap: the spectrum can be continuously flattened onto ± 1 , resulting in an effective Hermitian Hamiltonian. (b) Imaginary line gap: the spectrum can be flattened into an anti-Hermitian form, which is topologically equivalent to the Hermitian case via the transformation $H \mapsto -iH$. (c) Point gap: the invertible Hamiltonian is flattened into a unitary matrix.

form loops that wind around the reference point. Such winding structures have no counterpart in Hermitian systems and give rise to genuinely non-Hermitian topological phenomena.

Importantly, the existence of a line gap automatically implies the existence of a point gap. However, the converse is not true: a point gap can exist even in the absence of any line gap. This hierarchical structure implies that point-gap topology provides a more general framework for classifying non-Hermitian phases.

Accordingly, it is useful to distinguish between two types of non-Hermitian topology. Extrinsic topology refers to phases protected by a line gap, which can be adiabatically connected to Hermitian topological phases. Intrinsic topology, on the other hand, relies solely on the point gap and has no Hermitian counterpart. As discussed below, this intrinsic point-gap topology underlies phenomena unique to non-Hermitian systems, most notably the NHSE.

1.2.4 Spectral flattening

To classify topological phases, it is often convenient to simplify the Hamiltonian while preserving its essential topological features. This procedure is known as spectral flattening. Depending on the type of spectral gap, different flattening schemes are employed in non-Hermitian systems.

For systems with a line gap, the topology can be reduced to that of Hermitian systems. In

the presence of a real line gap, the spectrum can be continuously flattened onto ± 1 without closing the gap, resulting in an effective Hermitian Hamiltonian satisfying $h^2 = 1$. Similarly, systems with an imaginary line gap can be flattened into anti-Hermitian Hamiltonians. Since an anti-Hermitian Hamiltonian can be mapped to a Hermitian one via the transformation $h \mapsto -ih$, both cases are topologically equivalent.

In contrast, systems with a point gap cannot, in general, be continuously deformed into Hermitian systems. Instead, the invertibility of the Hamiltonian allows one to perform a unitary flattening, which indicates that the Hamiltonian can be continuously deformed into a unitary matrix U without closing the point gap.

Crucially, the topological classification of U is equivalent to that of a chiral-symmetric Hermitian Hamiltonian constructed via Hermitian doubling [60, 61]:

$$\tilde{h}(k) = \begin{pmatrix} 0 & U(k)^\dagger \\ U(k) & 0 \end{pmatrix}. \quad (1.2.12)$$

Because $\tilde{h}$ anticommutes with the chiral operator $\tilde{\Gamma} = \text{diag}(1, -1)$, the intrinsic non-Hermitian topology, such as the winding number of the complex energy spectrum, is rigorously captured by the chiral topological invariants of this doubled system.

In this way, spectral flattening provides a unified framework for classifying both extrinsic and intrinsic non-Hermitian topological phases. While line-gap topology can always be reduced to Hermitian topology, point-gap topology defines genuinely non-Hermitian phases.

1.3 Non-Hermitian Skin effect

Among the phenomena unique to non-Hermitian systems, the non-Hermitian skin effect (NHSE) represents a fundamental obstruction to accessing the bulk spectrum under open boundary conditions (OBC) [60–87]. The NHSE is an intrinsic consequence of point-gap topology, in which an extensive number of bulk eigenstates become exponentially localized at the system boundaries, even in the thermodynamic limit. This macroscopic localization leads to an extreme sensitivity of the spectrum to boundary conditions.

A crucial implication of this macroscopic localization is the failure of the periodic boundary condition (PBC) approximation. In Hermitian systems, translational invariance allows the bulk spectrum under OBC to be faithfully approximated by the Bloch spectrum obtained under PBC. In non-Hermitian systems exhibiting the NHSE, however, this correspondence collapses: the OBC spectrum deviates qualitatively from the PBC spectrum even for infinitely large systems.

As a result, the computation of the bulk spectrum itself becomes a nontrivial problem. The standard momentum-space description no longer applies, while direct real-space diagonalization of large non-Hermitian matrices is numerically unstable and computationally

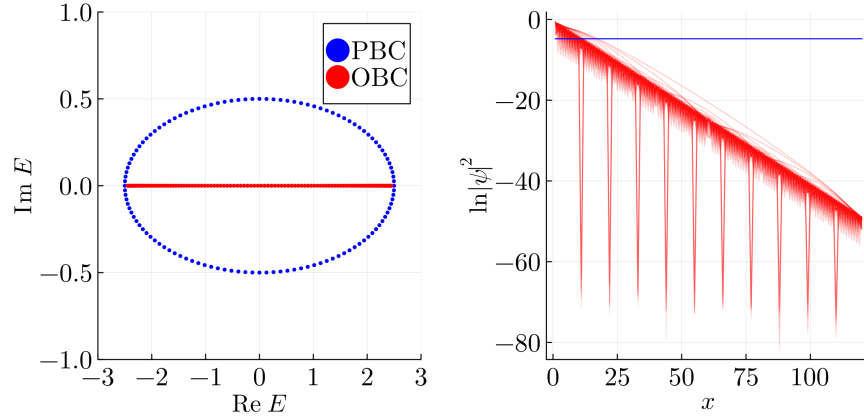


Figure 1.3: Numerical demonstration of the NHSE in the Hatano-Nelson model. The parameters are set to $t = 5/4$ and $g = -1/4$. Left: Blue and red points represent the energy spectra under PBC and OBC, respectively. Right: Blue and red curves represent the corresponding eigenstate profiles under PBC and OBC. While the PBC eigenstates are extended Bloch waves, the OBC bulk eigenstates are exponentially localized toward one boundary, illustrating the NHSE and the breakdown of the PBC approximation.

prohibitive [68]. This difficulty is not merely technical but reflects a fundamental limitation of conventional band theory in non-Hermitian systems.

In the following, we review the NHSE from both topological and analytical perspectives, emphasizing how the breakdown of the PBC approximation necessitates new theoretical frameworks for evaluating the OBC spectrum in a controlled manner.

1.3.1 Hatano-Nelson model

The most fundamental model to understand the NHSE is the Hatano-Nelson model, originally introduced to describe vortex pinning in superconductors [88–90].

The Hamiltonian takes the following form in a real-space representation,

$$\hat{H} = \sum_{x=1}^{N-1} \left[(t + g) \hat{c}_{x+1}^\dagger \hat{c}_x + (t - g) \hat{c}_x^\dagger \hat{c}_{x+1} \right], \quad (1.3.1)$$

where t represents the symmetric hopping and g introduces non-reciprocity ($g \neq 0$). Intuitively, g pushes particles to the boundary, resulting in localized bulk modes.

By assuming PBC and applying the Fourier transform, we obtain the Bloch Hamiltonian,

$$h(k) = (t + g)e^{ik} + (t - g)e^{-ik} = 2t \cos k + 2ig \sin k. \quad (1.3.2)$$

Under PBC, the Bloch Hamiltonian $h(k)$ yields an energy spectrum $E(k) = h(k)$ that traces an ellipse in the complex energy plane, which is a direct consequence of the asymmetric hopping amplitudes [Fig. 1.3].

In sharp contrast, the OBC spectrum exhibits a qualitatively different behavior. As shown by direct numerical diagonalization of the real-space Hamiltonian, the OBC spectrum is confined to the real axis and does not approach the PBC ellipse even in the thermodynamic limit. This striking discrepancy clearly demonstrates the failure of the PBC approximation in the presence of the NHSE.

The origin of this spectral deformation can be understood from the nature of the corresponding eigenstates. Under PBC, the eigenstates are Bloch waves and remain delocalized over the entire system. Under OBC, however, the bulk eigenstates become exponentially localized toward one boundary, reflecting the NHSE. As a result, the spectrum under OBC is governed by localized bulk modes rather than extended Bloch states.

From a spectral perspective, this behavior is closely related to the winding of the complex energy spectrum. The complex energy spectrum winds around a reference point E in the complex plane, giving rise to a nonzero spectral winding number,

$$W[E - h] = \int_0^{2\pi} \frac{dk}{2\pi i} \partial_k \ln[E - h(k)]. \quad (1.3.3)$$

This $W(E)$ is non-zero when E lies inside the PBC spectral loop, reflecting a point-gap in the complex energy plane.

The Hatano-Nelson model thus demonstrates that the central difficulty posed by the NHSE is not the identification of boundary modes, but the fundamental deviation of the OBC spectrum from its PBC counterpart.

1.3.2 Topological origin of the NHSE

The relationship between the spectral winding and the localized bulk modes observed in the Hatano-Nelson model is not a mere coincidence, but a universal feature dictated by point-gap topology.

Despite the breakdown of conventional Bloch band theory, the topological origin of the NHSE can be rigorously understood through the framework of Hermitian doubling [70, 91]. This procedure embeds a non-Hermitian Hamiltonian H and its Hermitian conjugate into a larger, chiral-symmetric Hermitian Hamiltonian $\tilde{H}$,

$$\tilde{H} = \begin{pmatrix} 0 & [E - H]^\dagger \\ E - H & 0 \end{pmatrix}, \quad (1.3.4)$$

whose chiral operator $\tilde{\Gamma}$ is defined as $\tilde{\Gamma} = \text{diag}(1, -1)$.

Within the Hermitian doubling framework, a non-Hermitian Hamiltonian in class A is mapped onto a chiral-symmetric Hermitian Hamiltonian belonging to class AIII. Through this mapping, the spectral winding number of the non-Hermitian system is identified with the one-dimensional topological invariant of the doubled Hermitian system.

This correspondence establishes a direct link between the bulk topology and the NHSE: the localized skin modes in the non-Hermitian system correspond one-to-one with the topological edge modes (zero-energy modes) of the extended Hermitian Hamiltonian, which are known to exhibit exponential localization near the boundary. This correspondence can be explicitly seen from the eigenvalue equations,

$$H |E\rangle_R = E |E\rangle_R \implies \tilde{H} \begin{pmatrix} |E\rangle_R \\ 0 \end{pmatrix} = 0 \begin{pmatrix} |E\rangle_R \\ 0 \end{pmatrix}, \quad \tilde{\Gamma} \begin{pmatrix} |E\rangle_R \\ 0 \end{pmatrix} = + \begin{pmatrix} |E\rangle_R \\ 0 \end{pmatrix}, \quad (1.3.5)$$

$$H^\dagger |E\rangle_L = E^* |E\rangle_L \implies \tilde{H} \begin{pmatrix} 0 \\ |E\rangle_L \end{pmatrix} = 0 \begin{pmatrix} 0 \\ |E\rangle_L \end{pmatrix}, \quad \tilde{\Gamma} \begin{pmatrix} 0 \\ |E\rangle_L \end{pmatrix} = - \begin{pmatrix} 0 \\ |E\rangle_L \end{pmatrix}. \quad (1.3.6)$$

Inheriting the exponential localization of the zero-energy edge modes in the doubled Hermitian system, the non-Hermitian skin modes are likewise exponentially localized at the boundaries. This correspondence establishes that the NHSE is not merely a boundary artifact, but a manifestation of the intrinsic non-Hermitian topology protected by the point gap.

1.3.3 Exact and quasi-zero modes and skin modes

As discussed above, the non-Hermitian skin modes can be understood as counterparts of topological edge modes through the framework of Hermitian doubling. This correspondence implies that the accumulation of bulk modes is governed by topological invariants defined for the complex energy spectrum.

At first sight, however, this picture appears to involve a conceptual tension. The parameter region in which the spectral winding number is nonzero is two-dimensional, corresponding to the interior of the PBC spectral loop, whereas the OBC spectrum forms only a one-dimensional curve in the complex energy plane. This apparent mismatch raises the question of why only a restricted subset of the point-gap topology is manifested in the OBC spectrum.

This issue can be resolved by distinguishing between exact and quasi-zero modes in the doubled Hermitian Hamiltonian [91]. In the thermodynamic limit, both types of modes collapse onto zero energy and become indistinguishable. For finite systems, however, their behaviors differ qualitatively. Exact zero modes remain pinned at zero energy, protected by topology and symmetry, while quasi-zero modes acquire an exponentially small energy gap controlled by their localization length.

The OBC skin modes correspond to the exact zero modes of the doubled Hamiltonian. While quasi-zero modes do not provide direct information about the OBC skin modes themselves, they play an essential role in the analysis developed in Chapter 5.

1.3.4 Symmetry-protected NHSE

Based on the correspondence between skin modes and exact zero modes discussed above, the topological classification of point gaps and the associated NHSE can be formulated through the symmetry classification of the extended Hermitian Hamiltonian. Crucially, this framework allows for the definition of symmetry-protected forms of the NHSE.

In the presence of certain symmetries, the spectral topology is no longer characterized by an integer winding number. Instead, symmetry constraints prohibit a nonzero winding while allowing for other topological invariants, such as a $\mathbb{Z}_2$ invariant, to remain well defined. Even when the winding number vanishes due to these symmetry constraints, the corresponding nontrivial $\mathbb{Z}_2$ topology protects exact zero modes in the doubled Hamiltonian and thereby induces a robust accumulation of bulk modes under OBC. This phenomenon is known as the $\mathbb{Z}_2$ skin effect [70].

A prominent example of a symmetry that gives rise to a $\mathbb{Z}_2$ skin effect is $\text{TRS}^\dagger$. Non-Hermitian Hamiltonians respecting this symmetry belong to class AII † . Within the Hermitian doubling framework, such non-Hermitian Hamiltonians are mapped onto Hermitian Hamiltonians in class DIII. As a result, the point-gap topology of class AII † systems is characterized not by an integer winding number, but by a $\mathbb{Z}_2$ topological invariant associated with class DIII.

This $\mathbb{Z}_2$ invariant protects exact zero modes of the doubled Hamiltonian, which appear as Kramers pairs due to the symmetry. In the original non-Hermitian system, these protected zero modes manifest themselves as skin modes localized at opposite boundaries of the system. This paired boundary localization is a characteristic feature of the $\mathbb{Z}_2$ skin effect.

Extending the Amoeba formulation to systems in class AII † constitutes one of the central objectives of this thesis. In particular, the multiband analysis developed in later chapters provides a systematic framework for evaluating the OBC spectrum in the presence of symmetry-protected skin effects.

1.3.5 NHSE in Multiband Systems and Higher Dimensions

While the Hermitian doubling framework successfully elucidates the topological origin of the NHSE in fundamental cases, the landscape of non-Hermitian phenomena becomes significantly more complex in multiband systems and higher dimensions. In these regimes, the simple picture of boundary modes with a fixed localization length often breaks down, giving rise to a "zoo" of exotic skin effects.

In multiband systems, the interplay between multiple bands can induce the critical skin effect [92]. Unlike the standard NHSE, where the localization length is an intrinsic property of the bulk determined by the point gap, the critical skin effect arises from the competition between modes with distinct localization tendencies. This competition can lead to scale-free localization, where the localization length scales with the system size, fundamentally

deviating from the standard exponential decay. Consequently, the OBC spectrum exhibits significant finite-size effects and a discontinuous jump with respect to system parameters. The scaling laws governing this criticality have been analysed in several studies [93–95].

The richness of non-Hermitian topology becomes even more pronounced in higher dimensions ($d \geq 2$). First, the correspondence between skin modes and topological edge modes can be generalized to higher-order topological structures [96–98]. This generalization leads to the higher-order skin effect, where bulk eigenmodes localize not merely at the boundaries (codimension-1) but at lower-dimensional boundaries such as corners or hinges (codimension-2 or higher). Through the lens of Hermitian doubling, these corner skin modes are mapped to the topological corner modes of higher-order topological insulators, demonstrating that the hierarchy of boundary physics extends into the non-Hermitian domain.

Furthermore, the skin effect in higher dimensions exhibits an inherent sensitivity to the spatial geometry of the boundaries, known as the geometry-dependent skin effect (GDSE) [99–108]. In contrast to the one-dimensional case, where the boundary consists of simple endpoints, higher-dimensional boundaries can possess complex shapes that dictate the spectral accumulation. Consequently, the specific shape of the open boundary (e.g., a square versus a parallelogram) fundamentally dictates not only the localization profile of eigenstates but also the energy spectrum in the thermodynamic limit. Recent studies have also revealed that the GDSE can exhibit scale-free localization similar to the critical skin effect [108, 109], indicating a deep connection between geometry and spectral instability.

Moreover, in higher dimensions, the interplay of non-Hermiticity can induce a dimensional transmutation of the momentum space. In this regime, an enormous number of skin modes compete, leading to algebraically localized skin modes rather than the conventional exponential localization [110, 111].

In summary, while significant progress has been made, the comprehensive classification of NHSE properties and the precise conditions for their emergence in higher dimensions remain an active area of ongoing research, leaving much to be explored in this burgeoning field.

1.3.6 Experimental realization of NHSE

The rapid theoretical developments in non-Hermitian topology have been closely paralleled by experimental breakthroughs across a diverse range of physical platforms. These experiments not only verify the existence of the NHSE but also demonstrate the universality of non-Hermitian physics beyond specific material systems.

In classical wave systems, photonic lattices [112, 113] and acoustic metamaterials [114–116] serve as versatile platforms where gain and loss can be engineered through optical pumping or acoustic attenuation. Similarly, mechanical metamaterials [117, 118] and active matter systems [119] have provided macroscopic visualizations of non-reciprocal dynamics and the skin effect. Beyond classical regimes, non-Hermitian topology has also been realized

in quantum systems. Notable examples include ultracold atoms [80, 120], where dissipation is controlled via atom loss, and photonic quantum walks [121–124], which allow for the observation of topological edge bursts and non-unitary time evolution [125].

Among these platforms, topoelectrical circuits stand out for their exceptional flexibility in lattice design [38, 39, 126–128]. By combining capacitors, inductors, and operational amplifiers, electric circuits can simulate a wide variety of tight-binding Hamiltonians with tunable hopping amplitudes and on-site potentials.

1.4 Non-Bloch band theory

The emergence of the NHSE leads to an extreme sensitivity to boundary conditions, where the physical properties under PBC and OBC fundamentally deviate from each other. Consequently, the conventional Bloch band theory, predicated on the translation invariance of infinite systems, breaks down, necessitating a new theoretical framework. In this section, we summarize the current status of the non-Bloch band theory, which provides a consistent description of non-Hermitian bulk systems.

1.4.1 Generalized Brillouin Zone

To overcome the breakdown of the conventional Bloch band theory, the concept of complex wavenumbers has been introduced. This generalization has established the generalized Brillouin zone (GBZ), which serves as the proper domain for the non-Bloch momentum $\beta = e^{ik}$ in the complex plane [62, 64].

In standard Hermitian band theory, the Hamiltonian h and its eigenstates are parameterized by the real momentum k within the Brillouin zone (BZ),

$$k \in \text{BZ} = [0, 2\pi) \rightarrow h(k) \quad (1.4.1)$$

However, in the presence of the NHSE, the OBC eigenstates are no longer Bloch waves but exhibit exponential localization. This can be formally captured by analytically continuing the momentum into the complex plane, $k \rightarrow k - i\mu$.

Taking the Hatano-Nelson model as a prototypical example, the non-Bloch Hamiltonian is expressed by shifting the momentum by an imaginary component μ :

$$h(k - i\mu) = (t + g)e^{\mu}e^{ik} + (t - g)e^{-\mu}e^{-ik}. \quad (1.4.2)$$

The hopping asymmetry is effectively neutralized, and the Hamiltonian becomes symmetrized when the following condition is met.

$$e^{2\mu} = \frac{t - g}{t + g}. \quad (1.4.3)$$

In this framework, the standard BZ (a unit circle $|\beta| = 1$) is replaced by the GBZ (a circle with radius $|\beta| = e^\mu$), such that

$$k - i\mu \in \text{GBZ} \subset \mathbb{C} \rightarrow h(k - i\mu), \quad (1.4.4)$$

where $\beta = e^{\mu+ik}$ parameterizes the OBC energy spectrum in the thermodynamic limit.

More generally, this approach can be extended to various multiband models, where the OBC spectra and eigenstates are parameterized by the complex momentum β on the GBZ. By replacing the unit circle of the conventional BZ with the GBZ, this framework provides a consistent description of non-Hermitian systems in the thermodynamic limit. This comprehensive formalism is known as the non-Bloch band theory.

1.4.2 Physical applications

Non-Bloch band theory has proven to be an indispensable framework for analyzing non-Hermitian phenomena across various physical contexts, bridging the gap between mathematical formalism and observable effects.

Topological Characterization and the BBC

The primary application of this theory is the rigorous establishment of the BBC [12, 62, 64, 129–135]. By evaluating topological invariants along the GBZ rather than the standard BZ, the framework correctly relates bulk topology to the existence and localization of boundary modes, even when the NHSE is present. This foundational concept has been successfully extended to Floquet systems to characterize time-dependent topological phases and their corresponding BBC [124, 136–139].

Non-Unitary Dynamics and Edge Phenomena

The non-Bloch perspective is equally crucial for understanding non-unitary dynamics. It enables accurate predictions of long-time evolution following a quench, anharmonic Rabi oscillations, and self-healing behaviors in topological modes [71, 122, 140–144]. A notable discovery directly predicted by this generalized formalism is the non-Hermitian edge burst [145], characterized by a rapid surge of loss or excitation at the boundary, a phenomenon that has recently been verified experimentally [146].

Theoretical Versatility in One-Dimensional Systems

This non-Bloch band theory has achieved remarkable success in one-dimensional systems, including symmetry-protected cases [147], providing a comprehensive framework for understanding non-Hermitian one-dimensional systems [71, 131, 134, 148–151].

Furthermore, the framework provides essential theoretical tools for concrete calculations: it allows for the precise evaluation of Green's functions [152, 153] and self-energy [154] in finite systems, and characterizes phenomena related to $\mathcal{PT}$ symmetry breaking [155, 156], entanglement distribution [157], and spectral braiding topology [158, 159]. Finally, the theory's versatility extends its utility to other localization problems, such as those found in Bosonic Bogoliubov–de Gennes systems, despite their Hermitian nature [75].

1.4.3 Generalizations to Broader Physical Systems

Beyond standard tight-binding models, the scope of non-Bloch band theory has been expanded to encompass a broader class of mathematical and physical systems.

One significant direction is the generalization to continuous systems, where the theory successfully describes the skin effect in optical or acoustic media governed by differential operators rather than difference equations [78, 160–162].

Another crucial frontier involves breaking translational invariance. In inhomogeneous systems, such as those with random onsite potentials, the interplay between the non-Hermitian skin effect and Anderson localization has been elucidated through modified GBZ theories [151, 163–166]. Similarly, the theory has been formulated for joint systems, clarifying the BBC at interfaces between distinct topological phases or materials [167].

Additionally, the framework has been adapted to generalized eigenvalue problems, which are essential for analyzing systems described by non-standard equations of motion, such as certain non-linear setups [168].

1.5 Non-Bloch Band Theory in Higher Dimensions: Amoeba Formulation and Open Challenges

In contrast to its success in one-dimensional systems, extending the non-Bloch band theory to higher dimensions has proved remarkably challenging [110, 111, 129, 169–172]. A fundamental difficulty arises from the fact that the GBZ in higher dimensions is generally not a simple product of one-dimensional GBZs, necessitating a paradigm shift in the theoretical construction. Consequently, a universal and computationally tractable framework for determining the higher-dimensional GBZ remains elusive, marking a significant gap in our current understanding of non-Hermitian topological phases.

In this context, the recently proposed Amoeba theory has significantly advanced the field [109, 173–180]. The Amoeba formulation defines the "electrical" potential of the spectrum, making it possible to access the density of states (DOS) under OBC for single-band class A systems in arbitrary dimensions. This framework provides a powerful bridge between the PBC dispersion and the thermodynamic limit of OBC systems via the concepts of the spectral

potential, offering a promising path toward a comprehensive theory of higher-dimensional NHSE.

However, the application of the amoeba formulation to multiband systems or symmetry-protected topological phases remains an open challenge. In this thesis, we demonstrate that the framework can be extended to one-dimensional multiband systems by employing the Wiener-Hopf factorization. By integrating this mathematical technique with the Amoeba formulation, we provide a method for determining the OBC spectra and localization properties that were previously difficult to access.

1.6 Outline of this thesis

The remainder of this paper is organized as follows. Chapter 2 reviews non-Bloch band theory and discusses the inherent difficulties in generalizing to higher dimensions.

Other existing formulations are also reviewed to clarify the theoretical context of the Amoeba formulation. Chapter 3 details the Amoeba formulation. This chapter introduces the central concept: the classical Szegő limit, which states that the OBC and PBC potentials correspond in topologically trivial regions. The analytical continuation of the Bloch Hamiltonian shifts the nontrivial regions and expands the domain where the OBC potential can be approximated by the PBC potential, resulting in the generalized Szegő limit theorem. Note that these theorems are mathematically established only for single-band systems.

To overcome the limitations in multiband systems, we establish the necessary mathematical frameworks in Chapter 4. Specifically, we review the Wiener-Hopf factorization and its application to the Szegő limit theorem [181]. We demonstrate that the partial indices of the symbol serve as a precise indicator of potential correspondence, extending the role of the conventional winding number. The partial indices of the Hermitian Bloch Hamiltonian possess a clear physical meaning: they correspond to the number of edge modes [182]. Furthermore, we show that the potential deviation in Hermitian topologically nontrivial systems can be evaluated by the localization length of these edge modes [183].

Chapter 5 generalizes the Amoeba formulation for multiband systems based on the framework developed in the previous chapters. First, we reformulate the generalized Szegő limit theorem using Hermitian doubling. We map the NHSE into topological edge modes via Hermitian doubling. Through this mapping, the partial indices of the non-Hermitian Bloch Hamiltonian precisely predict the edge modes in the corresponding Hermitian systems. This allows us not only to determine the robustness of the potential against boundary conditions but also to evaluate their deviation based on the localization length. Second, we generalized the Amoeba formulation under the $\text{TRS}^\dagger$. Since $\text{TRS}^\dagger$ protects Kramers pairs with opposite localization lengths, it renders the direct optimization of the Ronkin function inapplicable. In our modified formalism, we resolve this by separating Kramers pairs into symmetry-decomposed Ronkin functions.

Chapter 6 presents numerical results and verifies the theoretical discussions, and Chapter 7 concludes with a summary and outlook.

Appendix A reviews the mathematical properties of the Ronkin function. Appendix B provides the concrete algorithm for Wiener-Hopf factorization. Appendix C demonstrates a specific phase appearing under TRS[†]. Appendix D discusses the application of the Amoeba formulation to two-dimensional systems with reflection symmetry.

Chapter 2

Non-Bloch band theory

The non-Hermitian skin effect (NHSE) induces a macroscopic accumulation of bulk modes near the boundaries, signifying the breakdown of the conventional Bloch band theory. Unlike in Hermitian systems, the correspondence between periodic boundary conditions (PBC) and open boundary conditions (OBC) is lost. Without the simplification of translational invariance (PBC approximation), the eigenvalue problem, mathematically equivalent to analyzing the asymptotic behavior of large Toeplitz matrices, becomes highly nontrivial.

To resolve this discrepancy in one-dimensional systems, the non-Bloch band theory has been developed [62, 64]. This framework enables us to rigorously access the OBC spectral information by generalizing the concept of the Brillouin zone into the complex plane.

In this chapter, we provide a comprehensive review of the non-Bloch band theory. We begin by establishing the fundamental notations and definitions for non-Hermitian Toeplitz matrices. Second, we derive the generalized Brillouin zone (GBZ) condition for the generic symmetry class A and demonstrate how this condition is modified in the presence of internal symmetries, specifically in class AII[†] [147].

Finally, we discuss the fundamental difficulties in generalizing the GBZ condition to higher-dimensional systems, setting the stage for the subsequent chapters.

2.1 Notations and Matrix-valued polynomials

In this section, we introduce the notations related to matrix-valued polynomials and Toeplitz matrices. Our conventions are based on those established in Ref. [182] to ensure consistency with the algebraic structure of Toeplitz operators.

We consider an M -band Bloch Hamiltonian with finite hopping ranges,

$$h(e^{ik}) = \sum_{j=-p}^q h_j e^{ikj}, \quad (2.1.1)$$

where p and q denote the maximal hopping range to the left and right, respectively. Each h_j is an $M \times M$ matrix specifying the hopping amplitudes between orbitals separated by j sites. Substituting $\beta = e^{ik}$, we define the matrix-valued Laurent polynomial

$$h(\beta) = \sum_{j=-p}^q h_j \beta^j, \quad (2.1.2)$$

which we refer to as the non-Bloch Hamiltonian. In the context of Toeplitz operators, $h(\beta)$ is referred to as the symbol.

Constructed from the Fourier coefficients $\{h_j\}_j$, the real-space OBC Hamiltonian H_N on a lattice with N sites is given by

$$[H_N]_{x,y} = h_{x-y} \quad (1 \leq x, y \leq N), \quad (2.1.3)$$

where $[H_N]_{x,y}$ denotes the (x, y) -th $M \times M$ block acting on the internal space. The convention $h_\ell = 0$ is assumed for $\ell \notin [-p, q]$. Equivalently, the inverse Fourier transform of h generates an $MN \times MN$ block Toeplitz matrix $\mathcal{T}_N[h]$:

$$[H_N]_{x,y} = \mathcal{T}_N[h]_{x,y} = \int_0^{2\pi} \frac{dk}{2\pi} h(e^{ik}) e^{-ik(x-y)}. \quad (2.1.4)$$

We define Hermitian conjugate, transpose, and complex conjugation on matrix-valued polynomials $h(\beta)$ such that they commute with the Toeplitz operator map $\mathcal{T}_N$:

$$\mathcal{T}_N[h]^\dagger = \mathcal{T}_N[h^\dagger], \quad \mathcal{T}_N[h]^\top = \mathcal{T}_N[h^\top], \quad \mathcal{T}_N[h]^* = \mathcal{T}_N[h^*], \quad (2.1.5)$$

where the operations on the symbol are defined as

$$h^\dagger(\beta) = \sum_j h_j^\dagger \beta^{-j}, \quad h^\top(\beta) = \sum_j h_j^\top \beta^{-j}, \quad h^*(\beta) = \sum_j h_j^* \beta^j. \quad (2.1.6)$$

We stress that Hermitian conjugation and transposition reverse the Laurent powers ($\beta^j \mapsto \beta^{-j}$), reflecting the spatial inversion $x \rightarrow -x$ (or $k \rightarrow -k$), whereas complex conjugation acts only on the coefficients and leaves the powers β^j unchanged.

Consistent with the above definitions, the Hermiticity of the symbol $h(\beta)$ is determined by that of the operator $\mathcal{T}_N[h]$. Specifically, the self-adjoint condition $\mathcal{T}_N[h] = \mathcal{T}_N[h]^\dagger$ translates directly to the symmetry of the symbol: $h(\beta) = h^\dagger(\beta)$.

Following conventional band theory, we refer to the (right) eigenstates $u(\beta)$ and eigenvalues $\varepsilon(\beta)$ of the symbol $h(\beta)$ as Bloch states and the Bloch spectrum, respectively:

$$h(\beta)u(\beta) = \varepsilon(\beta)u(\beta). \quad (2.1.7)$$

We emphasize a notational distinction between $h^\dagger(\beta)$ and $h(\beta)^\dagger$. For a general matrix-valued function $A(\beta)$, we define

$$A(\beta)^\dagger = [A(\beta)]^\dagger, \quad A(\beta)^\top = [A(\beta)]^\top, \quad A(\beta)^* = [A(\beta)]^*, \quad (2.1.8)$$

which act pointwise on the functional value at fixed β . In contrast, $A^\dagger(\beta)$ denotes the Hermitian conjugate of the matrix-valued Laurent polynomial, involving the reciprocal transformation $\beta \mapsto \beta^{-1}$.

This distinction is essential for defining left eigenvectors in non-Hermitian systems. We define the left eigenvector $u_L(\beta)$ as a row vector satisfying

$$u_L(\beta)^\dagger h(\beta) = u_L(\beta)^\dagger \varepsilon(\beta)^*. \quad (2.1.9)$$

Here, the dagger acting on $u_L(\beta)$ and $\varepsilon(\beta)$ is understood as a pointwise operation on their functional values.

2.2 Non-Bloch band theory in class A

In this section, we derive the GBZ condition [62, 64], which characterizes the OBC spectrum based on the roots distribution of the characteristic equation.

The non-Bloch band structure is determined by the complex-valued wavenumbers β satisfying the characteristic equation. We define the characteristic polynomial as the determinant of the symbol $\sigma(\beta) = E - h(\beta)$.

The characteristic polynomial can be factorized in two complementary ways:

$$\det \sigma(\beta) = C_E \beta^{-Mp} \prod_{j=1}^{M(p+q)} (\beta - \beta_j(E)) = \prod_{m=1}^M (E - \varepsilon_m(\beta)) \quad (2.2.1)$$

where C_E is a β -independent constant. We assume that the β -roots, β_j , are ordered by their moduli: $|\beta_i(E)| \leq |\beta_j(E)|$ for $i \leq j$. Then, the E -roots, $\varepsilon_m(\beta)$, provide the energy band branches (EBB) [184]. For a model described by a rank- M matrix $h(\beta)$, there are M EBBs.

To construct the eigenstates under OBC, we seek a solution in the form of a generalized Bloch wave (or non-Bloch wave). The internal structure is determined by the Bloch states $u(\beta)$ defined in Eq. (2.1.7). Instead of assuming a plane-wave form e^{ikx} , we extend the domain to the complex plane $\beta \in \mathbb{C} \setminus \{0\}$. The spatial profile of the wavefunction at site x , denoted as $\psi(x)$, is constructed as a superposition of these modes:

$$\psi(x) = \sum_{j=1}^{M(p+q)} c_j \beta_j^x u(\beta_j), \quad (2.2.2)$$

where the summation runs over the roots $\{\beta_j\}_j$ of the characteristic equation $\det[h(\beta) - E] = 0$ for a fixed energy E .

The boundary conditions impose constraints on the coefficients $\{c_j\}$.

$$\psi(-p+1) = \cdots = \psi(0) = 0, \quad \psi(N+1) = \cdots = \psi(N+q) = 0. \quad (2.2.3)$$

For a system of length N , this leads to a linear system of equations $\mathcal{M}(E)\mathbf{c} = 0$:

$$\begin{pmatrix} f_1(\beta_1, E) & \cdots & f_1(\beta_{M(p+q)}, E) \\ \vdots & \vdots & \vdots \\ f_{Mp}(\beta_1, E) & \cdots & f_{Mp}(\beta_{M(p+q)}, E) \\ g_1(\beta_1, E)\beta_1^N & \cdots & g_1(\beta_{M(p+q)}, E)\beta_{M(p+q)}^N \\ \vdots & \vdots & \vdots \\ g_{Mq}(\beta_1, E)\beta_1^N & \cdots & g_{Mq}(\beta_{M(p+q)}, E)\beta_{M(p+q)}^N \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{Mp} \\ c_{Mp+1} \\ \vdots \\ c_{M(p+q)} \end{pmatrix} = 0 \quad (2.2.4)$$

where f_i and g_i denote the boundary constraints at the left and right ends, respectively. The condition for the existence of a nontrivial solution determines the OBC energy spectrum,

$$\det \mathcal{M}(E) = 0. \quad (2.2.5)$$

In the thermodynamic limit $N \rightarrow \infty$, the determinant is dominated by the terms containing the highest powers of β . The leading and next dominant term in the Laplace expansion is obtained by

$$\beta_{M(p+q)}^N \cdots \beta_{M(p+2)}^N \beta_{M(p+1)}^N \quad \text{and} \quad \beta_{M(p+q)}^N \cdots \beta_{M(p+2)}^N \beta_{Mp}^N. \quad (2.2.6)$$

For the boundary condition to hold, these two dominant terms must be of comparable magnitude to cancel each other out. Comparing their moduli leads to the condition:

$$|\beta_{Mp}| = |\beta_{Mp+1}|. \quad (2.2.7)$$

This is precisely the GBZ condition for multiband systems.

The GBZ is defined as the set of roots satisfying this condition. In class A systems, the GBZ for band m is characterized as:

$$\text{GBZ}_m = \{\beta \in \mathbb{C}; |\beta_{Mp} \circ \varepsilon_m(\beta)| = |\beta_{Mp+1} \circ \varepsilon_m(\beta)|\}, \quad (2.2.8)$$

where $\circ$ denotes a function composition [62, 64, 68]. The spectrum of band m is then obtained by mapping the GBZ_m through ε_m .

Since searching the entire complex energy plane is computationally prohibitive, we need a systematic approach to extract the eigenvalues E satisfying the GBZ condition. To this end, an efficient algebraic method based on the resultant was established in Ref. [68].

2.3 Non-Bloch band theory in class AII[†]

In this section, we elucidate how symmetries modify the GBZ condition [147]. We first discuss the topological invariants and the symmetry constraints on eigenvectors that give rise to Kramers pairs. We then demonstrate that this Kramers pair structure necessitates a modified GBZ condition.

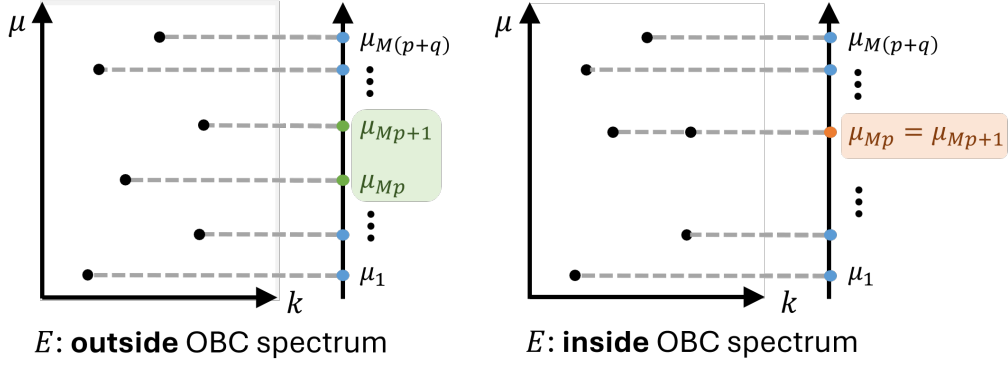


Figure 2.1: Schematic illustration of the GBZ condition. The variables $\mu_j = \ln |\beta_j|$ represent the logarithmic moduli of the roots of the characteristic equation, sorted in ascending order $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{M(p+q)}$. The continuum of the OBC spectrum is determined by the condition $\mu_{Mp} = \mu_{Mp+1}$, which corresponds to the trajectory of the roots on the complex β -plane where the decay rates of the two middle modes coincide. This condition replaces the conventional Bloch condition $\mu_j = 0$ (i.e., $|\beta| = 1$) in non-Hermitian systems.

2.3.1 Symmetry and Constraints on characteristic polynomial

The transposed-type time-reversal symmetry ($\text{TRS}^\dagger$), defined in Eq. (1.2.8), imposes the following constraint on the non-Bloch Hamiltonian $h(\beta)$:

$$U_T h^\top(\beta) U_T^{-1} = h(\beta), \quad U_T U_T^* = \pm 1, \quad (2.3.1)$$

where U_T is a unitary matrix. This symmetry constrains the maximal hopping range to be symmetric: $p = q$.

The sign of $U_T U_T^*$ determines the symmetry class. The case with $U_T U_T^* = +1$ (i.e., $U_T^\top = U_T$) is classified as class $\text{AI}^\dagger$ (orthogonal class), whereas the case with $U_T U_T^* = -1$ (i.e., $U_T^\top = -U_T$) constitutes class $\text{AII}^\dagger$ (symplectic class).

Since the characteristic polynomial is defined as $\det \sigma(\beta) = \det[E - h(\beta)]$, this symmetry implies invariance under the reciprocal transformation $\beta \rightarrow \beta^{-1}$:

$$\det \sigma(\beta) = \det \sigma(\beta^{-1}). \quad (2.3.2)$$

This symmetry ensures that the roots of the characteristic equation appear in reciprocal pairs: for each root $\beta_i(E)$, its reciprocal $\beta_i^{-1}(E)$ is also a root.

$$\det \sigma(\beta) = C_E \beta^{-Mp} \prod_{j=1}^{Mp} [\beta - \beta_j(E)] [\beta - \beta_j^{-1}(E)] = \prod_{m=1}^M [E - \varepsilon_m(\beta)], \quad (2.3.3)$$

where the roots can be ordered by their magnitude as

$$|\beta_{Mp}^{-1}| \leq \dots \leq |\beta_1^{-1}| < 1 < |\beta_1| \leq \dots \leq |\beta_{Mp}|. \quad (2.3.4)$$

Furthermore, the set of eigenvalues is invariant under the reciprocity transformation $\beta \rightarrow \beta^{-1}$. Thus, by appropriately choosing the branch cuts, we can define the single-valued dispersion relation satisfying:

$$\varepsilon(\beta) = \varepsilon(\beta^{-1}). \quad (2.3.5)$$

2.3.2 Kramers degeneracy and Eigenstates

To understand the spectral consequences of $\text{TRS}^\dagger$, we examine the constraints on the eigenstates in both real space and momentum space.

Real-space degeneracy

First, we consider the finite system described by the Toeplitz operator $\mathcal{T}_N[h]$. Let ϕ_R and ϕ_L be the right and left eigenstates of $\mathcal{T}_N[h]$ with eigenenergies E and E^* , respectively.

$$\mathcal{T}_N[h]\phi_R = E\phi_R, \quad \mathcal{T}_N[h]^\dagger\phi_L = E^*\phi_L. \quad (2.3.6)$$

Due to the symmetry constraint, $U_T\phi_L^*$ is a right eigenstate with energy E .

$$\mathcal{T}_N[h](U_T\phi_L^*) = E(U_T\phi_L^*). \quad (2.3.7)$$

Thus, ϕ_R and $U_T\phi_L^*$ are degenerate partners. To determine whether these states are distinct, we check their biorthogonality. The overlap is given by

$$\phi_L^\dagger U_T\phi_L^* = (\phi_L^\dagger U_T\phi_L^*)^\top = \phi_L^\dagger U_T^\top \phi_L^* = \pm \phi_L^\dagger U_T\phi_L^*. \quad (2.3.8)$$

This relation reveals a crucial distinction between the symmetry classes. In class $\text{AI}^\dagger$ ($U_T U_T^* = +1$), the relation becomes an identity, allowing for a non-vanishing overlap. Consequently, the symmetry-related partner $U_T\phi_L^*$ may be proportional to ϕ_R itself, and thus no degeneracy is enforced. In contrast, for class $\text{AII}^\dagger$ ($U_T U_T^* = -1$), the relation implies $\phi_L^\dagger(U_T\phi_L^*) = -\phi_L^\dagger(U_T\phi_L^*)$, which forces the overlap to vanish identically: $\phi_L^\dagger(U_T\phi_L^*) = 0$. This orthogonality guarantees that the two eigenstates are linearly independent, thereby enforcing the biorthogonal Kramers degeneracy.

Momentum-space degeneracy

A parallel argument applies to the Bloch states in the complex momentum plane. Let $u_R(\beta)$ and $u_L(\beta)$ be the right and left eigenstates of the symbol $h(\beta)$, defined by

$$h(\beta)u_R(\beta) = \varepsilon(\beta)u_R(\beta), \quad h^\dagger(\beta^{-1})u_L(\beta) = \varepsilon(\beta)^*u_L(\beta). \quad (2.3.9)$$

Note that the left eigenstate is defined involving the reciprocal variable β^{-1} , consistent with the definition of the Hermitian conjugate for the symbol.

Using the symmetry relation, we can construct a partner state for the right eigenvector. Direct substitution shows that the state $U_T u_L^*(\beta^{-1})$ satisfies the right eigenvalue equation with energy $\varepsilon(\beta)$:

$$h(\beta)[U_T u_L^*(\beta^{-1})] = U_T \varepsilon(\beta^{-1}) u_L^*(\beta^{-1}) = \varepsilon(\beta)[U_T u_L^*(\beta^{-1})]. \quad (2.3.10)$$

This implies that the eigenstates form pairs related by the symmetry, generalizing the concept of Kramers pairs to the complex-valued momentum domain,

$$u_R(\beta) = U_T u_L^*(\beta^{-1}), \quad u_L(\beta) = -U_T u_R^*(\beta^{-1}). \quad (2.3.11)$$

To determine whether this pairing enforces degeneracy, we examine the biorthogonal overlap between the relevant states, analogous to the real-space analysis. The overlap satisfies the relation,

$$[u_L(\beta)]^\dagger U_T u_L^*(\beta^{-1}) = \pm [u_L(\beta)]^\dagger U_T u_L^*(\beta^{-1}). \quad (2.3.12)$$

Since the sign is determined by $U_T U_T^* = \pm 1$, this leads to distinct spectral consequences for the two symmetry classes.

In class $\text{AI}^\dagger$ ($U_T U_T^* = +1$), the overlap is generally non-zero. Importantly, in this symmetry class, the spectral winding number usually vanishes due to the symmetry constraint, prohibiting the formation of a nontrivial point gap. The absence of a topological winding number implies that the NHSE does not occur, and the bulk eigenstates remain delocalized on the unit circle ($|\beta| = 1$). This is consistent with the conventional GBZ condition $|\beta_1| = |\beta_1^{-1}|$ (i.e., $|\beta| = 1$).

In contrast, for class $\text{AII}^\dagger$ ($U_T U_T^* = -1$), the skew-symmetry forces the overlap to vanish identically. This orthogonality enforces the formation of biorthogonal Kramers pairs throughout the complex plane. Although the total winding number vanishes ($W_{\text{total}} = 0$), this pair structure accommodates the $\mathbb{Z}_2$ skin effect characterized by the $\mathbb{Z}_2$ topological invariant [70]. This necessitates the generalized Szegő limit theorem and the modified GBZ condition derived in the following section.

2.3.3 GBZ condition in class $\text{AII}^\dagger$

Using the symmetry relations derived in the previous section, the bulk modes can be expanded in terms of Kramers pairs. The general solution for the right eigenstate $\phi_R(x)$ and the left eigenstate $\phi_L(x)$ is given by:

$$\phi_R(x) = \sum_{j=1}^{2Mq} [c_{+,j} \beta_j^x u_R(\beta_j) + c_{-,j} \beta_j^{-x} u_R(\beta_j^{-1})], \quad (2.3.13)$$

$$\phi_L(x) = \sum_{j=1}^{2Mq} [c_{+,j}^* (\beta_j^*)^{-x} u_L(\beta_j) + c_{-,j}^* (\beta_j^*)^x u_L(\beta_j^{-1})]. \quad (2.3.14)$$

Here, we explicitly used the reciprocal pairs of roots. As shown in Eq. (2.3.8), the Kramers partners are orthogonal, $\phi_L^\dagger U_T \phi_L^* = 0$.

We impose the OBC on the right eigenstate:

$$\phi_R(-p+1) = \cdots = \phi_R(0) = 0, \quad \phi_R(N+1) = \cdots = \phi_R(N+p) = 0. \quad (2.3.15)$$

This leads to a linear system of equations for the coefficients $\mathbf{c}$, represented by $\mathcal{M}(E)\mathbf{c} = 0$,

$$\begin{pmatrix} f_1(\beta_1) & \cdots & f_1(\beta_{2Mp}) & f_1(\beta_1^{-1}) & \cdots & f_1(\beta_{2Mp}^{-1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{2Mp}(\beta_1) & \cdots & f_{2Mp}(\beta_{2Mp}) & f_{2Mp}(\beta_1^{-1}) & \cdots & f_{2Mp}(\beta_{2Mp}^{-1}) \\ \beta_1^N g_1(\beta_1) & \cdots & \beta_{2Mp}^N g_1(\beta_{2Mp}) & \beta_1^{-N} g_1(\beta_1^{-1}) & \cdots & \beta_{2Mp}^{-N} g_1(\beta_{2Mp}^{-1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^N g_{2Mp}(\beta_1) & \cdots & \beta_{2Mp}^N g_{2Mp}(\beta_{2Mp}) & \beta_1^{-N} g_{2Mp}(\beta_1^{-1}) & \cdots & \beta_{2Mp}^{-N} g_{2Mp}(\beta_{2Mp}^{-1}) \end{pmatrix} \begin{pmatrix} c_1^{(+)} \\ \vdots \\ c_{2Mp}^{(+)} \\ c_1^{(-)} \\ \vdots \\ c_{2Mp}^{(-)} \end{pmatrix} = 0. \quad (2.3.16)$$

A crucial consequence of the $\text{TRS}^\dagger$ symmetry is that the standard GBZ construction fails. Consider the boundary conditions for the partner state ϕ_L . Due to the symmetry relation $u_L(\beta) \propto u_R(\beta^{-1})$, the condition $\phi_L(x) = 0$ at the boundaries imposes constraints that effectively replace $\beta_j \rightarrow \beta_j^{-1}$ in the matrix blocks. This symmetry leads to the vanishing of the sub-determinant formed by the standard set of roots:

$$\det \begin{pmatrix} f_1(\beta_1) & \cdots & f_1(\beta_{2Mp}) \\ \vdots & \ddots & \vdots \\ f_{2Mp}(\beta_1) & \cdots & f_{2Mp}(\beta_{2Mp}) \end{pmatrix} = 0. \quad (2.3.17)$$

A similar constraint holds for the right boundary condition blocks g_j .

Consequently, the leading term in the Laplace expansion of $\det \mathcal{M}$, which would normally involve the product of the largest $2Mp$ roots, vanishes identically. To satisfy the condition $\det \mathcal{M} = 0$, we must look at the next dominant terms in the expansion. The relevant terms involve mixing the largest roots with their reciprocal partners to avoid the zero-determinant constraint. The two dominant terms are given by:

$$\beta_{2Mp}^N \cdots \beta_2^N \beta_1^{-N} \quad \text{and} \quad \beta_{2Mp}^N \cdots \beta_2^N \beta_2^{-N}, \quad (2.3.18)$$

Here, we have replaced the largest root β_1 (which is forbidden by the determinant constraint) with its inverse β_1^{-N} in the first term, and similarly for β_2^{-N} in the second term. The condition for destructive interference between these terms implies

$$|\beta_1| = |\beta_2|, \quad (2.3.19)$$

which is the modified GBZ condition [147].

This leads to the definition of the GBZ in class AII[†] as the set of roots satisfying this condition:

$$\text{GBZ}_{m,\pm} = \{\beta \in \mathbb{C} \mid |\beta_1^{\pm 1} \circ \varepsilon_m(\beta)| = |\beta_2^{\pm 1} \circ \varepsilon_m(\beta)|\}. \quad (2.3.20)$$

This equation signifies that the GBZ splits into two distinct branches corresponding to the reciprocal pairs of the β -roots.

2.4 Non-Bloch band theory in higher-dimensional systems

As discussed in the previous sections, the non-Bloch band theory has achieved remarkable success in one-dimensional systems, covering various symmetry classes. In stark contrast, extending this framework to higher-dimensional systems presents fundamental challenges. GBZ conditions analogous to Eq. (2.2.8) have not yet been established for general higher-dimensional cases. We highlight three fundamental difficulties in establishing a higher-dimensional GBZ condition.

- **Extensive scaling of boundary conditions:**

First, the number of boundary conditions scales extensively with the system size. In d -dimensional systems, the characteristic equation involves a multivariate polynomial:

$$\det \sigma(\beta_1, \dots, \beta_d) = \det[E - h(\beta_1, \dots, \beta_d)]. \quad (2.4.1)$$

Consequently, the set of solutions $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d)$ forms a complex algebraic hypersurface in $\mathbb{C}^d$, rather than a discrete set of points. A general wavefunction ansatz is constructed as a superposition of a continuum of Bloch states residing on this hypersurface

$$\psi(\mathbf{x}) = \sum_{\boldsymbol{\beta}} c_{\boldsymbol{\beta}} \boldsymbol{\beta}^{\mathbf{x}} u(\boldsymbol{\beta}), \quad (2.4.2)$$

where $\boldsymbol{\beta}^{\mathbf{x}} \equiv \beta_1^{x_1} \dots \beta_d^{x_d}$.

Determining the dominant contributions from this continuum is fundamentally difficult because the number of boundary sites increases in the thermodynamic limit. In one dimension, the number of boundary sites is fixed (e.g., 2), and the boundary constraint matrix $\mathcal{M}$ has a finite dimension independent of the system size N . In contrast, in higher dimensions, the number of boundary sites scales as $\mathcal{O}(N^{d-1})$. This implies that the dimension of the boundary constraint matrix diverges in the thermodynamic limit. Consequently, the standard asymptotic expansion techniques used in one dimension become inapplicable.

- **Geometry-dependent skin effect (GDSE):**

Second, the NHSE induces the macroscopic accumulation of bulk modes at the boundaries. This prevents us from neglecting boundary effects even in the thermodynamic limit, rendering the bulk spectrum sensitive to the specific geometry of the open boundaries. This phenomenon is known as the geometry-dependent skin effect (GDSE).

Although the GDSE reveals the complex interplay between boundary conditions and non-Hermitian topology, characterizing it in general geometries remains a formidable challenge. Current studies have been largely restricted to specific domain shapes, such as polygons with rational slopes. Among the various manifestations of the GDSE, scale-free localization presents a particular difficulty. In this regime, the inverse localization length does not converge to a constant but depends on the system size or the aspect ratio of the boundaries. This behavior is physically analogous to the critical skin effect discussed in one-dimensional multiband systems, where the localization length diverges or scales anomalously.

- **Algebraic Skin Effect (ASE) and dimensional transmutation:**

A further complication in higher-dimensional systems is the emergence of the Algebraic Skin Effect (ASE) [110, 113]. The ASE has attracted significant attention as it induces a mismatch between the spatial dimension and the effective momentum-space dimension, a phenomenon known as dimensional transmutation.

From a physical viewpoint, the ASE arises directly from the continuum nature of the eigenstate superposition discussed in the first point. As obtained in Eq. (2.4.2), an eigenstate is written as a superposition of non-Bloch modes with the same energy. In one-dimensional systems, the number of such contributing modes is finite, leading to exponential localization characterized by a well-defined localization length.

In higher-dimensional systems, however, the situation changes qualitatively. Since the eigenstate is constructed from an integral (or infinite superposition) of modes on the hypersurface, the spatial profile can exhibit algebraic (power-law) rather than exponential localization. From this perspective, the geometry of the set of complex wave numbers sharing the same energy, often referred to as a generalized Fermi surface, plays a crucial role in determining the localization properties. This observation highlights an intrinsic limitation of the naive hierarchical adoption of the one-dimensional non-Bloch band theory and motivates the development of genuinely higher-dimensional formulations.

A unified theory successfully describing these novel skin effects in arbitrary dimensions has yet to be established, and further theoretical developments are anticipated.

To overcome the difficulties described above, several formulations have been proposed to extend the non-Bloch band theory to higher dimensions. In the rest of this section, we review some of the prominent approaches.

2.4.1 Dimensional reduction

One of the most straightforward approaches is the mapping of two-dimensional systems to one-dimensional systems [170]. The presence of specific symmetries, such as inversion symmetry, can prohibit the NHSE along the corresponding directions. Consequently, applying standard Bloch band theory along these symmetry-preserving directions allows us to parameterize the system with a real wavenumber k , effectively reducing the dimensionality of the problem to a family of one-dimensional non-Hermitian Hamiltonians.

This formulation is particularly effective when the Hamiltonian is separable into independent x and y components, where the problem trivially reduces to decoupled one-dimensional chains. Furthermore, in systems where the non-Hermiticity can be removed via a similarity transformation (imaginary gauge transformation), the problem can be mapped to a Hermitian one, allowing for exact analytical solutions [185].

2.4.2 Hierarchical adoption of one-dimensional theory

A more general strategy involves reinterpreting the spatial structure of the system. Although a d -dimensional lattice model formally possesses spatial dimension d with a fixed number of internal degrees of freedom, it can be viewed as a one-dimensional chain (spatial dimension 1) with extensively enlarged internal degrees of freedom corresponding to the transverse ($d - 1$) dimensions. This perspective enables the analysis using the framework of the one-dimensional multiband non-Bloch band theory developed in the previous chapters.

To make this procedure explicit, we denote the Toeplitz construction along the j -direction by $\mathcal{T}_{N_j}^{(j)}$. In two-dimensional systems, treating the y -direction as an internal degree of freedom corresponds to the successive application of the Toeplitz operators,

$$\mathcal{T}_{N_x}^{(x)} \mathcal{T}_{N_y}^{(y)}[h] = \mathcal{T}_{N_x}^{(x)} [\mathcal{T}_{N_y}^{(y)}[h]]. \quad (2.4.3)$$

Here, the inner operator $\mathcal{T}_{N_y}^{(y)}$ generates an effective one-dimensional Hamiltonian with a large internal dimension N_y . Applying the one-dimensional non-Bloch band theory along the x -direction then leads to the GBZ condition

$$|\beta_{x,j}(E, \beta_y)| = |\beta_{x,j+1}(E, \beta_y)|. \quad (2.4.4)$$

where $\beta_{x,i}$ are the roots of the characteristic equation for the effective Hamiltonian.

The choice of the direction to be regarded as the spatial dimension is arbitrary, and this ambiguity provides a useful perspective for understanding the GDSE [108]. Crucially, this hierarchical approach has been proposed as a valid framework to describe the ASE [113]. Since the ASE induces a dimensional transmutation where skin modes localize to lower-dimensional submanifolds, the hierarchical reduction correctly captures the dominant decay channels by effectively integrating out the transverse dimensions.

However, this formulation is computationally demanding in practice. Since the effective non-Bloch bands are defined for a fixed energy E , understanding the localization properties of individual eigenstates is feasible, but reconstructing the entire OBC spectrum requires collecting solutions over a wide range of energies in the complex plane. This procedure becomes numerically prohibitive, especially in higher dimensions where the effective internal dimension grows rapidly.

At present, it remains unclear to what extent the Amoeba formulation, discussed in the following chapter, can be systematically applied to these phenomena. Clarifying the applicability and limitations of the Amoeba formulation to GDSE and ASE is an important open problem.

Chapter 3

Amoeba formulation for single band systems

In this chapter, we review the Amoeba formulation [173] as a framework for determining the energy spectrum of non-Hermitian systems under OBC.

We begin by introducing the concept of the spectral potential and reviewing the classical Szegő limit theorem. By extending this theorem to topologically nontrivial regimes via similarity transformations, we derive the Generalized Szegő limit theorem, which determines the OBC potential through the optimization of the Ronkin function. Next, we elucidate the geometric structure underlying this optimization using the concepts of Amoeba. We demonstrate that the "central hole closing" condition serves as a unified spectral indicator valid in arbitrary dimensions and confirm that it reproduces the established GBZ condition in one-dimensional systems exactly.

Finally, we address the fundamental difficulties encountered when generalizing this formulation to multiband systems. We discuss how the symmetry-induced cancellations obstruct the direct application of global optimization. Furthermore, we demonstrate that a naive decomposition into band-resolved Ronkin functions is mathematically ill-defined due to the loss of analyticity. These limitations highlight the need for a robust topological indicator beyond the winding number and an alternative methodology for potential evaluation, motivating the introduction of the Wiener-Hopf factorization in the subsequent chapters.

3.1 Basic idea: Spectral potential

The Amoeba formulation focuses not on the spectra themselves but on comparing the spectral potentials [173] under different boundary conditions. In particular, the OBC spectrum is determined through the density of states (DOS) defined over the complex energy plane.

We first introduce the spectral potential. For an $N \times N$ matrix A with eigenvalues $\{E_j\}$,

the DOS is defined using Dirac's delta functions as

$$\rho_A(E) = \frac{1}{N} \sum_j \delta(E - E_j) = \frac{1}{N} \text{Tr} \delta(E - A). \quad (3.1.1)$$

Interpreted as a two-dimensional charge density on the complex energy plane, the DOS defines the spectral potential $\phi_A(E)$ through the Poisson equation,

$$\rho_A(E) = \frac{1}{2\pi} \Delta \phi_A(E), \quad (3.1.2)$$

where $\Delta = \partial^2 / \partial(\text{Re } E)^2 + \partial^2 / \partial(\text{Im } E)^2$. Solving this equation yields the explicit expression,

$$\phi_A(E) = \frac{1}{N} \sum_j \ln |E - E_j| = \frac{1}{N} \ln |\det(E - A)|. \quad (3.1.3)$$

Our goal is to evaluate the spectral potential of the OBC Hamiltonian in the thermodynamic limit. Consider a d -dimensional lattice with linear dimension N , i.e., there are N^d sites. The OBC potential is defined as

$$\phi_{\text{OBC}}(E) = \lim_{N \rightarrow \infty} \frac{1}{N^d} \ln |\det \mathcal{T}_N[E - h]|. \quad (3.1.4)$$

where $\mathcal{T}_N$ denotes the block-Toeplitz matrix generated by the symbol h under OBC. Analogously, the potential for the PBC Hamiltonian is defined by the integral over the d -dimensional Brillouin zone,

$$\phi_{\text{PBC}}(E) = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln |\det[E - h(e^{ik})]|, \quad (3.1.5)$$

where $e^{ik} = (e^{ik_1}, \dots, e^{ik_d})$.

In the subsequent discussions, we aim to establish a correspondence between the OBC and PBC spectral potentials by utilizing the limit theorem for Toeplitz determinants.

3.2 Potential correspondence via Szegő limit theorem

In the absence of the NHSE, the OBC spectrum converges to the PBC spectrum in the thermodynamic limit, implying $\phi_{\text{OBC}}(E) = \phi_{\text{PBC}}(E)$. However, the NHSE induces extreme sensitivity to the boundary condition, breaking this correspondence. Nevertheless, even in the presence of NHSE, the potentials remain equivalent in the regions where the winding number vanishes for all directions:

$$W_j[\sigma] = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} (-i \partial_{k_j}) \ln \det \sigma(e^{ik}), \quad (3.2.1)$$

where $\sigma(\beta) = E - h(\beta)$ be the symbol of the system. In such a topologically trivial regime, the classical Szegő limit theorem [181, 186, 187] yields

$$\lim_{N \rightarrow \infty} \frac{1}{N^d} \ln \det \mathcal{T}_N[\sigma] = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln \det \sigma(e^{ik}), \quad (3.2.2)$$

which recovers the equality of the potentials, $\phi_{\text{OBC}}(E) = \phi_{\text{PBC}}(E)$. Therefore, we can substitute the PBC potential for the OBC potential in the region with a vanishing winding number. A heuristic proof of the Szegő limit theorem following Ref. [173] is provided in Appendix A.

Since the Szegő limit theorem necessitates a vanishing winding number, the potentials coincide only outside the spectral loop of the PBC Hamiltonian. In the Amoeba formulation, this theorem is extended to topologically nontrivial regions by the generalized Szegő limit theorem.

3.3 Similarity transformation and Generalized Szegő limit theorem

As discussed in the previous section, the classical Szegő limit theorem provides a powerful tool for evaluating the asymptotic behavior of Toeplitz determinants. However, its application is fundamentally restricted to the topologically trivial regime where the winding number vanishes. When the energy E lies within the spectral loop of the PBC Hamiltonian, the non-zero winding number ($W[\sigma] \neq 0$) violates the condition for the theorem, preventing the direct evaluation of the OBC potential.

To overcome this limitation, we introduce a diagonal similarity transformation

$$D_N^{(\mu)} \mathcal{T}_N[\sigma] [D_N^{(\mu)}]^{-1} = \mathcal{T}_N[\sigma_\mu], \quad (3.3.1)$$

where $[D_N^{(\mu)}]_{x,y} = \delta_{x,y} e^{\mu \cdot x}$ and $\sigma_\mu(e^{ik}) = \sigma(e^{\mu+ik})$, maps the Toeplitz matrix $\mathcal{T}_N[\sigma]$ to one defined by the analytically continued symbol σ_μ . Since the determinant is invariant under similarity transformations, $\det \mathcal{T}_N[\sigma] = \det \mathcal{T}_N[\sigma_\mu]$.

Crucially, the similarity transformation modifies the winding number $W_j[\sigma_\mu]$. Hence, if there exists a μ such that the winding numbers vanish in all directions, $W_j[\sigma_\mu] = 0$, the classical Szegő limit theorem Eq. (3.2.2) can be applied to σ_μ :

$$\lim_{N \rightarrow \infty} \frac{1}{N^d} \ln |\det \mathcal{T}_N[\sigma]| = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln |\det \sigma_\mu(e^{ik})|. \quad (3.3.2)$$

The right-hand side is a function of μ , which is known as the Ronkin function [188]:

$$R_\sigma(\mu) = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln |\det \sigma_\mu(e^{ik})|. \quad (3.3.3)$$

The central question in this approach is whether it is always possible to find a $\boldsymbol{\mu}$ such that the winding numbers vanish. To answer this question, we introduce the concept of the Amoeba and investigate the geometric properties of the Ronkin function in the next section.

3.4 Geometric Interpretation via Amoeba

Having established the relationship between the OBC potential and the Ronkin function in the previous section, we now elucidate the geometric structure inherent in this function by introducing the concept of the Amoeba.

3.4.1 Analytical properties of Ronkin function

We now investigate the analytic properties of the Ronkin function to clarify the geometry underlying the determination of the valid parameter $\boldsymbol{\mu}$.

The Ronkin function associated with a multivariate Laurent polynomial $f(\boldsymbol{\beta})$ is defined by

$$R_f(\boldsymbol{\mu}) = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln |f_{\boldsymbol{\mu}}(e^{i\mathbf{k}})| = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln |f(e^{\boldsymbol{\mu} + i\mathbf{k}})|, \quad (3.4.1)$$

which is convex with respect to $\boldsymbol{\mu}$ over the entire space $\mathbb{R}^d$. See Appendix A for a detailed proof of convexity.

The gradient of the Ronkin function is directly related to the winding numbers of the analytically continued symbol,

$$\frac{\partial R_f}{\partial \mu_j}(\boldsymbol{\mu}) = \text{Re} \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial \mu_j} \ln f(e^{\boldsymbol{\mu} + i\mathbf{k}}) = W_j[f_{\boldsymbol{\mu}}], \quad (3.4.2)$$

where $f_{\boldsymbol{\mu}}$ denotes the analytically continued polynomial $f_{\boldsymbol{\mu}}(\boldsymbol{\beta}) = f(e^{\mu_1} \beta_1, \dots, e^{\mu_d} \beta_d)$. This winding number may be viewed as the average phase winding along the k_j -direction, with the remaining momenta serving as parameters:

$$W_j[f_{\boldsymbol{\mu}}] = \int_0^{2\pi} \frac{d^{d-1} \mathbf{k}_{\perp}}{(2\pi)^{d-1}} w_j(\mathbf{k}_{\perp}). \quad (3.4.3)$$

Here, $w_j(\mathbf{k}_{\perp})$ is the one-dimensional winding number calculated for a fixed set of transverse momenta $\mathbf{k}_{\perp}$,

$$w_j(\mathbf{k}_{\perp}) \equiv \int_0^{2\pi} \frac{dk_j}{2\pi i} \partial_{k_j} \ln f(e^{\boldsymbol{\mu} + i\mathbf{k}}). \quad (3.4.4)$$

In one-dimensional systems, the winding number characterizes the analytic structure of f via the Argument Principle. Specifically, it is equal to the number of roots ($f = 0$) minus

the number of poles ($f = \infty$) enclosed by the integration contour $|\beta| = e^\mu$. This implies that the winding number is entirely determined by the geometric distribution of the zeros and poles relative to the contour. Crucially, by varying μ , we can effectively scan the β plane with the integration contour; the value of the winding number changes only when the contour crosses a zero or a pole. Thus, the gradient of the Ronkin function encapsulates the geometric information of the zero locus in the complex plane. Provided the path does not encounter any zeros, the winding number is strictly quantized to an integer.

This geometric perspective extends to higher dimensions. Analogous to the one-dimensional case, the winding numbers W_j reflect the distribution of the zero locus hypersurface $f(\beta) = 0$ relative to the integration torus defined by $|\beta_j| = e^{\mu_j}$. However, unlike the one-dimensional case, these winding numbers are not always quantized. This is because W_j is given by the average of the one-dimensional winding numbers $w_j(\mathbf{k}_\perp)$ over the transverse momenta.

Geometrically, the breakdown of quantization occurs when the zero locus hypersurface intersects the integration torus. At such intersection points, the integrand $\ln |f|$ exhibits logarithmic singularities. As $\mathbf{k}_\perp$ varies, the one-dimensional integration path may cross the zero locus, causing $w_j(\mathbf{k}_\perp)$ to jump between different integers. Consequently, the total winding number W_j , obtained by averaging these values, becomes a continuous, non-integer value. Therefore, the regions in μ -space where the gradient is non-integer correspond directly to the configurations where the zero locus intersects the integration torus.

3.4.2 Amoeba and Ronkin function

For this reason, the distribution of the zero locus $f = 0$ in the μ -space serves as a crucial geometric guide. This projection is formalized as the mathematical object known as the Amoeba [189].

The Amoeba of f is the projection of the zero locus of f onto the space of logarithmic moduli $\mathbb{R}^d$:

$$\mathcal{A}_f = \{(\ln |\beta_1|, \dots, \ln |\beta_d|) \in \mathbb{R}^d \mid f(\beta) = 0\}. \quad (3.4.5)$$

Figure 3.1 shows the Amoeba for the polynomial $f(x, y) = 1/xy + a + x + y$. Geometrically, the Amoeba $\mathcal{A}_f$ appears as a closed set with tentacle-like structures extending to infinity. The complement $\mathbb{R}^d \setminus \mathcal{A}_f$ consists of a finite number of open connected components, which are referred to as the holes of the Amoeba.

The utility of the Amoeba lies in the behavior of the Ronkin function within these complementary components. Inside each hole, the integration torus does not intersect the zero locus. Consequently, the Ronkin function becomes affine linear, meaning its gradient $\nabla_\mu R_f$ is constant throughout the component. Remarkably, this constant gradient vector is strictly quantized to an integer vector $\mathbf{W} \in \mathbb{Z}^d$:

$$\nabla_\mu R_f(\mu) = \mathbf{W} = \text{const.} \quad (\text{for } \mu \in \text{hole}). \quad (3.4.6)$$

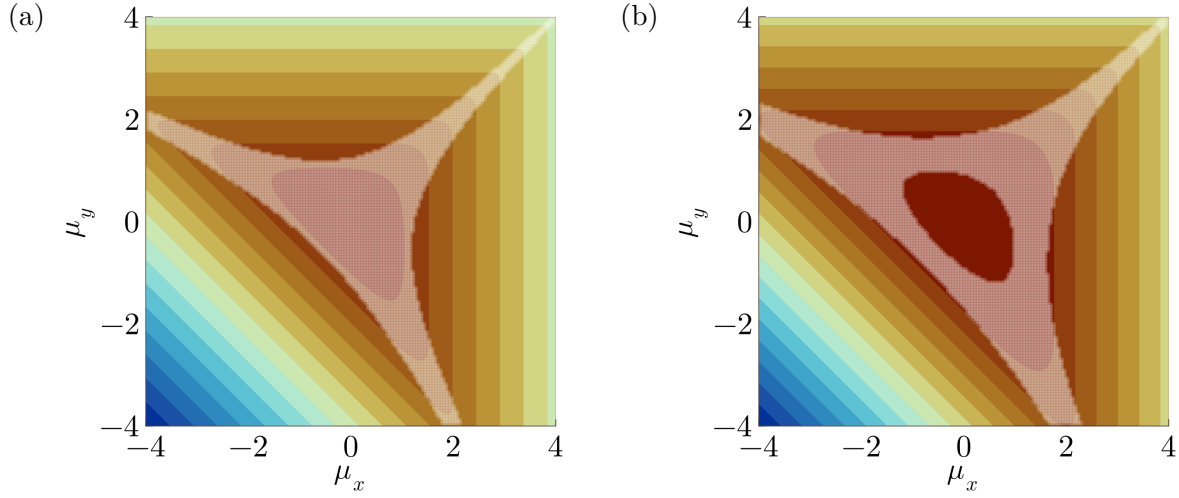


Figure 3.1: Geometric structure of the Amoeba and the Ronkin function for the polynomial $f(x, y) = 1 + axy + x^2y + xy^2$: (a) $a = -2$ and (b) $a = -4$. The light gray shaded regions correspond to the Amoeba $\mathcal{A}_f$, while the superimposed contours represent the values of the Ronkin function R_f .

Thus, each hole is uniquely labeled by this integer vector $\mathbf{W}$, which corresponds to the topological winding numbers. Crucially, the component characterized by a vanishing gradient $\nabla_{\mu} R_f = \mathbf{0}$ is termed the central hole.

Figure 3.1 visually confirms these properties. Inside the holes, the contours appear as equally spaced straight lines, serving as a clear signature of the affine linearity of the Ronkin function. Notably, the figure exhibits a striking difference around the origin: in Fig. 3.1(b), a central hole is clearly open, whereas in Fig. 3.1(a), this hole is closed.

It is noteworthy that the presence or absence of the central hole is governed by the single parameter a . While the convexity of the Ronkin function guarantees the existence of a global minimum where the gradient vanishes, this fact does not ensure the existence of a central hole. This implies that the global structure of the Amoeba can change drastically depending on the value of a , determining whether the minimum lies within a gapped region. Consequently, the condition for the central hole plays a central role in our subsequent discussions.

3.4.3 Amoeba and Generalized Szegő limit theorem

We now apply the geometric framework established above to the characteristic polynomial of the Hamiltonian, $\det \sigma = \det[E - h]$. This perspective allows us to reinterpret the Szegő limit theorem in terms of the Amoeba structure.

The classical Szegő limit theorem applies only when the winding numbers vanish in all

directions. In the language of Amoeba theory, for a topologically trivial energy E , this requirement is equivalent to demanding that the origin $\boldsymbol{\mu} = \mathbf{0}$ lies within the central hole of the Amoeba $\mathcal{A}_{\det \sigma}$. However, in the presence of the NHSE, the origin is typically located in a region with non-zero winding numbers, rendering the classical theorem inapplicable.

The generalized Szegő limit theorem overcomes the inapplicability of the classical theorem by employing a similarity transformation $\boldsymbol{\mu}$. Recall that the winding numbers W_j are equivalent to the components of the gradient of the Ronkin function:

$$W_j = \frac{\partial R_\sigma}{\partial \mu_j}. \quad (3.4.7)$$

Therefore, the requirement that the winding numbers vanish in all directions ($W_j = 0$ for all j) is mathematically equivalent to finding a stationary point of the Ronkin function, $\nabla R_\sigma(\boldsymbol{\mu}) = \mathbf{0}$.

Crucially, due to the inherent convexity of the Ronkin function over $\mathbb{R}^d$, any such stationary point necessarily corresponds to the global minimum. This implies that the minimization of the Ronkin function uniquely determines the similarity transformation required to satisfy the condition for Szegő limit theorem. The resulting OBC spectral potential is thus given by:

$$\phi_{\text{OBC}}(E) = \min_{\boldsymbol{\mu}} R_\sigma(\boldsymbol{\mu}). \quad (3.4.8)$$

Geometrically, this variational principle implies that the reference point $\boldsymbol{\mu}$ automatically seeks the topologically trivial region. If the central hole is open, the minimization shifts the reference point into it, thereby recovering the classical Szegő condition. More importantly, this formulation remains valid even when the central hole is closed. In such critical regimes, the minimum is attained on the boundary of or inside the Amoeba. Thus, the generalized theorem provides a unified description of the OBC spectrum, irrespective of the geometric phase transition of the Amoeba.

3.5 GBZ condition and Amoeba

The preceding discussions have established that the OBC potential is intrinsically linked to the Amoeba structure via the Ronkin function. Crucially, this geometric framework remains valid over the entire complex plane, allowing us to evaluate the OBC spectral potential directly from the properties of the bulk characteristic polynomial.

Building upon this foundation, we now aim to generalize the GBZ condition to higher dimensions. Our goal is to establish a geometric criterion to determine whether a given energy E belongs to the OBC spectrum based on the characteristic polynomial $\det \sigma = \det[E - h]$.

We demonstrate that the "Amoeba condition" (closing of the central hole) provides a unified spectral indicator valid in arbitrary dimensions, and show that the conventional GBZ condition corresponds to a specific instance of this geometric framework.

3.5.1 GBZ condition and Amoeba in one dimension

To derive this criterion, let us first revisit the conventional GBZ condition in one-dimensional systems. This explicitly connects the algebraic condition introduced in Chapter 2 with the geometric structure of the Amoeba.

Let the determinant of the symbol be factorized as

$$\det \sigma(\beta) = C_E \beta^{-p} \prod_{j=1}^{p+q} [\beta - \beta_j(E)], \quad (3.5.1)$$

where p and q are the orders of the poles at the origin and infinity, respectively, and the roots are sorted by their magnitude: $|\beta_1| \leq \dots \leq |\beta_{p+q}|$. In the standard non-Bloch band theory, the continuum of the GBZ is determined by the condition, Eq. (2.2.8):

$$|\beta_p| = |\beta_{p+1}|. \quad (3.5.2)$$

We now reinterpret this condition from the perspective of the Amoeba theory. In one dimension, the Amoeba $\mathcal{A}_f$ consists of a discrete set of points on the real axis,

$$\mathcal{A}_{\det \sigma} = \{\ln |\beta_j(E)| \mid j = 1, \dots, p+q\} \subset \mathbb{R}. \quad (3.5.3)$$

The "holes" of the Amoeba correspond to the open intervals between these log-moduli.

The integral in the Ronkin function can be performed explicitly using the factorization of the characteristic polynomial. Substituting the factorization into the definition of the Ronkin function, we obtain a compact form,

$$R_\sigma(\mu) = \oint_{\ln |\beta| = \mu} \frac{d\beta}{2\pi i \beta} \ln |\det \sigma(\beta)| \quad (3.5.4)$$

$$= \ln |C_E| - p\mu + \sum_{j=1}^{p+q} \max(\mu, \mu_j), \quad (3.5.5)$$

where $\mu_j \equiv \ln |\beta_j|$. The slope of this function represents the winding number of the symbol along the circle with radius e^μ :

$$\partial_\mu R_\sigma(\mu) = W[\sigma_\mu] = -p + \sum_{j=1}^{p+q} \theta(\mu - \mu_j), \quad (3.5.6)$$

where $\theta(x)$ is the Heaviside step function. This derivative counts the number of roots inside the contour minus the number of poles at the origin, which precisely aligns with the definition of the winding number.

The central hole of the Amoeba corresponds to the stable region where the winding number vanishes, $W[\sigma_\mu] = 0$. From the slope formula, this condition is satisfied when exactly p roots lie inside the contour (i.e., $\mu > \mu_p$) and the remaining q roots lie outside

(i.e., $\mu < \mu_{p+1}$). Therefore, the central hole corresponds to the interval (μ_p, μ_{p+1}) on the logarithmic axis.

The central hole closing implies that this topologically trivial interval shrinks to a single point. This occurs when the boundaries of the hole collide:

$$\mu_p = \mu_{p+1} \iff |\beta_p(E)| = |\beta_{p+1}(E)|. \quad (3.5.7)$$

This is precisely the GBZ condition established in non-Bloch band theory [62, 64], which we introduced algebraically in Chapter 2.

Thus, in one-dimensional systems, the conventional GBZ condition precisely corresponds to the central hole closing of the Amoeba.

3.5.2 Central hole closing as GBZ condition in higher dimensions

Motivated by the rigorous correspondence in one dimension, we propose that the "central hole closing" serves as the fundamental geometric criterion for the GBZ in arbitrary dimensions:

$$E \in \text{OBC spectrum} \iff \text{central hole of } \mathcal{A}_{\det \sigma} = \emptyset. \quad (3.5.8)$$

We justify this criterion by analyzing the geometric relationship between the zero locus of the symbol and the Brillouin zone torus.

First, consider the case where the vector $\boldsymbol{\mu}$ resides within a hole. By definition, if $\boldsymbol{\mu} \notin \mathcal{A}_{\det \sigma}$, the zero locus of the characteristic polynomial, $\det \sigma(\boldsymbol{\beta}) = 0$, does not intersect the torus defined by $\ln |\beta_j| = \mu_j$. Consequently, there exists no real wavenumber $\mathbf{k}$ such that

$$\det \sigma(e^{\boldsymbol{\mu} + i\mathbf{k}}) = 0. \quad (3.5.9)$$

This implies that for the given energy E , the configuration specified by the decay rate $\boldsymbol{\mu}$ cannot support any bulk-like eigenstate, effectively placing the system in a gapped phase with respect to this reference frame.

Conversely, let us consider the scenario where the central hole is closed. In our thermodynamic framework, the spectral properties are determined by the "optimal" decay rate $\boldsymbol{\mu}$ that minimizes the Ronkin function. If the central hole does not exist, the optimization process inevitably drives $\boldsymbol{\mu}$ onto the boundary or the interior of the Amoeba. This geometric collapse implies that the zero locus necessarily intersects the optimal torus $\ln |\beta| = \mu_j$. At these intersection points, there exist solutions to the characteristic equation, meaning that $\boldsymbol{\mu}$ can support valid eigenstates. Physically, this corresponds to a non-zero DoS in the complex energy plane.

Therefore, the transition from an open central hole to a closed one marks the emergence of valid eigenstates under the optimal boundary conditions. This establishes the central hole closing as the rigorous generalization of the GBZ condition for higher-dimensional non-Hermitian systems.

Furthermore, this geometric framework provides a natural and unified interpretation of the localization length in higher dimensions. In one dimension, the radius $|\beta|$ on the GBZ governs the spatial decay of eigenstates, corresponding to the inverse localization length. By analogy, in higher dimensions, the vector $\boldsymbol{\mu} = (\ln |\beta_1|, \dots, \ln |\beta_d|)$ at the collapsed central hole characterizes the anisotropic inverse localization lengths of the skin modes. Unlike the one-dimensional case, the skin modes in higher dimensions can exhibit complex decay profiles depending on the spatial direction. The components of the vector $\boldsymbol{\mu}$ encode these direction-dependent decay rates, allowing us to quantitatively describe the corner or edge localization characteristic of the higher-dimensional NHSE.

3.6 Difficulties in multiband systems

While the Amoeba formulation provides a rigorous framework for single-band systems, generalizing it to multiband systems presents fundamental difficulties regarding the topological characterization of the spectrum.

3.6.1 Topological perspective

In single-band systems, the condition of vanishing winding number, $W = 0$, uniquely identifies the "central hole" of the Amoeba, which corresponds to the topologically trivial region outside the spectrum. In multiband systems, however, the total winding number is given by the sum of the winding numbers of individual bands, $W_{\text{total}} = \sum_{n=1}^M W_n$. Consequently, the condition $W_{\text{total}} = 0$ is no longer sufficient to guarantee that the system is in a topologically trivial state. Individual bands can possess non-zero winding numbers that cancel each other out ($W_n = -W_m \neq 0$), resulting in a vanishing total winding number. Geometrically, this cancellation implies that the overlapping of Amoebae from different bands can create "fake" central holes. In these regions, the total Ronkin function appears trivial, making it indistinguishable from the true central hole, where all bands are actually trivial.

This cancellation issue is not merely accidental but can be enforced by system symmetries. A prominent example is the class $\text{AII}^\dagger$ system with $\text{TRS}^\dagger$, where the topology is characterized by a $\mathbb{Z}_2$ invariant rather than a non-zero total winding number. As discussed in Sec. 1.6, this symmetry pairs eigenstates (Kramers pairs) with opposite localization tendencies. Consequently, their contributions to the total winding number and the Ronkin function cancel out perfectly, enforcing the global minimum to be at $\mu = 0$ [175]. This leads to the physical paradox where the theory predicts delocalization ($\mu = 0$) despite the presence of the NHSE driven by the nontrivial $\mathbb{Z}_2$ topology [70, 147, 175].

These observations suggest that the total Ronkin function fails to capture the correct topology and localization properties due to inter-band cancellation. To resolve this, it is necessary to decompose the system and assess the contributions of individual bands.

3.6.2 Failure of Band-decomposed Ronkin function

To address the issues of overlapping and symmetry-induced cancellation, a natural strategy is to decompose the total Ronkin function into band-resolved components:

$$R_\sigma(\mu) = \sum_{m=1}^M R_\sigma^{[m]}(\mu), \quad (3.6.1)$$

$$R_\sigma^{[m]}(\mu) = \int_0^{2\pi} \frac{dk}{2\pi} \ln |E - \varepsilon_m(e^{ik})|, \quad (3.6.2)$$

where $\varepsilon_m(\beta)$ represents the individual energy bands. Ideally, optimizing each component would yield the correct spectral potential $\Phi_m(E)$ and localization length μ_m for the corresponding band.

In one-dimensional systems, specifically, we fundamentally expect such a valid spectral potential to possess robust mathematical properties: convexity, piecewise linearity, and integer-quantized derivatives corresponding to spectral winding numbers. However, this naive decomposition fails to preserve these essential properties. The failure stems from the multi-valued nature of the energy spectrum in multiband systems. Unlike the characteristic polynomial, the individual energy bands $\varepsilon_m(\beta)$ are algebraic functions defined as roots of the polynomial. Defining them as single-valued functions necessitates the introduction of branch cuts, which destroys the analytic structure underlying the Amoeba formulation. These branch cuts have two destructive consequences:

- **Loss of Periodicity and Quantization:** As detailed in the context of spectral braiding [12, 49, 158, 190], branch cuts fracture the closed spectral loops into open arcs. Consequently, the band-resolved functions lose the periodicity required for integer-quantized derivatives (winding numbers) and piecewise linearity.
- **Loss of Convexity:** As mathematically proven in Appendix A, the loss of analyticity along the branch cuts violates the condition for subharmonicity. Therefore, unlike the total Ronkin function, band-resolved Ronkin functions are generally not convex.

Since the resulting functions are neither convex nor physically interpretable via winding numbers, this naive approach is mathematically ill-behaved. This fundamental limitation necessitates an alternative approach that can decompose the system while preserving the analytic structure.

The failure of both the total determinant (due to cancellation) and the naive eigenvalue decomposition (due to branch cuts) highlights the necessity for a fundamentally different approach. We require a formulation that can: (1) capture the topological contributions of internal degrees of freedom, avoiding cancellation, and (2) preserve the analytic structure (convexity and periodicity) by avoiding branch cuts.

To overcome these obstacles, we must move beyond scalar functions and treat the matrix structure directly. In the following sections, we introduce a unified approach based on the Wiener-Hopf factorization and Hermitian doubling, which rigorously fulfills these criteria and establishes a consistent Amoeba formulation for multiband systems.

3.7 Comment on Geometry-Dependent Skin Effect

Throughout this chapter, we have developed the Amoeba formulation for single-band systems, focusing on geometry-independent spectral accumulation governed solely by the bulk Hamiltonian. In this framework, the Amoeba spectrum coincides with the uniform spectrum derived from the complex Bloch spectrum under PBC, and the central hole condition is equivalently characterized by this spectral set [179].

Since the uniform spectrum is uniquely determined by the Hamiltonian, the resulting Amoeba formulation does not incorporate any information about the geometry of the open boundary. Consequently, the formulation presented in this chapter captures only geometry-independent skin effects and does not describe the geometry-dependent skin effect (GDSE), where spectral accumulation depends explicitly on the boundary shape.

Nevertheless, recent studies have suggested that GDSE may be partially captured by suitable extensions of the Amoeba formulation, in which boundary geometry is incorporated as an additional ingredient [109]. Such extensions are beyond the scope of the present chapter and will be briefly discussed in Chapter D.

Chapter 4

Mathematical foundations

In this chapter, we establish the mathematical foundations essential for resolving the difficulties encountered in the multiband Amoeba formulation discussed in the previous chapter. While the naive application of the Amoeba theory fails due to the multivalued nature of the band structure and symmetry constraints, the Wiener-Hopf factorization (WHF) provides a rigorous algebraic framework to decompose matrix-valued functions while preserving their analytic and topological structures [182].

We begin by defining the WHF for matrix-valued Laurent polynomials and introducing its symmetry-constrained variant, the Symmetric Wiener-Hopf factorization (SWHF). We define the partial indices, which play a central role as topological invariants in our framework. Next, we review the application of the SWHF to Hermitian band topology, demonstrating how partial indices rigorously characterize topological phases and edge modes.

Finally, we connect the WHF to the asymptotic behavior of Toeplitz determinants [183]. We review the classical and modified Szegő limit theorems, elucidating how non-vanishing partial indices lead to deviations in the spectral potential. This mathematical groundwork forms the basis for the generalized Amoeba theory in multiband systems developed in the next chapter.

4.1 Wiener-Hopf factorization

In this section, we provide the formal definition of the Wiener-Hopf factorization (WHF) and introduce the concept of partial indices for matrix-valued Laurent polynomials. We first present the general definition of WHF for generic invertible matrices. Subsequently, we discuss the Symmetric Wiener-Hopf factorization (SWHF), which is specifically tailored for Hamiltonians satisfying internal symmetries.

4.1.1 Wiener-Hopf factorization for matrix Laurent polynomials

In this section, we consider a matrix-valued Laurent polynomial $A(\beta) = \sum_{j=-p}^q a_j \beta^j$, defined on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

A Wiener-Hopf factorization of A is a decomposition of the form

$$A(\beta) = A_+(\beta)D(\beta)A_-(\beta). \quad (4.1.1)$$

where the factors are given by

$$A_+(\beta) = \sum_{j=0}^q a_{+,j} \beta^j, \quad A_-(\beta) = \sum_{j=-p}^0 a_{-,j} \beta^j \quad (4.1.2)$$

and

$$D(\beta) = \text{diag}(\beta^{\kappa_1}, \dots, \beta^{\kappa_M}). \quad (4.1.3)$$

Here, $A_+(\beta)$ must be analytical and invertible for $|\beta| \leq 1$ (including the origin), while $A_-(\beta)$ must be analytical and invertible for $|\beta| \geq 1$ (including ∞). The integers κ_j are the partial indices of A , and we order them in descending order. We denote the set of partial indices of a symbol σ as

$$\mathcal{K}[\sigma] = (\kappa_1, \dots, \kappa_M). \quad (4.1.4)$$

We note that a WHF of A exists if and only if $A(\beta)$ is invertible on the unit circle, that is, $\det A(\beta) \neq 0$ for $|\beta| = 1$. This condition ensures that $\det A(\beta)$ has no zeros on the unit circle, so that analytic continuations of A_+ and A_- can be defined uniquely up to an invertible constant matrix. Given an invertible matrix $U(\beta)$ with $\det U(\beta) = \text{const}$, another valid factorization can be found as

$$A_+ D A_- = (A_+ U) D (D^{-1} U^{-1} D A_-) = A'_+ D A'_-, \quad (4.1.5)$$

where $A'_+ = A_+ U$ and $A'_- = D^{-1} U^{-1} D A_-$. However, after fixing an ordering of the partial indices $\{\kappa_j\}$, the diagonal factor D is unique, so that the partial indices are uniquely defined.

We will often partition the index set $\{1, \dots, M\}$ into M_+ , M_0 , and M_- corresponding to positive, zero, and negative partial indices, respectively. The corresponding partial indices are referred as $\mathcal{K}_+$, $\mathcal{K}_0$, and $\mathcal{K}_-$.

4.1.1.1 Geometric interpretation on the Riemann sphere

To grasp the physical mechanism of the Wiener-Hopf factorization, it is instructive to visualize the decomposition on the Riemann sphere as illustrated in Fig. 4.1. The factorization $A(\beta) = A_-(\beta)D(\beta)A_+(\beta)$ can be understood as a separation of the analytic properties of the symbol into distinct spatial domains.

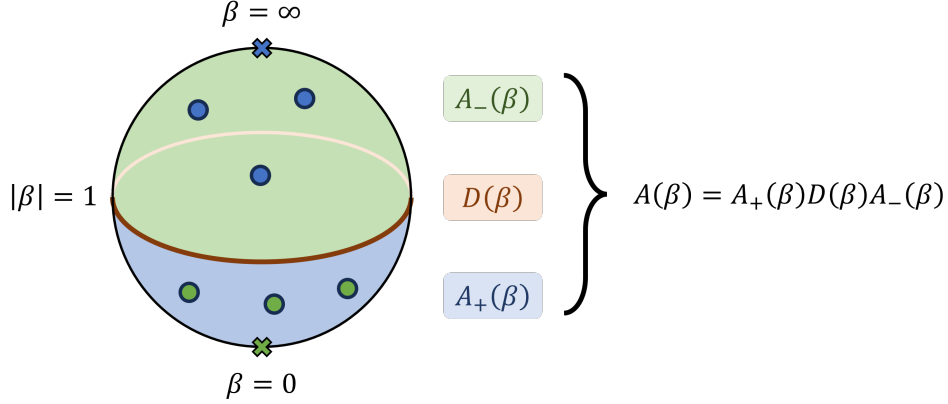


Figure 4.1: Schematic illustration of the Wiener-Hopf factorization on the Riemann sphere. The symbol $A(\beta)$ is gapped on the integration contour $|\beta| = 1$ (orange curve). The factors $A_+(\beta)$ and $A_-(\beta)$ are analytic and invertible in the blue region ($|\beta| \leq 1$) and the green region ($|\beta| \geq 1$), respectively. Consequently, the zeros of $A_+(\beta)$ (blue dots) are excluded from the blue region, and the zeros of $A_-(\beta)$ (green dots) are located inside the blue region.

Geometric Decomposition

The factors $A_+(\beta)$ and $A_-(\beta)$ are defined to be analytic and invertible in the interior ($|\beta| \leq 1$) and exterior ($|\beta| \geq 1$) of the unit circle, respectively. On the Riemann sphere, these domains correspond to the "southern hemisphere" (including the origin $\beta = 0$) and the "northern hemisphere" (including the point at infinity $\beta = \infty$). Crucially, the condition of invertibility implies that $A_+(\beta)$ has neither poles nor zeros in the southern hemisphere, and similarly, $A_-(\beta)$ is free of poles and zeros in the northern hemisphere.

Topological Constraints via Residue Theorem

This separation imposes a strong topological constraint. According to the argument principle, the winding number of a function along a contour counts the net number of zeros minus poles inside the contour. For a function to be invertible and analytic everywhere inside a region, its accumulated winding number within that region must be strictly zero. Consequently, both $A_+(\beta)$ and $A_-(\beta)$ carry zero topological charge (winding number) along the integration contour, $|\beta| = 1$.

The Role of the Diagonal Factor $D(\beta)$

However, the original symbol $A(\beta)$ can generally possess a non-zero winding number $W \neq 0$ reflecting the nontrivial topology of the system. This creates a fundamental mismatch: the "neutral" factors $A_\pm(\beta)$ cannot reproduce the nontrivial topology of $A(\beta)$. This is where the middle term $D(\beta) = \text{diag}(\beta^{\kappa_1}, \dots, \beta^{\kappa_N})$ plays the critical role. The term β^κ possesses a zero

of order κ at the origin and a pole of order κ at infinity. The diagonal factor $D(\beta)$ essentially acts as a "topological source/sink," absorbing the imbalance of zeros and poles that cannot be accommodated by the analytic factors. Mathematically, the partial indices $\{\kappa_i\}$ precisely account for the winding numbers that are "expelled" from the domains of $A_{\pm}(\beta)$, ensuring the consistency of the residue theorem on the entire Riemann sphere.

4.1.2 Symmetric Wiener-Hopf factorization

We now consider the WHF of Hermitian Bloch Hamiltonians. As recently shown in Ref. [182], a modified WHF, referred to as the symmetric Wiener-Hopf factorization (SWHF), is closely related to band topology. Motivated by applications to non-Hermitian topology, we focus on classes AIII and DIII, although analogous statements hold for other symmetry classes.

For a gapped Hermitian Bloch Hamiltonian, $h(\beta)$ admits a WHF

$$h(\beta) = h_+(\beta)D(\beta)h_-(\beta), \quad (4.1.6)$$

which can be chosen to respect Hermiticity,

$$h_+^\dagger = h_-, \quad D^\dagger = D. \quad (4.1.7)$$

We denote the D -factor satisfying this Hermitian condition by D_{Herm} .

As defined in Eq. (2.1.6), the Hermitian conjugation reverses the signs of the partial indices, $\kappa_j \mapsto -\kappa_j$. Together with the uniqueness of the partial indices, this implies that the set of partial indices of σ , denoted $\mathcal{K}[\sigma]$, is symmetric, so that every $\kappa_m > 0$ has a corresponding partner $-\kappa_m$. Thus, we can arrange D in a Hermitian form

$$\begin{cases} D_{M+1-m,m} = \beta^{\kappa_m} & m \in M_+ \\ D_{m,M+1-m} = \beta^{-\kappa_m} & m \in M_+ \\ D_{m,m} = s_m & m \in M_0 \end{cases} \quad (4.1.8)$$

with $s_m \in \pm 1$ arranged in ascending order. Here, the block assignment realizes a symmetric placement of positive/negative indices. We denote the D -factor satisfying this Hermitian condition by D_{Herm} . This type of factorization is called the symmetric Wiener-Hopf factorization (SWHF)[182].

In the presence of additional symmetries, the SWHF assumes a symmetry-constrained (modified) form. A Hamiltonian in class AIII satisfies the chiral symmetry, which allows h to be written as

$$h = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}. \quad (4.1.9)$$

Then, the WHF of A , $A = A_+ D A_-$, yields a SWHF of h in the form of

$$\begin{aligned} h &= h_+ D_{\text{Herm}} h_- \\ &= \begin{pmatrix} A_-^\dagger & \\ & A_+ \end{pmatrix} \begin{pmatrix} & D^\dagger \\ D & \end{pmatrix} \begin{pmatrix} A_- & \\ & A_+^\dagger \end{pmatrix}, \end{aligned} \quad (4.1.10)$$

consistent with Eq. (4.1.7).

Similarly, Hamiltonians in class DIII are block off-diagonal and admit an SWHF structurally equivalent to the AIII form introduced in Eq. (4.1.10). However, TRS ($\sigma_y h^* \sigma_y = h$) constrains A to satisfy the skew-symmetric condition $A = -A^\top$ in an appropriate basis. Choosing a suitable U , as introduced in Eq. (4.1.5), we can determine the factors as,

$$A_+^\top = A_-, \quad D^\top = -D \quad (4.1.11)$$

where D takes the skew-symmetric form

$$\begin{cases} D_{M+1-m,m} = -\beta^{\kappa_m} & m \in M_+ \\ D_{m,M+1-m} = \beta^{-\kappa_m} & m \in M_+ \\ D_{M+1-m,m} = 1 & m \in M_0, m \leq M/2 \\ D_{m,M+1-m} = -1 & m \in M_0, m \leq M/2 \end{cases}. \quad (4.1.12)$$

We denote the D -factor satisfying this skew-symmetric condition by D_{skew} .

Concrete algorithms for constructing both WHF and SWHF are provided in the Appendix of Ref. [182] and are also summarized in Appendix B of this thesis.

4.2 Applications to Hermitian band topology

In this section, we explore the connection between the SWHF and the topological classification of Hermitian Bloch Hamiltonians. We demonstrate that the partial indices appearing in the SWHF serve as fundamental topological invariants.

To justify this identification, we first establish the concept of spectral flattening within the SWHF framework. We show that any gapped Hermitian Hamiltonian can be adiabatically deformed into its diagonal factor $D(\beta)$ without closing the gap, implying that the topological information is entirely encoded in the partial indices. We then explicitly relate these indices to known topological invariants, including the winding number in symmetry class AIII and the $\mathbb{Z}_2$ invariant in class DIII.

Building on this correspondence, we derive how non-vanishing partial indices determine the existence and localization properties of zero-energy edge modes under semi-infinite boundary conditions. This result will later play a crucial role in understanding the NHSE via Hermitian doubling. Finally, we discuss the stability of the edge modes associated with the partial indices.

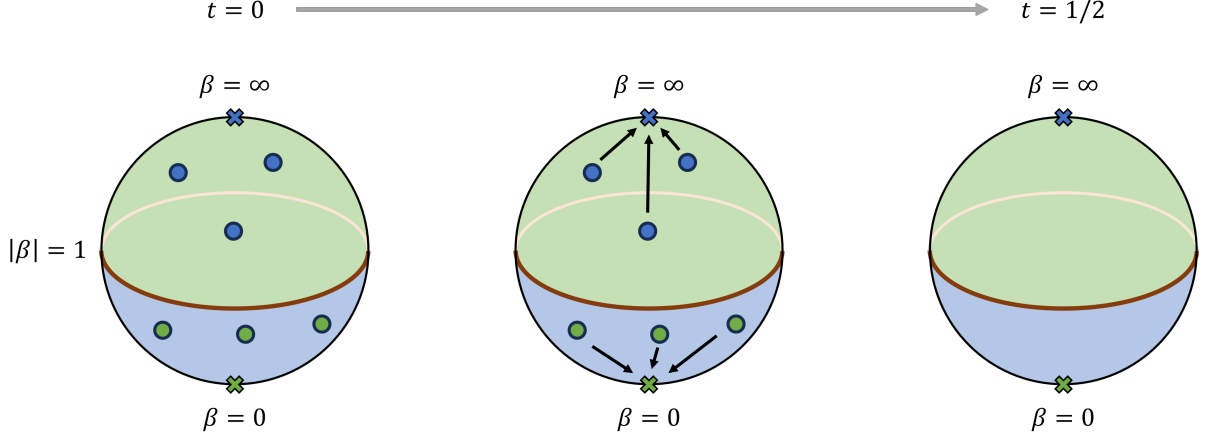


Figure 4.2: Schematic illustration of the spectral flattening process via SWHF on the Riemann sphere. Initially ($t = 0$), the singularities (zeros and poles) of the factors $h_+(\beta)$ and $h_+^\dagger(\beta)$ are distributed in the northern (blue dots) and southern (green dots) hemispheres, respectively. As the deformation process ($0 < t < 1/2$), the rescaling of the argument $\beta \rightarrow 0$ effectively drives the singularities of $h_+(\beta)$ towards the north pole ($\beta = \infty$) and those of $h_+^\dagger(\beta)$ towards the south pole ($\beta = 0$), as indicated by the arrows. In the flattened limit ($t \geq 1/2$), all singularities are expelled from the finite complex plane, resulting in constant unitary matrices. Consequently, the topological information of the system is now exclusively encoded in the diagonal factor $D(\beta)$ on the equator (orange line).

4.2.1 Spectral flattening via the SWHF

The topological classification of Bloch Hamiltonians is rigorously justified by the existence of an adiabatic transformation connecting the Hamiltonian to a canonical form. This process is often referred to as spectral flattening.

In the following discussion, we demonstrate that a Hermitian Bloch Hamiltonian $h(\beta)$, admitting the SWHF $h(\beta) = h_+(\beta)D(\beta)h_+^\dagger(\beta)$, is adiabatically connected to the diagonal factor $D(\beta)$. This connection implies that the topological invariants of $h(\beta)$ are entirely determined by the partial indices of $D(\beta)$. Since $D(\beta)$ already possesses the canonical form of topological equivalence classes, the proof reduces to showing that the factor $h_+(\beta)$ can be continuously deformed to a constant unitary matrix while maintaining invertibility.

We construct a homotopy $h_+^{(t)}(\beta)$ parameterized by $t \in [0, 1]$ as follows:

$$h_+^{(t)}(\beta) = \begin{cases} h_+((1-2t)\beta) & 0 \leq t \leq 1/2 \\ S_+^{2(1-t)}U_+ & 1/2 \leq t \leq 1 \end{cases} \quad (4.2.1)$$

where $h_+(0) = S_+U_+$ denotes the polar decomposition of the constant matrix $h_+(0)$, with S_+ being Hermitian positive-definite and U_+ unitary. By construction, this homotopy satisfies the boundary conditions $h_+^{(0)}(\beta) = h_+(\beta)$ and $h_+^{(1)}(\beta) = U_+$. Note that invertibility is

preserved throughout the path $t \in [0, 1]$ as discussed below.

In the first stage ($0 \leq t \leq 1/2$), we rescale the argument β to zero. Since $h_+(\beta)$ is invertible inside the unit disk ($|\beta| \leq 1$), the rescaling $\beta \rightarrow (1 - 2t)\beta$, which scales the roots of h_+ as $\beta_j \rightarrow \beta_j/(1 - 2t)$, ensures that the determinant remains non-zero throughout the process. At $t = 1/2$, the factor reduces to the constant matrix $h_+(0)$. In the second stage ($1/2 \leq t \leq 1$), we deform the positive-definite part S_+ to the identity matrix. Since S_+ is positive-definite, S_+^α is well-defined for any real α , ensuring a smooth path to the unitary matrix U_+ .

Consequently, the intermediate Hamiltonian is defined as $h^{(t)}(\beta) = h_+^{(t)}(\beta)D(\beta)[h_+^{(t)}(\beta)]^\dagger$. Since this construction explicitly preserves the SWHF structure $h_- = h_+^\dagger$ and $D = D^\dagger$ at every step t , the Hamiltonian remains Hermitian and maintains the same topological class as the diagonal factor $D(\beta)$. Since the final factor U_+ is a constant unitary matrix, it carries no winding number, confirming that the topology is exclusively encoded in the partial indices of $D(\beta)$.

4.2.2 Partial indices and Topological invariants

For both classes AIII and DIII, the relevant topological invariants can be expressed in terms of the off-diagonal block A introduced in Eq. (4.1.9).

For class AIII, the topological invariant is the winding number of A . Considering the WHF $A = A_+DA_-$, the factors A_+ and A_- are analytic and invertible for $|\beta| \leq 1$ and $|\beta| \geq 1$, respectively. Since $A_\pm$ have neither zeros nor poles in their corresponding regions, their winding numbers vanish, $W[A_\pm] = 0$. Consequently, the winding number of A equals that of D ,

$$W[A] = W[D] = \sum_{\kappa \in \mathcal{K}[A]} \kappa. \quad (4.2.2)$$

Hence, the winding number is given by the sum of the partial indices. Furthermore, the number of right [left] localized modes is equivalent to the sum of positive [negative] partial indices.

Next, we consider the $\mathbb{Z}_2$ topological invariant for class DIII. Conventionally, it is defined by the Pfaffian of the skew-symmetric matrix $A(\beta)$:

$$(-1)^{\nu[A]} = \text{sgn} \left[\frac{\text{Pf } A(-1)}{\text{Pf } A(1)} \exp \left\{ -\frac{1}{2} \int_0^\pi dk \partial_k \ln \det A(e^{ik}) \right\} \right]. \quad (4.2.3)$$

Here, the sign function $\text{sgn}(x)$ takes the value 1 for $x > 0$ and -1 for $x < 0$. Note that the integral is performed over half of the Brillouin zone, reflecting the TRS.

By applying the SWHF in Eqs. (4.1.11) and (4.1.12), this integral formula is drastically simplified. Similar to the class AIII case, the analytic factors $A_\pm$ do not contribute to the

invariant, and the topological information is solely determined by the central factor D_{skew} . Consequently, the invariant is reduced to the parity of the partial indices:

$$(-1)^{\nu[A]} = \frac{\text{Pf } D_{\text{skew}}(-1)}{\text{Pf } D_{\text{skew}}(1)} = (-1)^{\sum_{\kappa \in \mathcal{K}_+[A]} \kappa}. \quad (4.2.4)$$

In both classes, the topological invariant is thus encoded in the partial indices of the off-diagonal block A .

4.2.3 Edge modes and localization properties

We relate the edge modes and partial indices under semi-infinite boundary conditions, where the system is defined on a semi-infinite lattice $x \in \{0, 1, 2, \dots\}$. Let ψ be a state vector in the Hilbert space, represented by a sequence of vectors $\psi_x \in \mathbb{C}^M$ acting on the internal degrees of freedom.

We define the translation (shift) operator T acting on the spatial basis as the left shift operator:

$$(T\psi)_x = \psi_{x+1}. \quad (4.2.5)$$

The adjoint operator $T^\dagger$ acts as the right shift:

$$(T^\dagger\psi)_x = \begin{cases} \psi_{x-1} & (x > 1) \\ 0 & (x = 1). \end{cases} \quad (4.2.6)$$

Consequently, we have the algebraic relations:

$$TT^\dagger = I, \quad T^\dagger T = I - P_0, \quad (4.2.7)$$

where P_0 denotes the projection onto the boundary site $x = 0$. The powers of the translation operator are defined as T^n and $T^{-n} = (T^\dagger)^n$ for $n > 0$.

We map the matrix-valued polynomial $h(\beta)$ to the Toeplitz operator $\mathcal{T}[h]$ via the Fourier transform, replacing the variable β with the shift operator T :

$$h(\beta) = \sum_{j=-p}^q h_j \beta^j \quad \rightarrow \quad \mathcal{T}[h] = \sum_{j=-p}^q h_j T^j. \quad (4.2.8)$$

Using the symmetric Wiener-Hopf factorization, the Toeplitz operator can be decomposed as:

$$\mathcal{T}[h] = \mathcal{T}[h_+] \mathcal{T}[D] \mathcal{T}[h_+]^\dagger. \quad (4.2.9)$$

Here, $\mathcal{T}[h_+]$ corresponds to the analytic factor. Since $h_+(\beta)$ is invertible inside the unit circle, the corresponding operator $\mathcal{T}[h_+]$ is invertible. Similarly, $\mathcal{T}[h_+]^\dagger$ is invertible. Therefore, the kernel and cokernel of $\mathcal{T}[h]$ are isomorphic to those of the central factor $\mathcal{T}[D]$.

The diagonal factor corresponds to a block-diagonal operator:

$$\mathcal{T}[D] = \text{diag}(T^{\kappa_1}, \dots, T^{\kappa_M}). \quad (4.2.10)$$

The zero modes are determined by the partial indices κ_m . For a positive partial index $\kappa_m > 0$, the block T^{κ_m} acts as a left shift of order κ_m . This operator has a nontrivial kernel spanned by the first κ_m sites:

$$\ker \mathcal{T}[D] \cong \bigoplus_{m \in \mathcal{K}_+} \text{span}\{\mathbf{e}_{x,m} \mid 0 \leq x \leq \kappa_m\}, \quad (4.2.11)$$

where $\mathbf{e}_{x,m}$ denotes the basis vector localized at site x with internal component m . Conversely, for a negative partial index $\kappa_m < 0$, the block corresponds to the right shift $(T^\dagger)^{|\kappa_m|}$. This operator has a trivial kernel but a nontrivial cokernel, corresponding to the image deficiency at the boundary:

$$\text{coker} \mathcal{T}[D] \cong \bigoplus_{m \in \mathcal{K}_-} \text{span}\{\mathbf{e}_{x,m} \mid 0 \leq x \leq |\kappa_m|\}. \quad (4.2.12)$$

Thus, the partial indices directly count the number of localized modes at the boundaries.

4.2.4 Stability of edge modes

As established in the previous subsection, the kernel of $\mathcal{T}[h]$ is isomorphic to that of the diagonal factor $\mathcal{T}[D]$. This implies that the partial indices provide a rigorous count of the net number of zero-energy edge modes protected by the topology. Crucially, while the net topological index guarantees the existence of robust zero-energy edge modes, it does not protect edge modes in excess of the topological invariant. Such excess edge modes are therefore unstable and can be gapped out by symmetry-preserving perturbations through hybridization.

For the class AIII Hamiltonian, let us consider the case with partial indices $\mathcal{K}[h] = (+1, -1)$. Since the total winding number vanishes, this phase is topologically trivial, and no robust zero modes should exist in the bulk limit. However, the non-vanishing partial indices imply the existence of spatially separated boundary states: one mode η_+ localized at the right boundary ($\kappa = +1$) and one mode η_- at the left boundary ($\kappa = -1$). Due to their spatial separation, they cannot hybridize locally. In a finite system of length, however, a finite-size tunneling couples these modes, effectively serving as a symmetry-preserving perturbation:

$$\Delta H_{\text{AIII}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (4.2.13)$$

where the basis is (η_+, η_-) . This coupling lifts the degeneracy, resulting in the energy splitting $E = \pm 1$.

For the class DIII Hamiltonian, let us consider the case with a positive partial index $\mathcal{K}_+[h] = 2$. This corresponds to two Kramers pairs, $\eta_1 = (\eta_{1,+}, \eta_{1,-})$ and $\eta_2 = (\eta_{2,+}, \eta_{2,-})$. In contrast to the AIII case, these modes coexist at the same boundary and can be gapped out by a local symmetry-preserving perturbation,

$$\Delta H_{\text{DIII}} = \begin{pmatrix} & & 1 \\ & -1 & \\ -1 & & \\ 1 & & \end{pmatrix}, \quad (4.2.14)$$

where the basis is $(\eta_{1,+}, \eta_{1,-}, \eta_{2,+}, \eta_{2,-})$. This off-diagonal coupling explicitly mixes the two Kramers pairs while respecting TRS, thereby lifting the degeneracy and demonstrating why the $\kappa = 2$ phase is unstable.

4.3 Wiener-Hopf factorization and Szegő limit theorem

In this section, we consolidate the theoretical groundwork connecting the Szegő limit theorem and the Wiener-Hopf factorization (WHF) as a preparation for our later analysis of multiband non-Hermitian systems. We first revisit the classical Szegő limit theorem, interpreting it in terms of the partial indices of the WHF [181, 186, 187]. This review clarifies the conditions under which the OBC and PBC spectral potentials coincide and identifies how nonvanishing partial indices lead to deviations in the OBC potential. Next, we focus on Hermitian topologically nontrivial systems and review the modified Szegő limit theorem [183]. These insights then form the basis for our generalization of the Szegő limit theorem to multiband systems in the following section.

4.3.1 Classical strong Szegő limit theorem and partial indices

Let σ be a matrix-valued Laurent polynomial (Hermitian or non-Hermitian). The classical strong Szegő limit theorem [181, 186, 187] concerns the asymptotic behavior of the determinant ratio E_N ,

$$\lim_{N \rightarrow \infty} E_N[\sigma] = \lim_{N \rightarrow \infty} \frac{\det \mathcal{T}_N[\sigma]}{G[\sigma]^{N+1}}, \quad (4.3.1)$$

where $G[\sigma]$ is the exponential of the Ronkin function for $\mu = 0$, corresponding to the PBC spectral potential. It is written as

$$G[\sigma] = \exp \left[\int_0^{2\pi} \frac{dk}{2\pi} \ln \det \sigma(e^{ik}) \right] = e^{R_\sigma(0)}. \quad (4.3.2)$$

This limit converges when σ is gapped on the unit circle and its winding number vanishes. Thus, the OBC spectral potential can be expressed as

$$\frac{1}{N} \ln \det \mathcal{T}_N[\sigma] = \frac{N+1}{N} R_\sigma(0) + \frac{1}{N} \ln E_N[\sigma], \quad (4.3.3)$$

indicating that its deviation from the PBC spectral potential is controlled by $E_N[\sigma]$. The applicability of the Szegő limit theorem can be expressed in terms of the partial indices of the WHF of σ [181]. When all partial indices of σ vanish, the ratio $E_N[\sigma] \rightarrow C \neq 0$ converges to a nonzero constant, and the OBC and PBC spectral potentials coincide up to $\mathcal{O}(1/N)$ corrections. Deviations from this condition arise when one or more partial indices are nonzero, in which case $E_N[\sigma]$ in Eq. (4.3.1) decays to zero, and the OBC and PBC spectral potentials no longer coincide. In this case, the convergence rate of $E_N[\sigma]$,

$$E_N[\sigma] = e^{-\alpha N + \mathcal{O}(N)}, \quad (4.3.4)$$

determines the deviation of the OBC potential by $-\alpha$ [183]. The convergence rate α is related to the localization length μ , on which we will focus later.

For single-band Hamiltonians, the winding number equals the partial index. Thus, optimizing the Ronkin function is equivalent to finding a μ for which the partial index of σ_μ is zero, and thus

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \det \mathcal{T}_N[\sigma] = \min_{\mu} R_\sigma(\mu). \quad (4.3.5)$$

This is the generalized Szegő limit theorem proposed in Ref. [173].

In multiband systems, however, the total winding number is the sum of all partial indices, so a vanishing winding number no longer guarantees that each index individually vanishes. Consequently, the naive Amoeba formulation can fail unless there exists a μ for which all partial indices become zero. If such a μ exists, the classical Szegő limit theorem still holds for σ_μ ; otherwise, the correspondence between OBC and PBC potentials breaks down.

4.3.2 Modified Szegő limit theorem

As a next step, we review the modified Szegő limit theorem established for topologically nontrivial Hermitian Hamiltonians [182, 183], which governs the convergence behavior of $E_N[\sigma]$. We further summarize how these deviations are connected to boundary modes in such systems, thereby establishing the conceptual bridge used in the following section.

Let $\tilde{\sigma}$ be a Hermitian two-band class AIII Hamiltonian, written in the off-diagonal form of Eq. (4.1.9):

$$\tilde{\sigma} = \begin{pmatrix} & A^\dagger \\ A & \end{pmatrix}. \quad (4.3.6)$$

Because of chiral symmetry, the winding number of the full Hamiltonian satisfies $W[\tilde{\sigma}] = 0$. However, the topology of $\tilde{\sigma}$ is characterized by the winding number of its off-diagonal block, $W[A] = \kappa$. When $\kappa = 0$, the system is topologically trivial and the determinant ratio $E_N[\tilde{\sigma}]$ converges to a finite, nonzero constant, as predicted by the classical Szegő limit theorem. For $\kappa \neq 0$, the Hamiltonian is topologically nontrivial, and $\mathcal{K}[\tilde{\sigma}]$ is $(|\kappa|, -|\kappa|)$. The resulting decay of $E_N[\tilde{\sigma}]$ to zero reflects the contribution of boundary-localized modes in $\mathcal{T}_N[\tilde{\sigma}]$.

It has been shown in Ref. [183] that the spectrum of $\mathcal{T}_N[\tilde{\sigma}]$ is well approximated by the PBC spectrum together with a pair of boundary-localized modes. However, due to finite-size hybridization, these two edge modes acquire small but finite energies $\pm\varepsilon$ rather than lying exactly at zero. The splitting ε scales with the overlap between the edge modes, $\varepsilon \propto e^{-\mu N}$, where $\mu > 0$ is the inverse localization length. The modified Szegő limit theorem [183] then implies

$$E_N[\tilde{\sigma}] \propto \varepsilon^2 \propto e^{-2\mu N}. \quad (4.3.7)$$

The generalization to multiple pairs of edge modes is straightforward by

$$E_N[\tilde{\sigma}] \propto \prod_j e^{-2\mu_j N}, \quad (4.3.8)$$

where $\mu_j > 0$ denotes the inverse localization lengths of each pair.

As discussed in Ref. [182], the number of zero modes is determined by $\mathcal{K}[\sigma]$. A positive partial index of $\mathcal{K}[A]$ corresponds to a mode localized at the right boundary, while a negative partial index of $\mathcal{K}[A]$ (or equivalently, a positive partial index of $\mathcal{K}[A^\dagger]$) corresponds to a mode localized at the left boundary.

For example, if $\kappa = 1$ (i.e., $W[A] = 1$), the off-diagonal block A contributes one right-localized mode, while $A^\dagger$ contributes one left-localized mode. Thus, the full Hamiltonian $\mathcal{T}_N[\tilde{\sigma}]$ hosts one mode on each boundary. The localization length of these modes is determined by the root of $\det A$ with the largest magnitude inside the unit circle, yielding $\mu = \ln |\beta|$. This shows that the Szegő limit theorem for a topologically nontrivial Hermitian Hamiltonian must be modified by this decay exponent.

This argument extends to general nonzero κ . For positive (negative) κ , the block A contributes κ right- (left-) localized modes, governed by the κ roots of $\det A$ inside (outside) the unit circle with the largest (smallest) magnitudes. In either case, the total decay exponent is given by the sum of their inverse localization lengths.

The above discussion illustrates how nonzero winding numbers modify the asymptotic behavior of $E_N[\tilde{\sigma}]$. To further clarify the role of partial indices in multiband settings, we next consider a four-band class AIII Hamiltonian where the off-diagonal block A is a two-band matrix. In this case, even when $W[A] = 0$, the block A may still have nontrivial partial indices, e.g., $\mathcal{K}[A] = (+1, -1)$. Since $\tilde{\sigma}$ is constructed from A and $A^\dagger$, the partial indices $(+1, -1)$ of A lead to $\mathcal{K}[\tilde{\sigma}] = (+1, +1, -1, -1)$ for the full Hamiltonian $\tilde{\sigma}$. Thus, A supports

two edge modes localized at opposite boundaries, and $\mathcal{T}_N[\tilde{\sigma}]$ hosts two pairs of edge modes in total. These modes are controlled by the dominant roots of $\det A$: one root β_2 with the smallest absolute value outside the unit circle and one root β_1 with the largest absolute value inside. We denote their inverse localization lengths as $\mu_2 = \ln |\beta_2| > 0$ and $\mu_1 = \ln |\beta_1| < 0$, respectively. The contribution to $E_N[\tilde{\sigma}]$ then becomes

$$E_N[\tilde{\sigma}] \propto e^{-2\mu_2 N} e^{2\mu_1 N} = e^{-2(\mu_2 - \mu_1)N}. \quad (4.3.9)$$

We note that $\mu_2 - \mu_1$ is equivalent to the length of the interval where $W[\sigma_\mu]$ vanishes.

Chapter 5

Amoeba formulation for one-dimensional multiband systems

Building on the theoretical groundwork developed in the previous chapter, we extend the Amoeba framework to multiband systems. While the conventional Amoeba formulation has proven successful for single-band systems, its generalization to multiband systems is obstructed by topological cancellations and the breakdown of analyticity, as discussed in Chapter 3. To overcome these fundamental difficulties, we introduce a unified approach based on the WHF and Hermitian doubling.

We begin by mapping the non-Hermitian skin effect to topological edge modes in the doubled Hermitian system. This correspondence allows us to reformulate the generalized Szegő limit theorem physically, identifying the potential deviation with the localization lengths of edge modes. We bridge this physical picture with the conventional optimization formulation via the concept of residual partial indices.

Applying this general framework to specific symmetries, we focus on the symplectic class (class AII[†]). We first introduce a phenomenological formulation of symmetry-decomposed Ronkin functions to describe the $\mathbb{Z}_2$ skin effect. Finally, we rigorously validate this framework using the symplectic WHF and discuss the stability of the resulting topological phases.

5.1 Wiener-Hopf factorization and Hermitian doubling

The WHF-based classification, previously demonstrated for Hermitian systems via spectral flattening, naturally extends to non-Hermitian point-gap phases through the framework of Hermitian doubling [60, 61]. Mathematically, the spectral flattening of the doubled Hamiltonian, which preserves chiral symmetry, transforms its off-diagonal block into a unitary matrix. This operation is equivalent to the standard unitary flattening of the original non-Hermitian Hamiltonian [60], ensuring that the partial indices of the doubled system provide a complete

topological characterization consistent with established non-Hermitian classification.

Physically, this mapping establishes a rigorous correspondence between boundary states. Since the Hermitian doubling process maps the non-Hermitian skin modes of the original system to the topological edge modes of the doubled system [70, 91], non-vanishing partial indices directly indicate the presence of skin modes. Thus, the partial indices of the doubled Hamiltonian serve as a precise detector of nontrivial non-Hermitian topology and the associated skin effect.

We now apply this framework to specific symmetry classes. A class A non-Hermitian Hamiltonian maps to a class AIII Hermitian Hamiltonian via Hermitian doubling:

$$\tilde{\sigma} = \begin{pmatrix} & \sigma^\dagger \\ \sigma & \end{pmatrix}, \quad (5.1.1)$$

where $\sigma = E - h$. Crucially, a WHF of σ induces a SWHF of $\tilde{\sigma}$, meaning that the topological information of the non-Hermitian symbol σ is fully encoded in its partial indices. Thus, the topological invariant $W[\sigma]$ is simply the sum of all partial indices,

$$W[\sigma] = \sum_{\kappa \in \mathcal{K}[\sigma]} \kappa. \quad (5.1.2)$$

Similarly, a class AIII[†] non-Hermitian Hamiltonian is mapped to a class DIII Hermitian Hamiltonian, with

$$\tilde{\tau} = \begin{pmatrix} & \tau^\dagger \\ \tau & \end{pmatrix}, \quad (5.1.3)$$

where $\tau = (E - h)U_T$. Here, the multiplication by the unitary matrix U_T converts the TRS[†] constraint on h into the skew-symmetry condition $\tau^\top = -\tau$. In this case, the topological information is obtained from the partial indices of τ , which reflect the $\mathbb{Z}_2$ topology of the doubled system. The invariant is determined by the sum of the positive indices $\mathcal{K}_+[\tau]$,

$$\nu[\tau] = \sum_{\kappa \in \mathcal{K}_+[\tau]} \kappa \mod 2. \quad (5.1.4)$$

5.2 Hermitian doubling and classical Szegő limit theorem

Based on the correspondence between partial indices and skin modes, we provide a physical interpretation of the Szegő limit theorems. This interpretation clarifies why the OBC and PBC potentials coincide in the absence of topological winding, despite the drastic spectral changes caused by the NHSE.

Let us define the spectral potentials for the doubled system as:

$$\tilde{\phi}_{\text{OBC}}(0) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln |\det \mathcal{T}_N[\tilde{\sigma}]|, \quad \tilde{\phi}_{\text{PBC}}(0) = \int_0^{2\pi} \frac{dk}{2\pi} \ln |\det \tilde{\sigma}|. \quad (5.2.1)$$

Due to the block structure of $\tilde{\sigma}$ in Eq. (5.1.1), the determinant of the doubled operator factorizes as $|\det \mathcal{T}_N[\tilde{\sigma}]| = |\det \mathcal{T}_N[\sigma]|^2$. Here, the reference energy E is effectively absorbed into the definition of the doubled Hamiltonian $\tilde{\sigma}$. Consequently, the evaluation of the potential at E for the original system corresponds to evaluating the potential at the origin ($E' = 0$) for the doubled system. Thus, the potentials satisfy the relation:

$$\tilde{\phi}_{\text{OBC/PBC}}(0) = 2\phi_{\text{OBC/PBC}}(E). \quad (5.2.2)$$

This relation allows us to understand the behavior of the non-Hermitian potential $\phi(E)$ through the topology of $\tilde{\sigma}$. When all partial indices of σ vanish, the doubled Hamiltonian $\tilde{\sigma}$ is in a topologically trivial phase. In this regime, the doubled system hosts no topological edge modes at zero energy. Since there are no near-zero eigenvalues to exponentially suppress the determinant $\det \mathcal{T}_N[\tilde{\sigma}]$, the determinant is governed solely by the bulk spectrum. Consequently, the OBC and PBC spectral potentials coincide up to finite-size corrections $\tilde{\phi}_{\text{OBC}}(0) = \tilde{\phi}_{\text{PBC}}(0)$, resulting in zero potential deviation for the original system: $\phi_{\text{OBC}}(E) = \phi_{\text{PBC}}(E)$.

This physical picture allows us to reinterpret the classical Szegő limit theorem. From the perspective of the doubled Hermitian system, the validity of the theorem is equivalent to the absence of topological zero-modes. Translating this back to the original non-Hermitian system, this condition implies that the reference energy E resides in a topologically trivial point gap. Under these conditions, the boundary conditions become irrelevant for the evaluation of the potential at E , and it is a natural consequence that the OBC potential coincides with the PBC potential.

Conversely, when the reference energy E resides in a topologically nontrivial regime characterized by non-zero partial indices, the correspondence breaks down. From the perspective of the doubled system, this nontrivial topology guarantees the existence of zero-energy edge modes. These localized modes exponentially suppress the determinant, necessitating the Modified Szegő limit theorem, which explicitly accounts for their contribution.

5.3 Generalized Szegő limit theorem for multiband systems

In this section, we reformulate the generalized Szegő limit theorem for multiband systems by integrating the Hermitian doubling framework with the modified Szegő limit theorem. First, we demonstrate that the deviation between the OBC and PBC spectral potentials is

physically determined by the sum of the inverse localization lengths of the edge modes in the corresponding doubled Hermitian system. Next, we bridge this physical picture with the conventional optimization formulation of the Ronkin function. In this context, we show that the condition of a vanishing winding number is generalized to the condition of vanishing residual partial indices.

5.3.1 Potential deviation and localization lengths of edge modes

In non-Hermitian systems, topological phases often manifest not through isolated edge states but through the extensive localization of bulk modes near the boundaries, as seen in the NHSE. Through Hermitian doubling, each localized non-Hermitian bulk mode corresponds to a quasi-zero mode of a doubled Hermitian Hamiltonian sharing the identical localization length [70, 91]. This correspondence enables us to apply the modified Szegő limit theorem established for Hermitian systems to derive a generalized version valid for non-Hermitian Hamiltonians.

We first validate this approach in a single-band system and confirm that it reproduces the known results of the conventional Amoeba formulation. As an illustrative example, let σ be a non-Hermitian symbol whose determinant factorizes as

$$\det \sigma(\beta) = C\beta^{-p} \prod_{j=1}^{p+q} [\beta - \beta_j], \quad (5.3.1)$$

where $\beta_1, \dots, \beta_{p+1}$ are inside the unit circle ($|\beta_j| < 1$), and $\beta_{p+2}, \dots, \beta_{p+q}$ are outside the circle ($|\beta_j| > 1$). This root distribution corresponds to a winding number $W[\sigma] = 1$. The corresponding Hermitianized symbol $\tilde{\sigma}$ is defined in Eq. (5.1.1) and belongs to a two-band class AIII system. The edge mode of this Hamiltonian is governed by the excess root $\mu_{p+1} (< 0)$, which implies $E_N[\tilde{\sigma}] = \exp[2\mu_{p+1}N]$ [Eq. (4.3.7)]. Using the relation $E_N[\tilde{\sigma}] = |E_N[\sigma]|^2$, the modified Szegő limit theorem for $\tilde{\sigma}$ yields the deviation for the potential of σ :

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln |\det \mathcal{T}_N[\sigma]| = R_\sigma(0) + \mu_{p+1} = \ln |C| + \sum_{j=p+1}^{p+q} \mu_j, \quad (5.3.2)$$

which reproduces the generalized Szegő limit theorem for $W[\sigma] = 1$. This argument straightforwardly generalizes to arbitrary winding numbers, corresponding to the optimization form in Eq. (4.3.5).

This framework for evaluating potential deviations, based on the asymptotic behavior of $E_N[\tilde{\sigma}]$, naturally extends to multiband systems with non-vanishing partial indices. As another illustrative example, consider a two-band symbol σ with vanishing total winding number $W[\sigma] = 0$ but nonzero partial indices $\mathcal{K}[\sigma] = (1, -1)$. Let $\det \sigma$ factorize as

$$\det \sigma(\beta) = C\beta^{-2p} \prod_{j=1}^{2(p+q)} [\beta - \beta_j], \quad (5.3.3)$$

where $\beta_1, \dots, \beta_{2p}$ lie inside the unit circle ($|\beta_j| < 1$) and $\beta_{2p+1}, \dots, \beta_{2(p+q)}$ lie outside it ($|\beta_j| > 1$). This choice ensures $W[\sigma] = 0$. Using the asymptotic form of $E_N[\tilde{\sigma}]$ for the $(+1, -1)$ index structure, Eq. (4.3.9), we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln |\det \mathcal{T}_N[\sigma]| = R_\sigma(0) - (\mu_{2p+1} - \mu_{2p}), \quad (5.3.4)$$

where $\mu_{2p} = \ln |\beta_{2p}|$ is the logarithmic radius of the largest roots inside the unit circle and $\mu_{2p+1} = \ln |\beta_{2p+1}|$ that of the smallest root outside.

It is instructive to interpret the correction term in Eq. (5.3.4) from a "local" perspective using the roots of $\det \sigma$. For $W[\sigma] = 0$, the PBC spectral potential $R_\sigma(0)$ is given by the sum over roots outside the unit circle, $\ln |C| + \sum_{j=2p+1}^{2(p+q)} \mu_j$. The OBC spectral potential $\phi_{\text{OBC}}(E)$ in Eq. (5.3.4) thus becomes:

$$\phi_{\text{OBC}}(E) = \ln |C| + \mu_{2p} + \sum_{j=2p+2}^{2(p+q)} \mu_j \quad (5.3.5)$$

Thus, the contribution from the smallest outside root (μ_{2p+1}) is removed and replaced by that of the largest inside root (μ_{2p}). This result offers a clear interpretation: the non-vanishing partial indices quantify how many root contributions are exchanged across the unit circle. In this example, $\mathcal{K}[\sigma] = (1, -1)$ means that, even though $W[\sigma] = 0$, the correct OBC potential involves one such exchange, where the positive index brings in the innermost interior root and the negative index removes the outermost exterior root. These results can be straightforwardly generalized to cases with arbitrary nonzero partial indices and winding numbers.

5.3.2 Interpretation via residual partial indices

Finally, we integrate the physical insights from the previous subsection into the conventional optimization framework of the Ronkin function.

Recall that the OBC determinant is invariant under the similarity transformation given in Eq. (3.3.1). This invariance allows us to eliminate the non-zero global winding number by introducing an appropriate shift μ , effectively moving the integration contour. Even after tuning μ to eliminate the winding, however, the partial indices may remain nontrivial. By applying Eq. (5.3.4) to the shifted symbol σ_μ , we obtain a modified form of the generalized Szegő limit theorem that explicitly accounts for the non-vanishing partial indices, such as $(1, -1)$:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln |\det \mathcal{T}_N[\sigma]| = \min_{\mu} R_\sigma(\mu) - (\mu_{k+1} - \mu_k), \quad (5.3.6)$$

where μ_k and μ_{k+1} denote the logarithmic moduli of the roots located just inside and just outside the integration contour defined by the optimal μ .

Building on this observation, we propose that the validity of the Amoeba formulation is rigorously determined by the behavior of the partial indices after optimization. We formally define the residual partial indices,

$$\mathcal{K}^*[\sigma] = \mathcal{K}[\sigma_{\mu^*}], \quad (5.3.7)$$

where μ^* is the optimal shift chosen such that the global winding number vanishes, $W[\sigma_{\mu^*}] = 0$.

This definition generalizes the validity criterion for the Amoeba formulation. If all residual partial indices vanish individually, i.e., $\mathcal{K}^* = (0, 0, \dots)$, the conventional Amoeba optimization remains valid, and the standard generalized Szegő limit theorem applies. However, if one or more residual indices remain nonzero ($\mathcal{K}^* \neq 0$) despite the vanishing total winding, the simple Amoeba formulation is insufficient. In such cases, the generalized Szegő limit theorem requires additional correction terms dictated by $\mathcal{K}^*$. Consequently, the theorem takes the following compact, unified form,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln |\det \mathcal{T}_N[\sigma]| = \mathcal{M}_{\mathcal{K}^*}[\sigma], \quad (5.3.8)$$

where the functional $\mathcal{M}_{\mathcal{K}^*}[\sigma]$ depends on the residual partial indices:

$$\mathcal{M}_{(0,0)}[\sigma] = \min_{\mu} R_{\sigma}(\mu), \quad (5.3.9)$$

$$\mathcal{M}_{(+1,-1)}[\sigma] = \min_{\mu} R_{\sigma}(\mu) - (\mu_{k+1} - \mu_k), \quad (5.3.10)$$

where the indices k and $k+1$ identify the pair of roots closest to the integration contour $|\beta| = e^{\mu}$. This formulation clarifies that the condition "vanishing winding number" in single-band systems must be generalized to "vanishing residual partial indices" in multiband systems to correctly describe the OBC spectrum.

5.4 Phenomenological discussion for class AII[†]

The framework established in the previous sections, based on Hermitian doubling and the modified Szegő limit theorem, is universally applicable to non-Hermitian systems, regardless of their symmetry class. However, for class AII[†] systems, the underlying symplectic symmetry allows for an alternative formulation based on the decomposition of the Ronkin function. This formulation provides a more intrinsic understanding of the $\mathbb{Z}_2$ skin effect, distinct from the generic doubling approach.

In this and the subsequent section, we establish the generalized Szegő limit theorem for class AII[†] systems. This section introduces the symmetry-decomposed Ronkin functions from a phenomenological perspective, constructing them to be consistent with the established GBZ condition in Eq. (2.3.20). This approach leads to a conjecture for the generalized Szegő limit theorem in symplectic systems. These phenomenological results are then rigorously verified using the WHF in the next section.

5.4.1 Symmetry constraints on the Ronkin function

As discussed in Chapter 2, the determinant $\det \sigma(\beta)$ in class AII[†] is invariant under the transformation $\beta \rightarrow \beta^{-1}$. This symmetry imposes an even constraint on the Ronkin function, $R_\sigma(\mu) = R_\sigma(-\mu)$. Consequently, the Ronkin function is universally minimized at $\mu = 0$. This implies the absence of a global non-Hermitian skin effect, which seemingly contradicts the existence of the $\mathbb{Z}_2$ skin effect where boundary modes appear despite the vanishing global winding number.

To clarify this mechanism, we consider the factorized form of the determinant:

$$\det \sigma(\beta) = C_E \beta^{-2p} \prod_{j=1}^{2p} [\beta - \beta_j][\beta - \beta_j^{-1}]. \quad (5.4.1)$$

Here, the roots appear in reciprocal pairs (β_j, β_j^{-1}) due to the underlying symmetry. We define β_j as the roots outside the unit circle and order them by their magnitude,

$$|\beta_{2p}^{-1}| \leq \dots \leq |\beta_1^{-1}| < 1 < |\beta_1| \leq \dots \leq |\beta_{2p}|. \quad (5.4.2)$$

Performing the integration as Eq. (3.5.5), the Ronkin function can be expressed in the compact form:

$$R_\sigma(\mu) = \ln |C_E| - 2p\mu + \sum_{j=1}^{2p} [\max(\mu, -\mu_j) + \max(\mu, \mu_j)], \quad (5.4.3)$$

where $\mu_j = \ln |\beta_j|$ are defined to be positive.

This expression reveals that the total Ronkin function inevitably sums the contributions from both members of the Kramers pairs. Since this summation cancels out the directional information required to describe the $\mathbb{Z}_2$ topology, it is necessary to disentangle these contributions and construct symmetry-decomposed functions capable of independent optimization.

5.4.2 Constructing symmetry-decomposed Ronkin functions

As discussed in Sec. 3.6.2, the naive approach of using band-resolved Ronkin functions fails to conserve the essential analytic properties required for the theory. To overcome this limitation while respecting the underlying symplectic structure, we introduce an alternative method: the symmetry-decomposed Ronkin functions.

Our focus in this derivation is specifically on the topologically nontrivial regime where $\nu[\tau]$ is non-zero. The conventional Szegő limit theorem [Eq. (3.2.2)], which relies on optimizing the total Ronkin function, remains valid as long as the system is topologically trivial ($\nu[\tau] = 0$). Consequently, the modification of the theory is necessary if and only if the system exhibits the $\mathbb{Z}_2$ skin effect ($\nu[\tau] = 1$).

First, we discuss how the $\text{TRS}^\dagger$ constrains the decomposed functions. When a system satisfies $\text{TRS}^\dagger$, each state is doubly degenerate, forming Kramers pairs, and the states in each pair exhibit opposite localization lengths. Consequently, we require the symmetry-decomposed Ronkin functions to have the same optimal value and to satisfy the following relation:

$$R_\sigma(\mu) = R_\sigma^{(+)}(\mu) + R_\sigma^{(-)}(\mu), \quad R_\sigma^{(+)}(\mu) = R_\sigma^{(-)}(-\mu). \quad (5.4.4)$$

Next, we outline the mathematical properties that the symmetry-decomposed Ronkin function $R_\sigma^{(\pm)}$ must have. The symmetry-decomposed Ronkin functions should inherit the mathematical properties of the total Ronkin function R_σ : they must be convex, piece-wise linear, and have integer-quantized derivatives. Furthermore, we require that $R_\sigma^{(\pm)}$ exhibit central hole closing if E is in the OBC spectrum, analogous to the class A case. Considering Eq. (2.3.20), we assume that $R_\sigma^{(+)}[R_\sigma^{(-)}]$ remains constant in the domain $\mu_1 \leq \mu \leq \mu_2$ [$-\mu_2 \leq \mu \leq -\mu_1$].

To construct such proper symmetry-decomposed Ronkin functions, we use Eq. (5.4.3), which shows that the total Ronkin function is determined by the β -roots and the poles enclosed by the circle $|\beta| = e^\mu$. Therefore, we divide the roots and the order of the pole into two groups. We assume that $R_\sigma^{(+)} [R_\sigma^{(-)}]$ contains the roots whose moduli are larger [smaller] than 1. From the above assumption, we can deduce that the $R_\sigma^{(+)}[R_\sigma^{(-)}]$ must remain constant in the domain $\mu_1 \leq \mu \leq \mu_2$ [$-\mu_2 \leq \mu \leq -\mu_1$]. Thus, the order of poles must be divided into 1 for $R_\sigma^{(+)}$ and $2p - 1$ for $R_\sigma^{(-)}$.

Accordingly, the symmetry-decomposed Ronkin functions are given by,

$$\begin{aligned} R_\sigma^{(+)}(\mu) &= \frac{1}{2} \ln |C_E| - \frac{1}{2} \sum_{j=1}^{2p} \mu_j - \mu + \sum_{j=1}^{2p} \max(\mu_j, \mu) \\ R_\sigma^{(-)}(\mu) &= \frac{1}{2} \ln |C_E| + \frac{1}{2} \sum_{j=1}^{2p} \mu_j - (2p - 1)\mu + \sum_{j=1}^{2p} \max(-\mu_j, \mu) \end{aligned} \quad (5.4.5)$$

Here, the second term provides a constant shift, ensuring that the symmetry-decomposed Ronkin functions satisfy Eq. (5.4.4). The symmetry-decomposed Ronkin functions introduced here maintain the symmetry between bands while preserving the piecewise linearity and integer-valued derivatives of the Ronkin functions. One can readily verify that the sum of these symmetry-decomposed Ronkin functions reproduces the total Ronkin function.

5.4.3 Generalized Szegő limit theorem for class AII[†]

We reconstruct the total Ronkin function using this decomposition and demonstrate that the extrapolation of the total Ronkin function optimizes the symmetry-decomposed Ronkin functions, leading to a generalized Szegő's limit theorem for two-band class AII[†] systems.

The symmetry-decomposed Ronkin functions defined in Eq. (5.4.5) can be expanded around their minima as

$$R_{\sigma}^{(+)}(\mu) = a + \begin{cases} -(\mu - \mu_1) & -\infty \leq \mu \leq \mu_1 \\ 0 & \mu_1 \leq \mu \leq \mu_2 \\ \mu - \mu_2 & \mu_2 \leq \mu \leq \mu_3 \end{cases} \quad (5.4.6)$$

$$R_{\sigma}^{(-)}(\mu) = a + \begin{cases} -(\mu + \mu_2) & -\mu_3 \leq \mu \leq -\mu_2 \\ 0 & -\mu_2 \leq \mu \leq -\mu_1 \\ \mu + \mu_1 & -\mu_1 \leq \mu \leq \infty \end{cases} \quad (5.4.7)$$

Here a denotes the minimum of the symmetry-decomposed Ronkin functions,

$$a = \frac{1}{2} \ln |C_E| - \frac{1}{2} \mu_1 + \frac{1}{2} \sum_{j=2}^{2p} \mu_j, \quad (5.4.8)$$

which corresponds to the potential of each band.

Because the total Ronkin function is symmetric, we focus on $\mu \geq 0$ and can easily reconstruct the total Ronkin function as

$$R_{\sigma}(\mu) = 2a + \begin{cases} 2\mu_1 & 0 \leq \mu \leq \mu_1 \\ \mu + \mu_1 & \mu_1 \leq \mu \leq \mu_2 \\ 2\mu - (\mu_2 - \mu_1) & \mu_2 \leq \mu \leq \mu_3 \end{cases} \quad (5.4.9)$$

We observe that the minimum of the Ronkin function is shifted by $2\mu_1$ from the optimal value $2a$. Therefore, we can obtain the true potential by correcting this error. However, merely correcting this shift would result in the loss of hole-closing information, which plays a crucial role in the spectral detection for class A systems. Thus, we focus on $\mu_2 - \mu_1$, which represents the length of the region with a slope of 1 and works as a unit-slope hole closing indicator.

With this knowledge, we can now extrapolate the total Ronkin function to obtain the correct potential. By extrapolating the region with a slope of 2, we obtain the correct potential:

$$\phi_{\text{OBC}}(E) = 2a = \tilde{R}_{\sigma}^{(2)}(0) + (\mu_2 - \mu_1), \quad (5.4.10)$$

where $\tilde{R}_{\sigma}^{(2)}(\mu) = 2\mu + 2a - (\mu_2 - \mu_1)$. Combining the GBZ condition Eq. (2.3.20), the extrapolated value yields the appropriate potential for the OBC spectrum, and the localization length is obtained by $\mu_1 (= \mu_2)$.

Notably, this extrapolation method works even when the region with a slope of 2 vanishes. If $\mu_2 = \mu_3$ holds, we can consider the tangent line with a slope of 2 at $\mu = \mu_2$. This condition motivates us to reformulate Eq. (5.4.10) in terms of the Legendre transformation.

We introduce the Legendre transformation of the Ronkin function, which takes the form

$$Q_\sigma(m) = \min_{\mu} [R_\sigma(\mu) - m\mu]. \quad (5.4.11)$$

In this framework, we can define the correct potential of the generalized Szegő's limit theorem for two-band class AII[†] systems as

$$\phi_{\text{OBC}}(E) = Q_\sigma(2) - (Q'_\sigma(1^+) - Q'_\sigma(1^-)) \quad (5.4.12)$$

where Q'_σ denotes the derivative of Q_σ ¹.

We can finally include the trivial case and summarize our conjecture to compute the OBC potential as

$$\phi_{\text{OBC}}(E) = \begin{cases} Q_\sigma(0) & \nu[\tau] = 0 \\ Q_\sigma(2) - (Q'_\sigma(1^+) - Q'_\sigma(1^-)) & \nu[\tau] = 1 \end{cases}. \quad (5.4.13)$$

where $\nu[\tau]$ is the $\mathbb{Z}_2$ topological invariant defined in Eq. (4.2.3). The information about unit-slope hole closing in the total Ronkin function is inherited as the jump in the derivative Q'_σ at $m = 1$. Thus, the absence of such a jump serves as an indicator of the OBC spectrum.

Finally, it is instructive to express the spectral potential in a "local form" using the roots of the characteristic equation. Assuming the roots are ordered by their logarithmic moduli as $0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_{2p}$, the potential can be compactly written depending on the $\mathbb{Z}_2$ invariant $\nu[\tau]$:

$$\phi(E) = \ln |C_E| + \begin{cases} \mu_1 + \sum_{j=2}^{2p} \mu_j & (\nu[\tau] = 0) \\ -\mu_1 + \sum_{j=2}^{2p} \mu_j & (\nu[\tau] = 1) \end{cases} \quad (5.4.14)$$

$$= \ln |C_E| + (-1)^{\nu[\tau]} \mu_1 + \sum_{j=2}^{2p} \mu_j. \quad (5.4.15)$$

This formula explicitly demonstrates that the transition from the trivial to the topological phase corresponds to the exchange of the smallest positive root μ_1 with its reciprocal partner $-\mu_1$. This "root exchange" mechanism is the hallmark of the non-Hermitian skin effect in class AII[†], providing a clear physical picture of how the OBC potential deviates from the PBC result.

5.5 WHF-based discussion for class AII[†]

We now demonstrate that the WHF of a non-Bloch Hamiltonian in the two-band class AII[†] naturally leads to a decomposition of the Ronkin function, yielding the symmetry-decomposed forms defined in Eq. (5.4.5).

¹Since the Ronkin function is piecewise linear and has integer-quantized derivatives, the following equivalent expression exists: $\Phi(E) = Q_E(2) - (Q'_E(2^-) - Q'_E(0^+))$.

5.5.1 WHF and symmetry-decomposed Ronkin functions

We focus on two-band ($M = 2$) class AII[†] systems. Here, we define the total Ronkin function and its symmetry-decomposed counterparts using $\tau = (E - h)U_T$ instead of $\sigma = E - h$. Since U_T is a unitary matrix, R_τ is equivalent to R_σ . This redefinition aligns more naturally with the WHF framework for class AII[†] systems, enabling a symmetric decomposition of the Ronkin function.

As established in Sec. 4.1.2, the topological information is encoded in the partial indices of τ . The SWHF of τ can be arranged as shown in Eq. (4.1.11),

$$\tau = \tau_+ D_\kappa \tau_-, \quad (5.5.1)$$

where $\tau_+^\top = \tau_-$ and D_κ takes a form given by Eq. (4.1.12),

$$D_{\kappa=0} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \quad \text{or} \quad D_{\kappa>0} = \begin{pmatrix} & \beta^{-\kappa} \\ -\beta^\kappa & \end{pmatrix}. \quad (5.5.2)$$

We distinguish between the cases where the partial indices are $(0, 0)$ and $(+\kappa, -\kappa)$ with $\kappa > 0$. The $\mathbb{Z}_2$ invariant can be evaluated as

$$(-1)^{\nu[\tau]} = (-1)^\kappa. \quad (5.5.3)$$

Thus, the parity of κ determines the $\mathbb{Z}_2$ invariant.

We can rewrite the SWHF of τ , Eq. (5.5.1), in a symplectic form

$$\tau = \tau_+^{\text{symp}} J \tau_-^{\text{symp}}, \quad \tau_+^{\text{symp}} = \tau_+ S_\kappa, \quad \text{and} \quad J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \quad (5.5.4)$$

Here, $[\tau_+^{\text{symp}}]^\top = \tau_-^{\text{symp}}$. The factor S_κ appearing in τ_+^{symp} , takes the form

$$S_{\kappa=0} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \text{or} \quad S_{\kappa>0} = \begin{pmatrix} \beta^{-\kappa} & \\ & 1 \end{pmatrix}. \quad (5.5.5)$$

Using this factorization, we now define the decomposed Ronkin functions:

$$R_\tau^{(+)}(\mu) = \int_0^{2\pi} \frac{dk}{2\pi} \ln |\det[\tau_+^{\text{symp}}]_\mu|, \quad R_\tau^{(-)}(\mu) = \int_0^{2\pi} \frac{dk}{2\pi} \ln |\det[\tau_-^{\text{symp}}]_\mu|. \quad (5.5.6)$$

These quantities correspond to the contributions from τ_+^{symp} and τ_-^{symp} sectors, respectively. This WHF-based definition is applicable regardless of the topological phase, providing a unified framework.

Furthermore, this formulation is consistent with the phenomenological construction in Eq. (5.4.5). Since $\det J = 1$, it follows that

$$\det \tau = \det(\tau_+^{\text{symp}}) \det(\tau_-^{\text{symp}}) \quad (5.5.7)$$

Considering $\tau_+^{\text{symp}}[\tau_-^{\text{symp}}]$ has neither roots nor poles inside [outside] the unit circle, we find that

$$\det(\tau_+^{\text{symp}}) = \sqrt{C_E \prod_{j=1}^{2p} \beta_j^{-1} \times \beta^{-\kappa} \prod_{j=1}^{2p} [\beta - \beta_j]} \quad (5.5.8)$$

$$\det(\tau_-^{\text{symp}}) = \sqrt{C_E \prod_{j=1}^{2p} \beta_j \times \beta^{\kappa-2p} \prod_{j=1}^{2p} [\beta - \beta_j^{-1}]} \quad (5.5.9)$$

By integrating these expressions, we obtain the symmetry-decomposed Ronkin functions $R_\tau^{(\pm)}$, recovering the forms given in the Eq. (5.4.5) for the cases where $\kappa = 0$, and 1.

5.5.2 Generalized Szegő limit theorem for class AII[†]

We now provide a proof of the generalized Szegő limit theorem for class AII[†] using the WHF-based Ronkin functions.

The TRS[†] operator U_T is independent of β , which implies that $\mathcal{T}_N[U_T]$ is diagonal. Accordingly, the OBC spectral potential $\phi_{\text{OBC}}(E)$ can be rewritten as

$$\phi_{\text{OBC}}(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln |\det \mathcal{T}_N[\sigma]| = \lim_{N \rightarrow \infty} \frac{1}{N} \ln |\det \mathcal{T}_N[\tau]|. \quad (5.5.10)$$

Using the symplectic form of the WHF, Eq. (5.5.4), the spectral potential can be decomposed as

$$\phi_{\text{OBC}}(E) = \frac{2}{N} \ln |\det \mathcal{T}_N[\tau_+^{\text{symp}}]| + \frac{1}{N} \ln |\det \mathcal{T}_N[J]| + \mathcal{O}(N^{-1}). \quad (5.5.11)$$

The symplectic factor J does not contribute to the leading term since $\det \mathcal{T}_N[J] = (\det J)^N = 1$. The factor τ_+^{symp} itself does not preserve TRS[†] symmetry, since it corresponds to only one sector of the full WHF and therefore lacks the full symmetric structure. The Kramers-paired roots have been separated into τ_+^{symp} and τ_-^{symp} .

For $\kappa = 0$, τ_+ is invertible inside the unit circle, and $\tau_+^{\text{symp}} = \tau_+ S_0$ shares the same analytic domain. Thus, τ_+^{symp} admits a trivial WHF, $\tau_+^{\text{symp}} = \tau_+ S_0 \cdot 1 \cdot 1$, where both the central factor D and the right factor A_- are the identity, so all partial indices of τ_+^{symp} vanish. Consequently, the classical Szegő limit theorem applies directly to the symbol τ_+^{symp} :

$$\phi_{\text{OBC}}(E) = 2 \int_0^{2\pi} \frac{dk}{2\pi} \ln |\det \tau_+^{\text{symp}}| = \int_0^{2\pi} \frac{dk}{2\pi} \ln |\det \tau|. \quad (5.5.12)$$

For $\kappa > 0$, the factor S_κ contains negative powers of β . The WHF of $\tau_+^{\text{symp}} = \tau_+ S_\kappa$ takes the form $\tau_+^{\text{symp}} = \tau_+ \cdot S_\kappa \cdot 1$, where the central factor is S_κ , and hence $\mathcal{K}[\tau_+^{\text{symp}}] = (0, -\kappa)$.

Consequently, we must apply the generalized Szegő limit theorem, Eq. (5.3.8), which accounts for these nonzero partial indices

$$\phi_{\text{OBC}}(E) = 2\mathcal{M}_{\mathcal{K}^*}[\tau_+^{\text{symp}}], \quad (5.5.13)$$

where $\mathcal{K}^*$ denotes the residual partial indices of τ_+^{symp} , i.e., the partial indices of the similarity-transformed symbol $[\tau_+^{\text{symp}}]_\mu(e^{ik}) = \tau_+^{\text{symp}}(e^{\mu+ik})$ evaluated at the optimally chosen μ for which the total winding vanishes. If the partial indices of $[\tau_+^{\text{symp}}]_\mu$ vanish simultaneously, i.e., the residual partial indices are trivial, $\mathcal{K}^*[\tau_+^{\text{symp}}] = (0, 0)$, the simple optimization formulation is recovered:

$$\phi_{\text{OBC}}(E) = 2\mathcal{M}_{(0,0)}[\tau_+^{\text{symp}}] = 2 \min_{\mu} R_{\tau}^{(+)}(\mu). \quad (5.5.14)$$

If the partial indices of the optimized $[\tau_+^{\text{symp}}]_\mu$ do not vanish simultaneously, i.e., $\mathcal{K}^*[\tau_+^{\text{symp}}] = (-1, 1)$, the optimal value must be corrected according to Eq. (5.3.8):

$$\phi_{\text{OBC}}(E) = 2\mathcal{M}_{(+1,-1)}[\tau_+^{\text{symp}}] = 2(\min_{\mu} R_{\tau}^{(+)}(\mu) - (\mu_{k+1} - \mu_k)), \quad (5.5.15)$$

where μ_k and μ_{k+1} denote, respectively, the dominant roots of τ_+^{symp} located just inside and just outside the circle of radius e^μ .

Combining the results for $\kappa = 0$ and $\kappa > 0$, we arrive at a unified expression for the OBC spectral potential, valid for all $\kappa \geq 0$:

$$\phi_{\text{OBC}}(E) = 2\mathcal{M}_{\mathcal{K}^*}[\tau_+^{\text{symp}}] \quad (5.5.16)$$

This expression reproduces Eq. (5.4.14) for $\kappa = 0$ and 1, and extends it naturally to higher topological sectors ($\kappa \geq 2$).

5.5.3 Stability of the $\kappa = 2$ phase and modified GBZ condition

The rigorous derivation in the previous subsection serves as a mathematical justification for the phenomenological discussion in Sec. 5.4. Specifically, it confirms that the symmetry-decomposed Ronkin functions provide the correct potential for the standard $\kappa = 0$ and $\kappa = 1$ phases. However, the WHF framework also suggests the mathematical possibility of phases with higher partial indices, such as $\kappa \geq 2$, which exhibit anomalous spectral behaviors.

In a phase characterized by a partial index $\kappa = 2$, the optimization of the symmetry-decomposed Ronkin functions occurs in the interval (μ_2, μ_3) . Consequently, the localization length is determined by the condition $\mu_2 = \mu_3$. This stands in stark contrast to the conventional GBZ condition for class AII[†] systems, which is conventionally established as $\mu_1 = \mu_2$. This deviation implies that the $\kappa = 2$ phase possesses an inherent fragility compared to the standard $\kappa = 1$ phase.

We can understand this instability physically by revisiting the correspondence between partial indices and edge modes. As established in Sec. 4.2.4, the partial indices count the number of localized edge modes in the doubled Hermitian system. A partial index of $\kappa = 2$ corresponds to the existence of two pairs of Kramers degenerate edge modes at the boundary.

However, in the $\mathbb{Z}_2$ classification of class AII[†] (equivalent to class DIII for the doubled system), a system with an even number of Kramers pairs is topologically trivial ($\nu[\tau] = 0$). This means the system is not topologically protected against the annihilation of these pairs. Unlike the $\kappa = 1$ case, where a single Kramers pair is robust against TRS[†]-preserving perturbations, the two pairs in the $\kappa = 2$ phase coexist at the same boundary. As detailed in Sec. 4.2.4, local symmetry-preserving perturbations can induce hybridization between these inter-pair modes, lifting their degeneracy and opening a spectral gap. Therefore, the $\kappa = 2$ phase is generally unstable and is expected to collapse into the $\kappa = 0$ (fully gapped) phase under generic conditions.

Nevertheless, the $\kappa = 2$ phase can be realized if an additional mechanism forbids this hybridization. For instance, if the system possesses a hidden unitary symmetry that commutes with the TRS[†] operator, the two Kramers pairs may belong to different symmetry sectors, preventing them from mixing. In such specific cases, the "anomalous" GBZ condition $\mu_2 = \mu_3$ is physically realized, representing a $\kappa = 2$ phase.

Consequently, for Hamiltonians that do not exhibit the unstable $\kappa \geq 2$ phase in the complex energy plane (or where such phases have been removed by perturbations), the spectral properties are governed solely by the stable $\kappa = 0$ and $\kappa = 1$ phases. In the absence of the anomalous higher-order winding, the localization length is determined by the collision of the primary roots, ensuring that the standard GBZ condition $\mu_1 = \mu_2$ is maintained.

Under this condition, the potential is correctly captured by the symmetry-decomposed Ronkin function $R_\sigma^{(+)}(\mu)$. Recall that $R_\sigma^{(+)}(\mu)$ was constructed with a pole of order 1 [Eq. (5.4.5)]. This construction is specifically designed to detect the slope change associated with the first root crossing at $\mu = \mu_1$, which corresponds to the standard GBZ solution. Therefore, for generic class AII[†] systems, the general formula simplifies to:

$$\phi_{\text{OBC}}(E) = 2 \min_{\mu} R_\sigma^{(+)}(\mu) = 2 \min_{\mu} R_\tau^{(+)}(\mu). \quad (5.5.17)$$

This result justifies the phenomenological conjecture presented in Sec. 5.4, demonstrating that the optimization of the symmetry-decomposed function is the correct procedure for Hamiltonians free from the unstable high-winding phases.

Chapter 6

Numerical verification

In this chapter, we provide a comprehensive numerical and analytical validation of the WHF-based formulation developed in the preceding sections. Our primary objective is to demonstrate that the Wiener-Hopf approach successfully resolves the ambiguities inherent in the conventional amoeba formulation when applied to multiband systems.

First, we analyze two-band systems in class A characterized by non-vanishing partial indices. We demonstrate that the presence of these indices invalidates the naive optimization of the Ronkin function typically used in single-band systems. By comparing numerical OBC spectra with our analytical results, we show that the necessary correction is precisely captured by the generalized Szegő limit theorem as formulated in Eq. (5.3.8).

Next, we verify the WHF-based decomposition of the Ronkin function for systems possessing $\text{TRS}^\dagger$. By considering two representative parameter sets, we show that our decomposition method naturally leads to the emergence of symmetry-protected $\mathbb{Z}_2$ skin effect. Specifically, we demonstrate that the decomposed Ronkin functions based on WHF rigorously reproduce the phenomenological symmetry-decomposed Ronkin functions in the regimes where the topological index κ is 0 or 1.

Finally, we extend our investigation to a more complex parameter regime where a $\kappa = 2$ domain emerges. This analysis highlights the robustness of our formulation in higher topological sectors, where the interplay between multiple bands and boundary conditions becomes increasingly nontrivial.

6.1 Correction for non-vanishing partial indices in two-band class A systems

In this section, we numerically verify the correction to the generalized Szegő limit theorem arising from non-vanishing partial indices, as derived in Eq. (5.3.8). Our primary goal is to illustrate a specific regime where the conventional Amoeba formulation fails and demonstrate

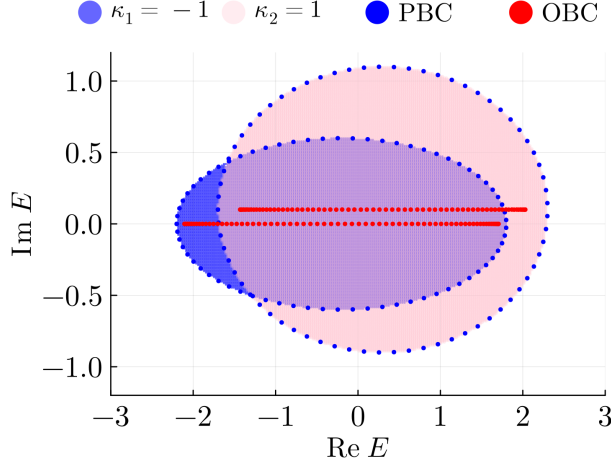


Figure 6.1: Comparison of OBC and PBC spectra for the two uncoupled Hatano-Nelson models defined in Eq. (6.1.1). Blue dotted lines show the PBC spectra of the individual blocks h_1 and h_2 , while red lines denote the full OBC spectrum for $N = 70$. Shaded regions in the complex energy plane indicate where the partial indices are nonzero. The blue-shaded region corresponds to $\kappa_1 = W[\sigma_1] = -1$ with $\kappa_2 = 0$, and the red-shaded region corresponds to $\kappa_2 = W[\sigma_2] = 1$ with $\kappa_1 = 0$. Their overlap (purple) represents the regime $\mathcal{K}[\sigma] = (-1, 1)$, where the conventional Amoeba optimization breaks down and the generalized formulation of Chapter 5 must be applied.

how our proposed correction restores the correct OBC potential.

To this end, we consider a model in which there is a parameter regime, where the partial indices $\mathcal{K}[\sigma_\mu]$ do not simultaneously vanish for any shift μ . The simplest example with this property is a block-diagonal two-band Hamiltonian composed of two uncoupled Hatano-Nelson chains:

$$\begin{cases} h(\beta) &= \begin{pmatrix} h_1(\beta) & \\ & h_2(\beta) \end{pmatrix} \\ h_1(\beta) &= 1.3 \beta^{-1} - 0.2 + 0.7 \beta \\ h_2(\beta) &= 0.5 \beta^{-1} + 0.3 + 0.1i + 1.5 \beta \end{cases} \quad (6.1.1)$$

We define the symbols of the blocks as $\sigma_{1,2} = E - h_{1,2}$ and the full symbol as $\sigma = \text{diag}(\sigma_1, \sigma_2)$. The partial indices of the full system, $\mathcal{K}[\sigma] = (\kappa_1, \kappa_2)$, are then given simply by the winding numbers of the uncoupled blocks, $\kappa_{1,2} = W[\sigma_{1,2}]$. Crucially, the total winding number of the system, which determines the point-gap topology in the conventional sense, is given by the sum of the winding numbers of each block, $W[\sigma] = W[\sigma_1] + W[\sigma_2] = \kappa_1 + \kappa_2$.

Figure 6.1 depicts the topological phases in the complex energy plane, showing the PBC (blue dotted lines) and OBC (red lines) spectra. The shaded regions indicate where the partial

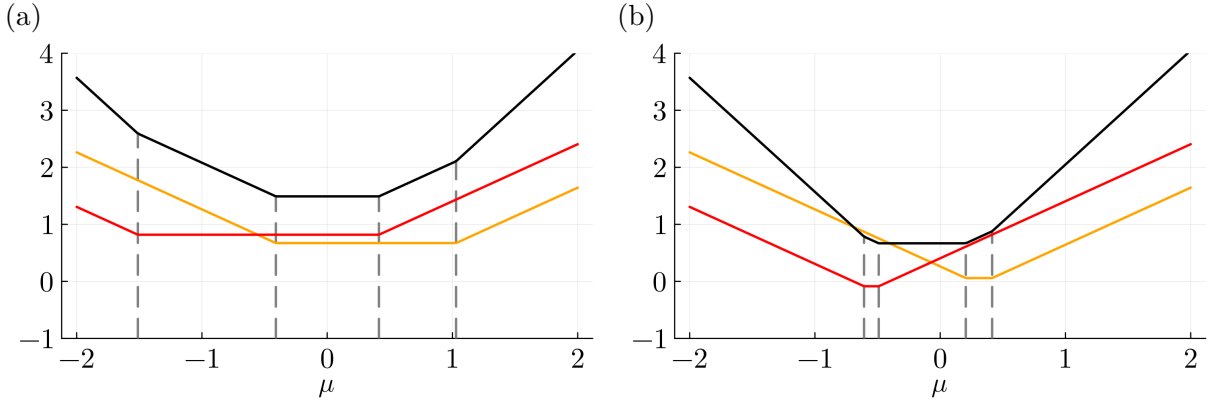


Figure 6.2: Ronkin functions for two representative reference energies E , selected from distinct topological domains in Fig. 6.1. The total Ronkin function R_σ is shown in black, while the partial Ronkin functions R_{σ_1} and R_{σ_2} are plotted in red and orange, respectively. (a) Trivial case ($E = -2.0 + 1.0i$): The reference point lies in the white region, where $\mathcal{K}[\sigma] = (0, 0)$. Both R_{σ_1} and R_{σ_2} can be minimized simultaneously, and the conventional optimization scheme applies. (b) Nontrivial case ($E = 0.0 + 0.2i$): The reference point lies in the purple region, where $\mathcal{K}[\sigma] = (-1, 1)$. Here, R_{σ_1} and R_{σ_2} cannot be minimized simultaneously, and the correction term in Eq. (5.3.8) is required to obtain the correct OBC potential.

indices (winding numbers) take nonzero values. In the blue-shaded area, $\mathcal{K}[\sigma] = (-1, 0)$; in the red-shaded area, $\mathcal{K}[\sigma] = (0, 1)$. The overlap between them, shown in purple, marks the regime where $\mathcal{K}[\sigma] = (-1, 1)$. In the white region, both partial indices vanish simultaneously, $\mathcal{K}[\sigma] = (0, 0)$.

Our analysis focuses on the regions where the total winding number vanishes ($W[\sigma] = 0$), corresponding to the white and purple regions in Fig. 6.1. Although $W[\sigma] = 0$ in both cases, the underlying partial indices behave differently. In the white region, both partial indices vanish ($\kappa_1 = \kappa_2 = 0$), and thus the conventional Amoeba formulation is expected to hold. In contrast, the purple region is characterized by "hidden" non-vanishing partial indices ($\kappa_1 = -1, \kappa_2 = 1$). As we demonstrate below, this nontrivial internal structure leads to the breakdown of the standard theory, necessitating the generalized correction.

Figure 6.2 verifies our theoretical prediction by analyzing the optimization landscape of the Ronkin function at two representative reference energies. The panels display the full Ronkin function R_σ (black) alongside the individual block contributions R_{σ_1} (red) and R_{σ_2} (orange), with vertical gray dashed lines marking the roots μ_j of the characteristic equation. Fig. 6.2(a) shows the trivial case at $E = -2.0 + 1.0i$ where the partial indices vanish simultaneously, i.e., $\mathcal{K}[\sigma] = (0, 0)$. Fig. 6.2(b) shows the nontrivial case at $E = 0.0 + 0.2i$ where the partial indices do not vanish simultaneously, i.e., $\mathcal{K}[\sigma] = (-1, 1)$. Since the total

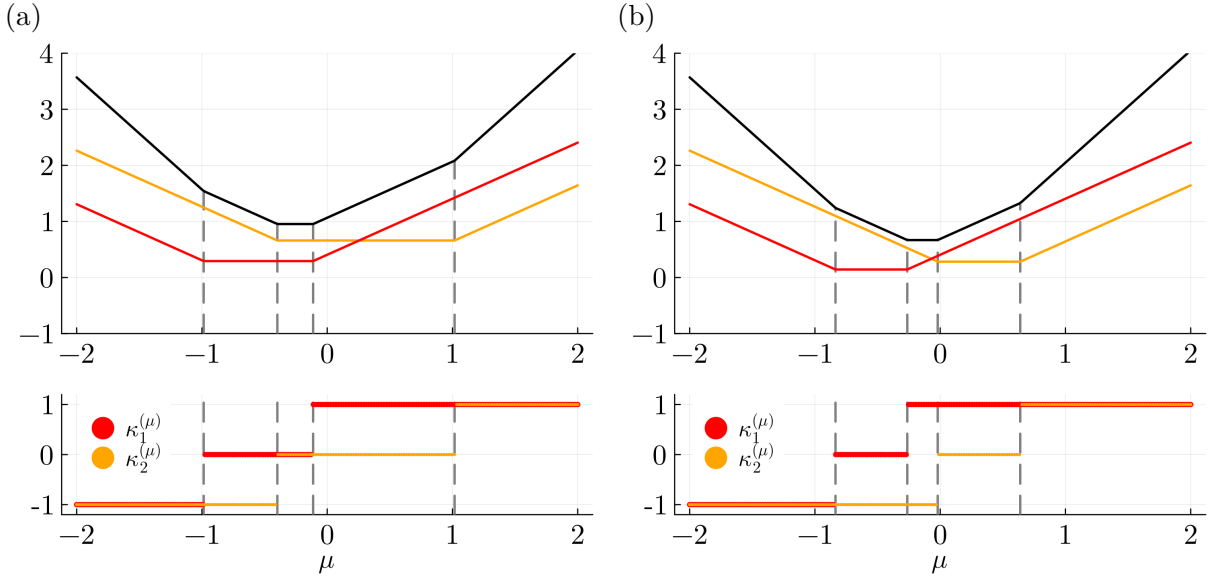


Figure 6.3: Ronkin functions and μ -dependent partial indices for two reference energies taken from the red region in Fig. 6.1. Each top panel shows the Ronkin functions, similar to Fig. 6.2. Each bottom panel displays the μ -dependent partial indices $\mathcal{K}[\sigma_\mu] = (\kappa_1^{(\mu)}, \kappa_2^{(\mu)})$. (a) $E = 2.2 + 0.1i$: The partial indices vanish after optimization, i.e., the residual partial indices are $\mathcal{K}^*[\sigma] = (0, 0)$, yielding the correct OBC potential through the standard optimization scheme. (b) $E = 0.5 + 0.6i$: The residual partial indices are nonzero even after optimization, $\mathcal{K}^*[\sigma] = (+1, -1)$, indicating that the correction in Eq. (5.3.8) is required to obtain the correct potential.

winding number $W[\sigma]$ vanishes at both points, the conventional optimization scheme would incorrectly predict the OBC potential to be $R_\sigma(0)$ in both cases.

For the case $E = -2.0 + 1.0i$ [Fig. 6.2(a)], the reference point lies in the white region of Fig. 6.1, and $\mathcal{K}[\sigma] = (0, 0)$. The Ronkin functions R_{σ_1} and R_{σ_2} admit a common minimization interval, so the simple optimization scheme applies. In contrast, for $E = 0.0 + 0.2i$ [Fig. 6.2(b)], the reference point lies in the purple region with nonzero partial indices, $\mathcal{K}[\sigma] = (-1, 1)$. As a result, there is no region where the Ronkin functions R_{σ_1} and R_{σ_2} can be jointly minimized, signaling the breakdown of the simple optimization scheme and requiring the correction term in Eq. (5.3.8).

For this specific model and parameter set, the correction term evaluates to $-(\mu_3 - \mu_2)$, corresponding to the modification $\min_\mu R_\sigma(\mu) - (\mu_{k+1} - \mu_k)$ in Eq. (5.3.8). This correction accounts for the exchange of contributions between the largest inner root (μ_2) and the smallest outer root (μ_3) at the minimum of R_σ .

The above discussion extends to points with non-vanishing winding number $W[\sigma] \neq 0$ by employing the similarity transformation and using μ -dependent partial indices $\mathcal{K}[\sigma_\mu] =$

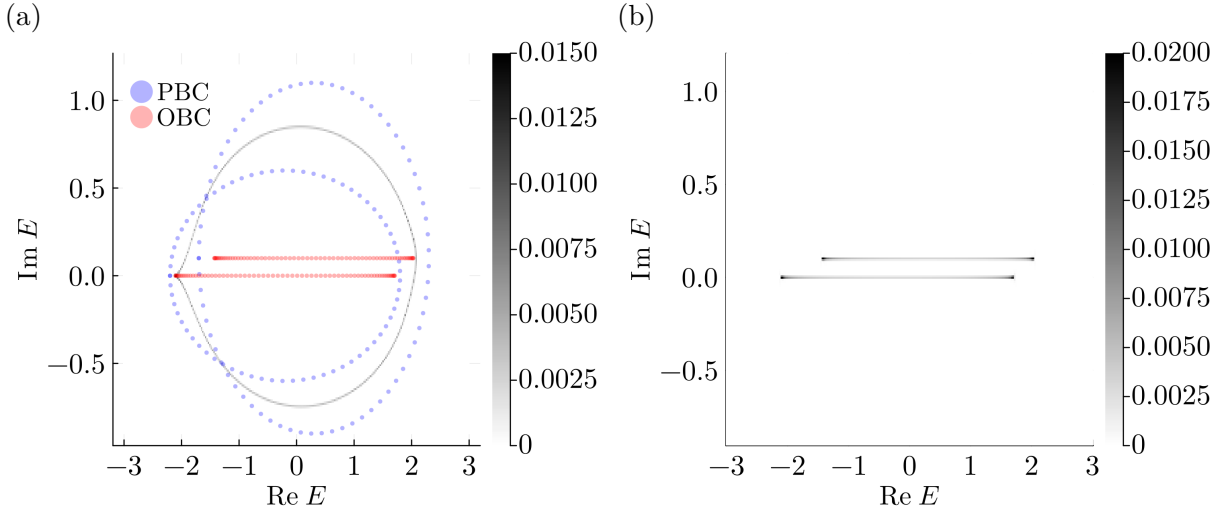


Figure 6.4: Comparison of the DOS evaluated by (a) the conventional formulation and (b) our WHF-based formulation. Values are scaled by the area of the energy cell $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 2.5 \times 10^{-4}$ due to the discretizations of the Hessian in Eq. (3.1.2). (a) The conventional simple optimization fails to reproduce the OBC spectrum (red line), as evident from the mismatch with the calculated DOS (grayscale). (b) The WHF-based formulation, which includes the correction for nonvanishing partial indices, accurately reproduces the OBC spectrum.

$(\kappa_1^{(\mu)}, \kappa_2^{(\mu)})$. Figure 6.3 illustrates this for two instructive reference energies from the red region in Fig. 6.1. In addition to the Ronkin functions, we show the partial indices $\kappa_1^{(\mu)}$ (red) and $\kappa_2^{(\mu)}$ (orange) separately. For $E = 2.2 + 0.1i$ [Fig. 6.3(a)], the partial indices share a common vanishing interval; equivalently, the residual partial indices vanish $\mathcal{K}^*[\sigma] = (0, 0)$. In this case, the simple optimization scheme applies, and the minimum of $R_\sigma(\mu)$ correctly yields the OBC potential. For the case $E = 0.5 + 0.6i$ [Fig. 6.3(b)], the partial indices never vanish simultaneously for any μ , so $\mathcal{K}^*[\sigma] = (+1, -1)$, and the correction in Eq. (5.3.8) is necessitated.

To test our prediction quantitatively, we compute the DOS, using the residual partial indices as a guide to determine where the correction term must be applied. Figure 6.4 compares the DOS obtained from (a) the conventional simple optimization and (b) our WHF-based formulation [Eq. (5.3.8)]. Because of the discretization of the Hessian in Eq. (3.1.2), the plotted values are scaled by the energy cell area $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 2.5 \times 10^{-4}$. As shown in Fig. 6.4(a), the simple optimization fails to reproduce the OBC DOS. By contrast, the corrected WHF-based formulation [Fig. 6.4(b)] accurately reproduces the OBC DOS, in excellent agreement with the spectrum shown in Fig. 6.1.

As a consequence, the corrected OBC potential reproduces the numerical eigenvalues in

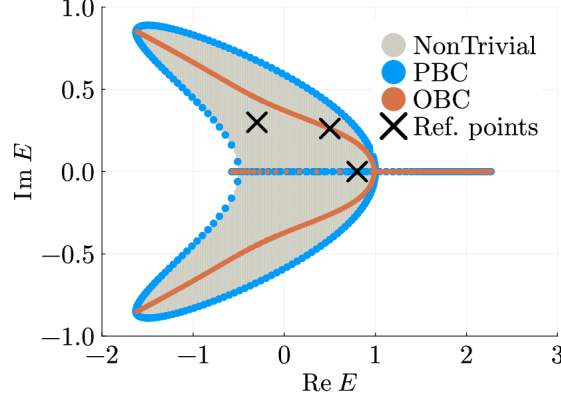


Figure 6.5: PBC and OBC ($N = 400$) spectrum: The Hamiltonian is defined in Eq. (6.2.1). Parameters are set to $t_1 = 0.3, t_2 = 0.8, \Delta_1 = 0.3, \Delta_2 = 0.2$, and $g_1 = 0.5$. The topologically nontrivial area is filled in gray. The reference points are $E = -0.3 + 0.3i, 0.5 + 0.265i, 0.8$. Reproduced from Fig. 1 of Ref. [191]. ©2025 by the American Physical Society.

the whole complex plane, whereas the conventional formulation fails to trace the spectral distribution. This result confirms that the generalized Szegő limit theorem with WHF-based corrections is essential for a precise description of the NHSE in multiband systems.

6.2 Verification of symmetry-decomposed Ronkin functions ($\kappa = 0, 1$)

In this section, we verify the WHF-based formulation for systems with $\text{TRS}^\dagger$ (class AII †). Unlike the block-diagonal model discussed in the previous section, the bands in symmetric systems are inherently coupled, making the spectral analysis more subtle.

Our validation follows two steps. First, we demonstrate that the band-resolved Ronkin functions, calculated by treating the two bands independently, fail to inherit the mathematical properties of the total Ronkin function. Second, we show that our WHF-based decomposition rigorously reproduces the symmetry-decomposed Ronkin functions introduced in Chapter 5. This consistency confirms that the WHF automatically identifies the symmetry-protected sectors without prior knowledge of the symmetry-decomposed basis.

To this end, we analyze a two-band symplectic Hatano-Nelson model with next-nearest-neighbor hopping:

$$h(\beta) = (t_2 - i\Delta_2 X + g_2 Z)\beta^{-2} + (t_1 - i\Delta_1 X + g_1 Z)\beta^{-1} + (t_1 + i\Delta_1 X - g_1 Z)\beta + (t_2 + i\Delta_2 X - g_2 Z)\beta^2 \quad (6.2.1)$$

where X, Y, Z are Pauli matrices. The $\text{TRS}^\dagger$ is implemented by the operator Y , which is independent of β , consistent with Eq. (1.2.8).

We first examine the parameter regime where the positive partial index is $\kappa = 0$ or 1 , setting the parameters as $t_1 = 0.3, t_2 = 0.8, g_1 = 0.5, \Delta_1 = 0.3$, and $\Delta_2 = 0.2$.

Figure 6.5 displays the spectrum obtained by diagonalizing the finite-size Hamiltonian on a chain with 400 sites. The blue and orange lines represent the spectra under PBC and OBC, respectively. The complex plane contains a region with a nontrivial $\mathbb{Z}_2$ topological invariant, as defined in Eq. (4.2.3) (shaded in gray).

We observe that while the OBC spectrum coincides with the PBC spectrum along the real axis, it deviates significantly in the complex plane, particularly for $\text{Re } E < 1$. Interestingly, while the PBC spectrum is gapless, the numerical OBC spectrum appears to exhibit a line gap around $E = 0.8$. However, this is a spurious gap arising from finite-size effects; we expect this gap to vanish and the spectrum to become continuous in the thermodynamic limit.

6.2.1 Failure of band-resolved Ronkin function

This part provides a mathematical analysis of why naive band-resolved Ronkin functions, defined in Eq. (3.6.2), fundamentally fail to preserve the essential mathematical properties of convexity, piecewise linearity, and integer-quantized derivatives. We demonstrate that the breakdown of these properties stems from the multi-valued nature of the energy band branches (EBBs), which necessitates branch cuts that destroy the analytic structure underlying the Amoeba formulation.

The EBBs of the Hamiltonian can be expressed as

$$E_{\pm}(\beta) = \frac{\text{Tr } h(\beta) \pm \sqrt{\Delta(\beta)}}{2}, \quad \Delta(\beta) = [\text{Tr } h(\beta)]^2 - 4 \det h(\beta), \quad (6.2.2)$$

where $\Delta(\beta)$ is the discriminant of the characteristic polynomial. The analyticity of the individual bands $E_{\pm}(\beta)$ is compromised at the exceptional points where $\Delta(\beta) = 0$.

The topological obstruction to a consistent band decomposition is encoded in the band vorticity $\chi(\mu)$ [12, 49, 158, 190]:

$$\begin{aligned} \chi(\mu) &= \frac{1}{2} \int_0^{2\pi} \frac{dk}{2\pi i} \partial_k \ln [E_+(e^{\mu+ik}) - E_-(e^{\mu+ik})]^2 \\ &= \frac{1}{2} \int_0^{2\pi} \frac{dk}{2\pi i} \partial_k \ln \Delta(e^{\mu+ik}). \end{aligned} \quad (6.2.3)$$

When $\chi(\mu)$ takes a half-integer value, the EBBs undergo a band-swapping (or spectral braiding) process: $E_{\pm}(e^{\mu+i \cdot 2\pi}) = E_{\mp}(e^{\mu+i \cdot 0})$. In this regime, the spectrum of $h(\beta)$ for a fixed μ forms a single concatenated loop rather than two disjoint ones. Consequently, any attempt to define individual Ronkin functions $R_E^{[\pm]}$ necessitates a branch cut that artificially "slices" this loop into open arcs. This non-analyticity destroys the piecewise linearity and the integer quantization of the derivatives for the partial functions, even though the total Ronkin function $R_E = R_E^{[+]} + R_E^{[-]}$ remains well-behaved and cut-independent.

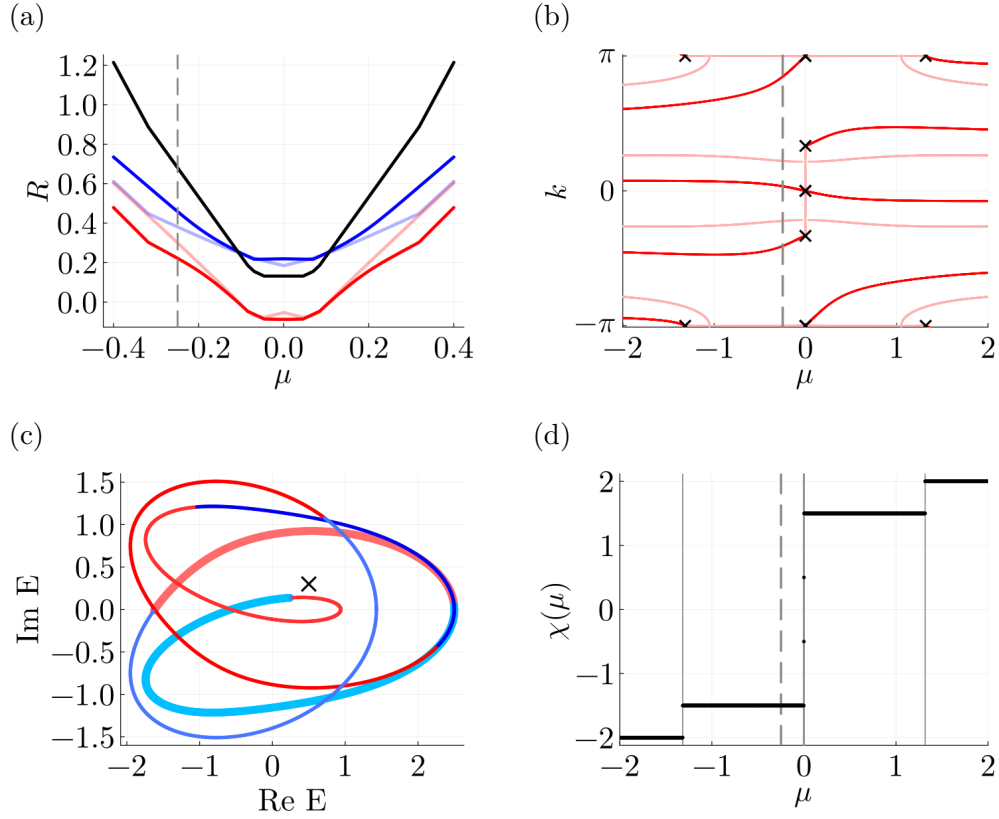


Figure 6.6: Breakdown of the mathematical properties of band-resolved Ronkin functions caused by branch-cut choice and eigenvalue braiding. (a) Band-resolved Ronkin functions $R_E^{[+]}$ (red) and $R_E^{[-]}$ (blue) at $E = 0.5 + 0.3i$. Branch cuts of the square root extend from the origin at angles of $5\pi/7$ (red/blue) and 0 (light red/light blue). Both branch cuts produce non-convex profiles with distinct minima, whereas their sum R_E (black) is cut-independent and convex. The vertical dashed line marks $\mu = -0.25$ for reference across all panels. (b) The two branch cuts in the (μ, k) -plane (red and light red), connecting the branch points (black $\times$). (c) Energy spectrum $E_{\pm}(\exp(\mu + ik))$ at $\mu = -0.25$ with branch cut $5\pi/7$. A single closed loop fractures into several open arcs (shown in different colors) due to the branch cuts. The black $\times$ marks the reference energy $E = 0.5 + 0.3i$. (d) The band vorticity $\chi(\mu)$ as a function of μ . Its value changes at the values of μ corresponding to the branch points shown in (b). For $\mu = -0.25$, $\chi(\mu)$ takes the half-integer value -1.5 , which results in a single spectral loop as shown in (c). Consequently, no global branch choice can recover the closed spectral loops, resulting in a loss of convexity and integer-quantized slopes for the individual resolved functions. Reproduced from Fig. 6 of Ref. [191]. ©2025 by the American Physical Society.

Figure 6.6 illustrates why the naive band-resolved Ronkin functions fail with numerical results. The EBBs are defined in Eq. (6.2.2), and their parameters are set as in Fig. 6.5. Figure 6.6(a) plots the total and the band-resolved Ronkin functions for $E = 0.5 + 0.3i$. The black, red and blue lines represent R_E , $R_E^{[+]}$ and $R_E^{[-]}$, respectively. The EBBs include the square root of a complex number and, thus, depend on the branch cut. We consider here two choices of branch cuts: a half-line from the origin to infinity at angles of $5\pi/7$ (red and blue) and 0 (light red and light blue). The corresponding branch cuts on (μ, k) -space are plotted in Fig. 6.6(b). The partial functions $R_E^{[\pm]}$ have different minimal values, and their shapes strongly depend on the choice of branch cuts, although the total Ronkin function is cut-independent and convex.

The spectrum with μ fixed at -0.25 forms a closed curve [Fig. 6.6(c)], resulting in the integer-quantized derivative in the total Ronkin function. However, since the EBBs are non-analytic in the vicinity of the branch cuts, this non-analyticity causes the spectral curve to split into several open arcs. Consequently, $R_E^{[\pm]}$ loses both its convexity and piecewise linearity, as demonstrated in Fig. 6.6(a). As $\chi(\mu)$ takes half-integer values [Fig. 6.6(d)], it becomes impossible to partition the spectrum into closed curves regardless of the branch cut choice. Consequently, the band-resolved Ronkin functions cannot recover their convexity and piecewise linearity.

These results confirm that any framework based on individual EBBs is fundamentally limited, necessitating the symmetry-based or WHF-based decomposition that restores the necessary mathematical rigor.

6.2.2 Analysis via symmetry-decomposed Ronkin function

In this subsection, we perform a detailed spectral analysis using the symmetry-decomposed Ronkin functions to elucidate how the NHSE behaves in the presence of $\text{TRS}^\dagger$. As established in Chapter 5, this decomposition provides a framework to partition the total Ronkin function into symmetry-protected sectors, $R_\pm(\mu)$, based on the internal structure of the Hamiltonian.

Our objective here is to verify that these decomposed Ronkin functions correctly capture the OBC potential and the localization properties of each sector. In the subsequent section, we demonstrate the consistency between this symmetry-based intuition and the rigorous analytical results obtained from the WHF-based formulation.

We show the total Ronkin functions and the symmetry-decomposed Ronkin functions in Fig. 6.7. The reference points are set as $E = -0.3 + 0.3i$ [(a)], $E = 0.5 + 0.265i$ [(b)], and $E = 0.8$ [(c)]. In each figure, the black, red, and blue lines represent R_E , $R_E^{(+)}$, and $R_E^{(-)}$, respectively. Due to $\text{TRS}^\dagger$ constraints, the total Ronkin function is symmetric around $\mu = 0$. The symmetry-decomposed Ronkin functions $R_E^{(\pm)}$ inherit the mathematical properties of the total function, i.e., they are convex, piece-wise linear, and have integer-quantized derivatives. Furthermore, they satisfy the symmetry relation in Eq. (5.4.4). The red and blue dashed lines

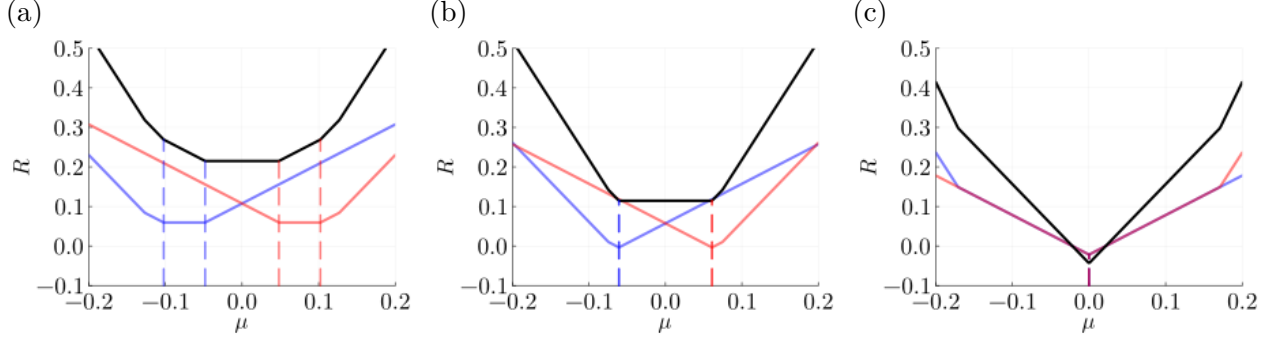


Figure 6.7: Total and symmetry-decomposed Ronkin functions evaluated at (a) $E = -0.3 + 0.3i$, (b) $E = 0.5 + 0.265i$, and (c) $E = 0.8$: The Hamiltonian and parameters are identical to those in Fig. 6.5. The black, red, and blue lines represent R_E , $R_E^{(+)}$ and $R_E^{(-)}$, respectively. The red and blue dashed lines indicate the central holes of $R_E^{(\pm)}$, which correspond to the unit-slope holes of R_E . The total Ronkin function is symmetric, and the symmetry-decomposed Ronkin functions $R^{(\pm)}$ satisfy the symmetry relation in Eq. (5.4.4). For points outside the OBC spectrum [(a)], both the central holes of $R_E^{(\pm)}$ and the unit-slope hole of R_E open. For points inside the OBC spectrum [(b) and (c)], the unit-slope holes of R_E vanish, accompanied by the closing of the central holes in the symmetry-decomposed Ronkin functions. Reproduced from Fig. 2 of Ref. [191]. ©2025 by the American Physical Society.

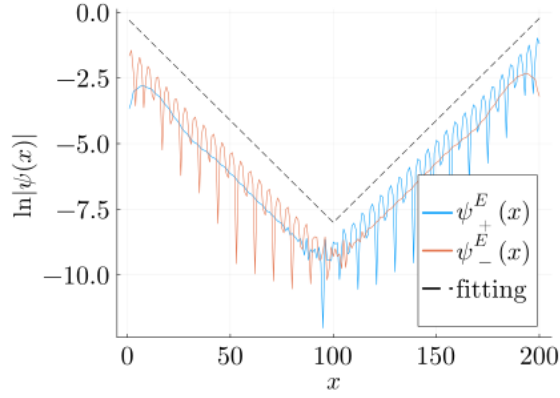


Figure 6.8: Fitting of wavefunctions using $\mu^* = Q_E^{-1}(1)$ for a system with $N = 200$. The reference point is set to be $E = -0.26 + 0.43i$, which is inside the spectrum. The blue and orange lines represent the +1 and -1 sectors of the Pauli matrix Z , respectively. Due to the symmetry, the wavefunction is exponentially localized at both boundaries, with a localization length of $1/\mu^*$. Reproduced from Fig. 3 of Ref. [191]. ©2025 by the American Physical Society.

indicate the unit-slope holes of the total Ronkin functions inherited by the central holes of the symmetry-decomposed Ronkin functions. For points outside the OBC spectrum [Fig. 6.7(a)], the central holes in the symmetry-decomposed Ronkin functions persist, which results in the opening of the unit-slope holes in the total Ronkin function. For reference points located outside the OBC spectrum [Fig. 6.7(a)], the symmetry-decomposed Ronkin functions $R_E^{(\pm)}$ possess central holes, which collectively manifest as the unit-slope holes in the total Ronkin function R_E . In contrast, for points inside the OBC spectrum [Figs. 6.7(b) and (c)], these unit-slope holes vanish, signaling the closing of the central holes in the decomposed functions. At these spectral points, the red and blue dashed lines indicate $\pm\mu^*$, where μ^* represents the inverse localization length. This behavior provides a clear numerical verification consistent with the GBZ criterion for class AII[†], as established in Eq. (2.3.20).

Figure 6.8 compares an eigenstate of the finite-size system ($N = 200$) with an exponentially localized wavefunction, using $\mu^* = Q_E^{-1}(1)$. We set the reference point as $E = -0.26 + 0.43i$, which is located within the spectrum. The blue [orange] line represents the $+1[-1]$ sector of the Pauli matrix Z . Due to the symmetry, the wavefunction is localized at both boundaries and exhibits linear behavior on a logarithmic scale, indicating exponential localization of the eigenstates with a localization length of $1/\mu^*$.

While the symmetry-decomposed Ronkin functions discussed here effectively determine the spectral boundaries and localization lengths, a complete description of the spectral distribution is more naturally formulated within the full machinery of the WHF-based framework. To avoid redundancy and to highlight the comprehensive power of the Wiener-Hopf approach, we will present the quantitative analysis of the DOS alongside the WHF-based results in the subsequent section. This combined presentation will demonstrate that our formulation not only identifies the skin effect's spatial profile but also reconstructs the entire spectral measure of the multiband system.

6.2.3 WHF framework

In this parameter regime, the system is characterized by the $\mathbb{Z}_2$ topological invariant $\nu[\tau]$, where $\tau = (E - h)U_T$, Eq. (4.2.3) [60]. As established in Chapter 5, $\nu[\tau]$ is equivalent to the parity of the sum of positive partial indices.

Figure 6.9 shows the positive partial index κ of $\tau = (E - h)U_T$ across the complex energy plane. Blue and red points indicate the PBC and OBC spectra, respectively, obtained by diagonalizing the Hamiltonian with 400 sites. This model exhibits two distinct phases: $\kappa = 0$ (white) and $\kappa = 1$ (shaded). The shaded $\kappa = 1$ region coincides with the topologically nontrivial region identified by the $\mathbb{Z}_2$ topological invariant [191]. This confirms that the parity of κ provides a consistent and physically equivalent characterization of the $\mathbb{Z}_2$ topology in class AII[†] systems.

Having confirmed that κ serves as the correct topological index, we now verify the central

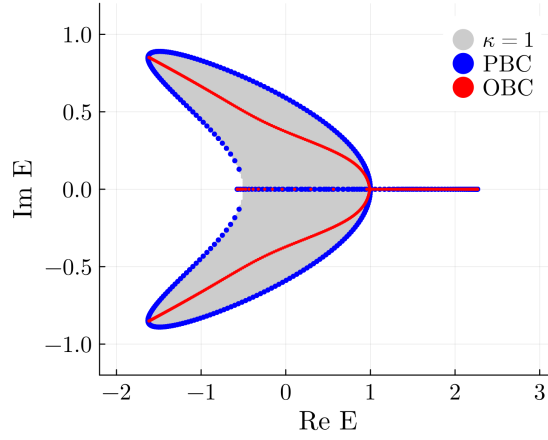


Figure 6.9: Validation of the partial index κ as the topological invariant. The Hamiltonian is defined in Eq. (6.2.1), with parameters $t_1 = 0.3$, $t_2 = 0.8$, $g_1 = 0.5$, $\Delta_1 = 0.3$, and $\Delta_2 = 0.2$. Blue and red points indicate the PBC and OBC spectra, respectively ($N = 400$). The shaded region denotes the domain where the WHF-derived partial index is $\kappa = 1$. This region coincides with the $\mathbb{Z}_2$ -nontrivial phase ($\nu[\tau] = 1$) obtained independently, confirming that the parity of κ correctly captures the $\mathbb{Z}_2$ topology.

mechanism of our WHF formulation [Eq. (5.5.16)], which defines the OBC potential through the symplectic form of the WHF, $\tau = \tau_+ S_\kappa J[\tau_+ S_\kappa]^\top$, as given in Eq. (5.5.4).

Figure 6.10 illustrates this test for three representative energies E . The top row [Fig. 6.10(a-c)] presents the WHF-based decomposed Ronkin functions $R_\tau^{(\pm)}$, while the bottom row [Fig. 6.10(d-f)] shows the corresponding partial indices of $[\tau_+^{\text{symp}}]_\mu = [\tau_+ S_\kappa]_\mu$. The reference points are chosen as $E = -0.5 + 0.3i$ ($\kappa = 1$) [(a,d)], $E = 0.5 + 0.265i$ ($\kappa = 1$) [(b,e)], and $E = 1.2 + 0.2i$ ($\kappa = 0$) [(c,f)]. The top panels (a-c) demonstrate that the WHF-derived $R_\tau^{(\pm)}$ perfectly reproduce the phenomenological decomposed Ronkin functions [Eqs. (5.4.5)]: they are convex, piecewise linear, and satisfy the symmetry relation $R_\tau^{(+)}(\mu) = R_\tau^{(-)}(-\mu)$ [Eq. (5.4.4)]. The bottom panels [Fig. 6.10(d-f)] display the partial indices $\mathcal{K}[[\tau_+^{\text{symp}}]_\mu] = (\kappa_1^{(\mu)}, \kappa_2^{(\mu)})$ of the factor $[\tau_+^{\text{symp}}]_\mu$. Their transposed counterparts follow $\mathcal{K}[[\tau_-^{\text{symp}}]_\mu] = (-\kappa_2^{(-\mu)}, -\kappa_1^{(-\mu)})$. Each index changes when μ crosses a root μ_j where τ closes its gap. The sum $\kappa_1^{(\mu)} + \kappa_2^{(\mu)}$ corresponds to the winding number of $[\tau_+^{\text{symp}}]_\mu$, which in turn equals the derivative of $R_\tau^{(+)}$.

Strikingly, we find that the partial indices $\kappa_{1,2}^{(\mu)}$ vanish simultaneously within the finite interval $\mu_1 \leq \mu \leq \mu_2$; thus, $\mathcal{K}^*[\tau_+^{\text{symp}}] = (0, 0)$. The emergence of this interval provides direct numerical evidence that the WHF framework is fully consistent with the non-Bloch band theory for class AIII[†] systems: it coincides precisely with the optimization window $[\mu_1, \mu_2]$ dictated by the GBZ condition [147]. From the perspective of the formulation based on the WHF, the condition $\mathcal{K}^*[\tau_+^{\text{symp}}] = (0, 0)$ precisely marks the regime in which the generalized Szegő limit theorem [Eq. (5.5.14)] holds exactly, without requiring any additional correction

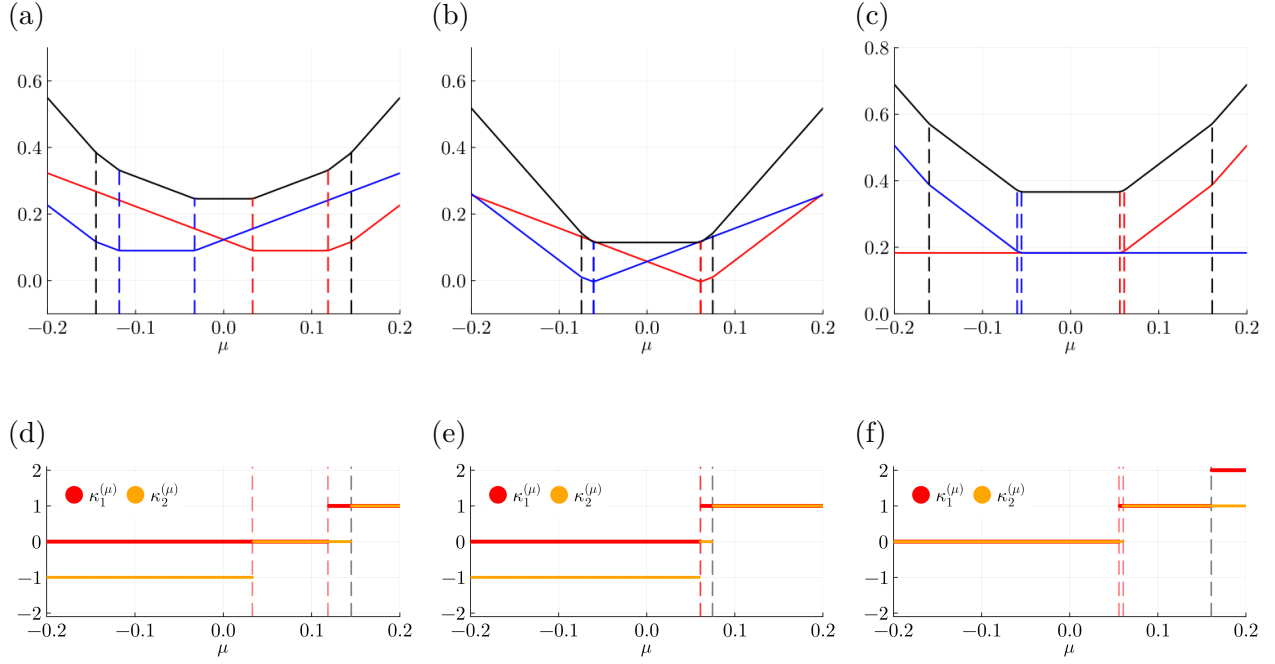


Figure 6.10: WHF-based decomposed Ronkin functions (top row, a-c) and partial indices of $[\tau_+^{\text{symp}}]_\mu$, $\mathcal{K}[[\tau_+^{\text{symp}}]_\mu] = (\kappa_1^{(\mu)}, \kappa_2^{(\mu)})$ (bottom row, d-f). The panels correspond to three reference energies: (a, d) $E = -0.5 + 0.3i$ ($\kappa = 1$), (b, e) $E = 0.5 + 0.265i$ ($\kappa = 1$), and (c, f) $E = 1.2 + 0.2i$ ($\kappa = 0$). (a-c): decomposed Ronkin functions $R_\tau^{(+)}$ (red), $R_\tau^{(-)}$ (blue), and total R_τ (black). Vertical dashed lines mark the root positions $\mu = \pm\mu_j$, where the colored lines correspond to the GBZ condition $\mu_1 = \mu_2$. Due to $\text{TRS}^\dagger$, R_τ is even in μ , and $R_\tau^{(\pm)}$ obey the symmetry relation in Eq. (5.4.4). (d-f): partial indices $\kappa_1^{(\mu)}$ (red) and $\kappa_2^{(\mu)}$ (orange), which change whenever μ crosses a root μ_j where σ_μ closes its gap. Their sum, $\kappa_1^{(\mu)} + \kappa_2^{(\mu)}$, equals the derivative of $R_\tau^{(+)}$. The WHF-based decomposition reproduces the phenomenological symmetry-decomposed Ronkin functions for $\kappa = 1$ and symmetrically decomposed R_τ for $\kappa = 0$.

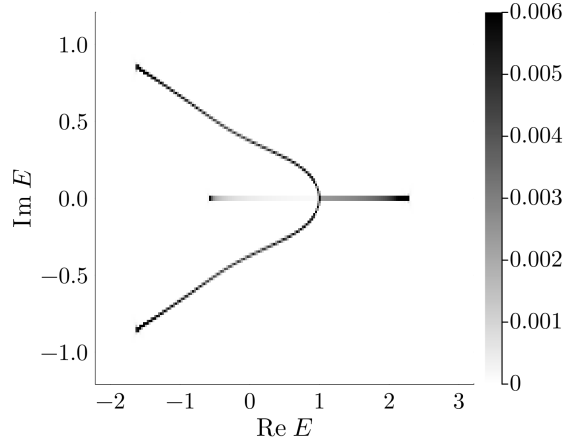


Figure 6.11: DOS multiplied by $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 4 \times 10^{-4}$, calculated by optimizing the WHF-based decomposed Ronkin functions defined in Eq (5.5.6). The Hamiltonian and parameters are equivalent to those in Fig. 6.9. The WHF-based formulation reproduces the OBC spectrum shown in Fig. 6.9.

terms.

Finally, as a comprehensive validation of our framework, we compute the OBC spectral DOS by evaluating the OBC potential through the WHF-based optimization [Eq. (5.5.6)] across the entire complex energy plane. The resulting distribution is shown in Fig. 6.11, where each point represents an energy cell of size $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 4 \times 10^{-4}$, used for discretizing the Hessian in Eq. (3.1.2). The WHF-based decomposition accurately reproduces the OBC spectrum, in full agreement with Fig. 6.9 and consistent with the results obtained from the symmetry-decomposed Ronkin functions. This correspondence confirms both the validity and the practical applicability of our method across the $\kappa = 0$ and $\kappa = 1$ regimes.

6.3 Analysis of anomalous phase and instability ($\kappa = 2$)

To demonstrate that Eq. (5.5.16) also holds in parameter regimes with partial index $\kappa = 2$, we analyze the same Hamiltonian using parameters $t_1 = 0.3$, $t_2 = 0.5$, $g_2 = 1.0$, $\Delta_1 = 0.0$, and $\Delta_2 = 0.9$.

Figure 6.12 shows the spectrum and partial index in the complex energy plane. The blue and red points denote the PBC and OBC spectra, respectively, obtained using $N = 200$ (PBC) and $N = 50$ (OBC) sites. The partial index κ of $\tau = (E - h)U_T$ takes the value 1 in the light-gray region, 2 in the dark-gray region, and 0 elsewhere. Because the numerical diagonalization of this two-band Hamiltonian under OBC becomes increasingly demanding with system size, we restrict the calculation to 50 sites for the OBC case, which still captures the essential spectral features.

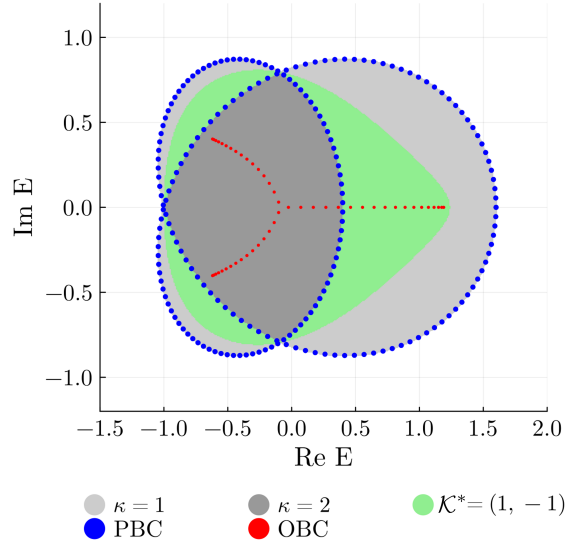


Figure 6.12: Map of the OBC and PBC spectra with WHF-derived partial indices of the regime, including a $\kappa = 2$ phase. Blue and red points denote the PBC ($N = 200$) and OBC ($N = 50$) spectra, respectively. Model parameters are $t_1 = 0.3$, $t_2 = 0.5$, $g_2 = 1.0$, $\Delta_1 = 0.0$, and $\Delta_2 = 0.9$. The shaded regions indicate the WHF-derived partial index: $\kappa = 1$ (light gray) and, notably, a $\kappa = 2$ phase (dark gray). The unshaded (white) region corresponds to the trivial $\kappa = 0$ phase. The green overlay marks a parameter domain where the partial indices do not vanish simultaneously, leading to nonvanishing residual partial indices $\mathcal{K}^* = (+1, -1)$.

Crucially, in the $\kappa = 2$ region (dark gray), the $\mathbb{Z}_2$ topological invariant [Eq. (4.2.3)] yields $\nu[\tau] = 0$, implying topological triviality from the $\mathbb{Z}_2$ viewpoint. Yet, this regime reveals a previously unexplored structure: despite its $\mathbb{Z}_2$ -trivial character, the partial indices of the symbol remain nonzero, indicating the presence of exponentially localized states under OBC, as typically associated with the NHSE. This sets the $\kappa = 2$ phase apart from both the conventional trivial ($\kappa = 0$) and topological nontrivial ($\kappa = 1$) cases. Further evidence for the presence of a skin effect in this regime is provided by the OBC spectrum, which differs markedly from the PBC spectrum and shows pronounced boundary sensitivity. The combination of nonvanishing partial indices (signaling localization) and the pronounced boundary sensitivity of the OBC spectrum relative to the PBC spectrum provides clear evidence that a skin effect occurs in the $\kappa = 2$ regime. This skin effect is not protected by $\text{TRS}^\dagger$ symmetry but instead originates from the nontrivial $\kappa = 2$ winding. The resulting $\kappa = 2$ skin modes are fragile: a small $\text{TRS}^\dagger$ -preserving perturbation (e.g., $\Delta_1 = 0.01$) can remove them entirely without closing the point gap, confirming that these skin modes lack symmetry protection and arise solely from the winding structure. For further details, see Appendix A.

Having established the existence of the $\kappa = 2$ phase, we now examine the optimization of

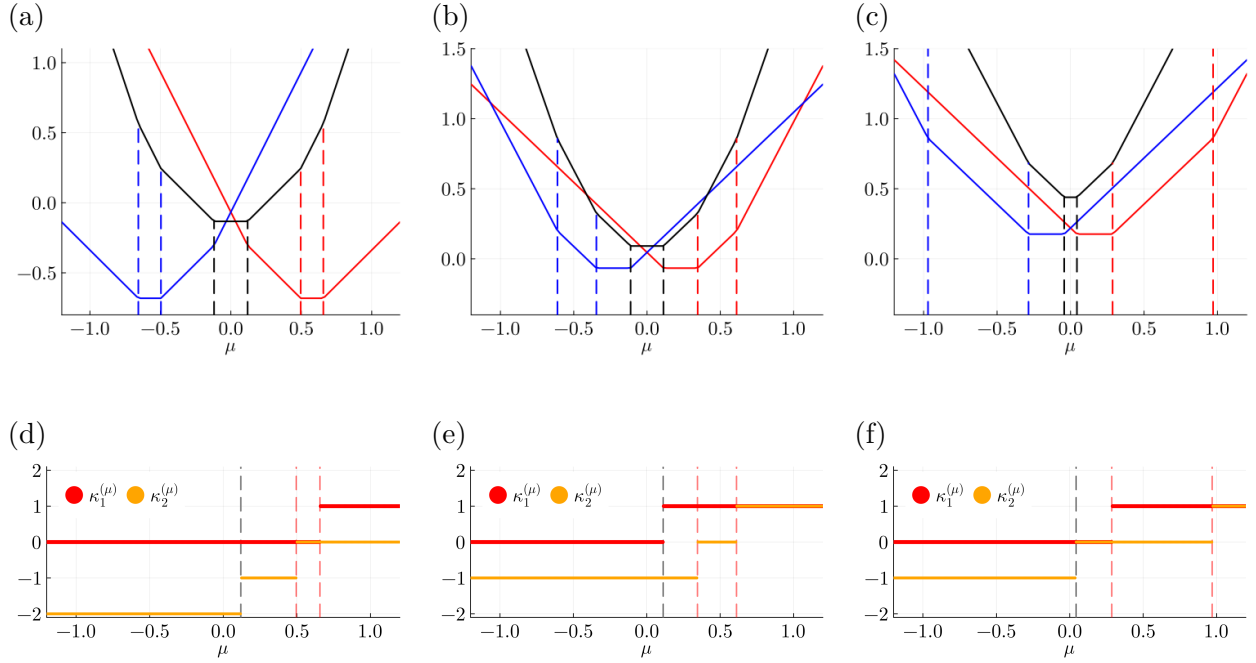


Figure 6.13: Partial indices of $[\tau_+^{\text{symp}}]_\mu$ for three representative energies: (a) $E = -0.5 + 0.3i$ with $\kappa = 2$, (b) $E = 0.6 + 0.2i$ with $\kappa = 1$, and (c) $E = 0.9 + 0.7i$ with $\kappa = 1$. Red and blue vertical lines correspond to the GBZ condition $\mu_2 = \mu_3$. Outside the green region in Fig. 6.12 [(a), (c)], $\kappa_1^{(\mu)}$ and $\kappa_2^{(\mu)}$ vanish simultaneously for an appropriate μ , resulting in vanishing residual partial indices. Inside the green region [(b)], however, the partial indices never vanish simultaneously, indicating the breakdown of the simple optimization formalism due to nonvanishing residual partial indices.

the factor $[\tau_+^{\text{symp}}]_\mu$ [from Eq. (5.5.4)], which plays a central role in decomposing the Ronkin function. Our WHF analysis reveals a complication: a parameter region where the partial indices of $[\tau_+^{\text{symp}}]_\mu$ do not vanish simultaneously, leaving residual partial indices $\mathcal{K}^*[\tau_+^{\text{symp}}] \neq (0, 0)$ even after finding the optimal μ , as indicated by the green area in Fig. 6.12. Within this green region, the optimization yields $\mathcal{K}^*[\tau_+^{\text{symp}}] = (1, -1)$ [or $(-1, 1)$], demonstrating that a straightforward optimization of the decomposed Ronkin function is insufficient and that a correction for the nonvanishing partial indices is required, analogous to the multiband class A case.

Figure 6.13 illustrates this behavior. The top panels [Fig. 6.13(a-c)] display the WHF-based decomposed Ronkin functions, while the bottom panels [Fig. 6.13(d-f)] show the μ -dependence of the partial indices, $\mathcal{K}[[\tau_+^{\text{symp}}]_\mu] = (\kappa_1^{(\mu)}, \kappa_2^{(\mu)})$. For reference points outside the green region in Fig. 6.12 [Figs. 6.13(a, c)], the decomposed Ronkin function $R_\tau^{(+)}$ (red line) is correctly minimized within the interval $[\mu_2, \mu_3]$, and the partial indices $\mathcal{K}[[\tau_+^{\text{symp}}]_\mu]$ vanish simultaneously [Figs. 6.13(d, f)], confirming that the simple optimization formalism

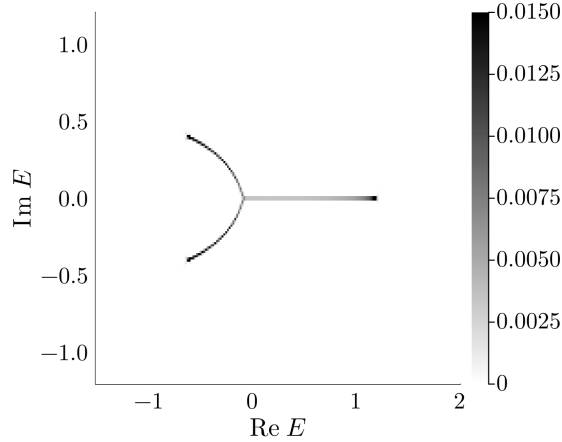


Figure 6.14: Density of states (DOS) multiplied by the surface element $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 3 \times 10^{-4}$. The Hamiltonian and parameters are identical to those in Fig. 6.12. The OBC spectral potential is obtained by optimizing the WHF-based decomposed Ronkin functions [Eq. (5.5.6)] together with the correction term accounting for nonvanishing partial indices [Eq. (5.5.15)]. The resulting WHF-based formulation accurately reproduces the OBC spectrum shown in Fig. 6.12.

remains valid. In contrast, for energies inside the green region [Figs. 6.13(b, e)], no interval exists where the partial indices vanish simultaneously, necessitating the correction term in Eq. (5.3.8).

The resulting OBC DOS, multiplied by the surface element $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 3 \times 10^{-4}$, is plotted in Fig. 6.14. The OBC potential is obtained by optimizing the decomposed Ronkin functions defined in Eq. (5.5.6), together with the correction in Eq. (5.5.15). By combining these two relations, the WHF-based formulation successfully reproduces the OBC spectrum shown in Fig. 6.12.

Our numerical analysis reveals that the conventional GBZ condition $\mu_1 = \mu_2$ [147] is replaced by $\mu_2 = \mu_3$. This modified GBZ condition is consistent with the WHF decomposed Ronkin functions and directly dictates the localization length of the eigenmodes. We interpret this behavior as a manifestation of the $\kappa = 2$ phase, which, as discussed in Appendix A, is generally fragile.

To verify the accuracy of the localization length, Fig. 6.15 shows fits of the OBC eigenstates ($N = 50$) using the functional form $e^{-\mu_2(x-25)} + e^{\mu_2(x-25)}$, where the values of $\mu_2 (= \mu_3)$ are obtained from the WHF-based optimization process. The red and blue lines represent the wave functions of the $+1$ and -1 sectors of the Pauli matrix Z , respectively. For $E = 1.02 + 0.06i$ [Fig. 6.15(b)], the partial index is $\kappa = 1$, and the corresponding eigenmode exhibits a $\text{TRS}^\dagger$ -protected $\mathbb{Z}_2$ skin mode. In contrast, for $E = -0.13 + 0.08i$ [Fig. 6.15(a)], the partial index is $\kappa = 2$, and the corresponding eigenmode realizes a non-protected skin

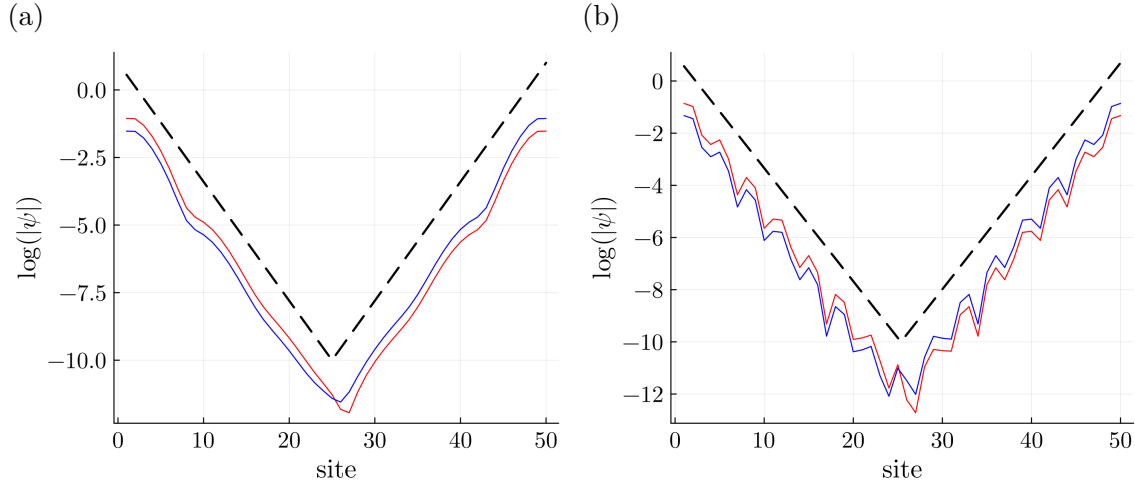


Figure 6.15: Fitting of wave functions for (a) $\kappa = 1$ and (b) $\kappa = 2$. The chain length of the OBC system is $N = 50$ sites. The corresponding eigenenergies are (a) $E = 1.02 + 0.06i$ and (b) $E = -0.13 + 0.08i$. The red and blue lines represent the wave functions of the $+1$ and -1 sectors of the Pauli matrix Z , respectively. The black dotted line shows the fitted profile using the localization lengths determined from the optimization, corresponding to $\mu_2 = \mu_3$.

mode. The fitting form, based on the optimized μ_2 , accurately reproduces the spatial profiles in both regimes.

Ultimately, these results demonstrate that even in the $\kappa = 2$ regime, our WHF-based formulation provides a precise description of the multiband NHSE, including its non-protected and fragile manifestations. Our framework thus establishes a comprehensive foundation for analyzing the NHSE in one-dimensional multiband systems, encompassing not only symmetry-protected topological phases but also higher-order winding-driven phenomena.

Chapter 7

Summary and outlook

In this thesis, we have established a comprehensive framework based on the Wiener-Hopf factorization (WHF) to generalize the Amoeba formulation to multiband non-Hermitian systems. Our work bridges the gap between the mathematical theory of Toeplitz operators and the physical understanding of the non-Hermitian skin effect (NHSE).

In Chapters 2 and 3, we reviewed the essential theoretical foundations: the non-Bloch band theory and the original Amoeba formulation. We highlighted that the generalized Brillouin zone (GBZ) condition in the former is equivalent to the closing of the central hole in the latter. However, we noted that the conventional Amoeba formulation, which relies on the classical Szegő limit theorem, remains fundamentally restricted to single-band systems.

In Chapter 4, we established the mathematical machinery required for the multiband extension. By employing the Wiener-Hopf factorization, we generalized the conventional winding number to the partial indices of the symbol, which provide a more granular topological characterization corresponding to the edge modes in Hermitian systems. Furthermore, we demonstrated that the potential deviation in topologically nontrivial systems can be quantitatively evaluated through the localization lengths of these modes, providing a rigorous analytical link between topology and spectral distribution.

Drawing upon these mathematical foundations, Chapter 5 presented the core contribution of this thesis: the modified Amoeba formulation for multiband systems. We reformulated the generalized Szegő limit theorem using WHF and Hermitian doubling, where the vanishing total winding number condition was successfully generalized to the vanishing of residual partial indices. Additionally, we extended this formalism to the $\mathbb{Z}_2$ skin effect protected by $\text{TRS}^\dagger$. Our modified approach allows for the decomposition of the Ronkin function into symmetry-protected sectors, effectively separating Kramers pairs and ensuring the mathematical consistency of the spectral potential.

Chapter 6 provided numerical validation of our theoretical framework. Through the analysis of various multiband models, including those in class A and class AII † , we confirmed that our WHF-based formulation accurately reproduces the OBC spectra, densities of states,

and localization lengths, even in "anomalous" regimes where conventional methods fail.

In conclusion, our results significantly extend the mathematical foundations of the Amoeba formulation beyond the single-band regime. By establishing a symmetry-resilient and mathematically rigorous framework, this work provides a powerful tool for studying the NHSE in one-dimensional and multiband non-Hermitian systems, paving the way for further exploration of non-Bloch physics in complex quantum and classical media.

Despite the comprehensive framework established in this thesis, several open questions and intriguing directions remain for future exploration:

- **Scale-free localization and the critical skin effect (CSE):** While our formulation primarily addresses the thermodynamic limit ($N \rightarrow \infty$), the critical skin effect is characterized by scale-free localization where the localization length scales with the system size N . Investigating whether the WHF-based Amoeba formulation can be extended to incorporate such finite-size effects is a crucial next step.
- **Generalization to higher dimensions:** The Wiener-Hopf factorization is currently well-established primarily for one-dimensional systems (Toeplitz operators). Generalizing this mathematical machinery to higher-dimensional operators (e.g., multivariable Wiener-Hopf factorization) is essential for a rigorous description of the NHSE in two or three dimensions without relying on specific symmetries.
- **Formulation of the Geometry-Dependent Skin Effect (GDSE):** The GDSE can often be understood by treating spatial dimensions as internal degrees of freedom. A promising direction is to examine whether our residual partial indices can serve as the fundamental topological invariants to fully characterize the GDSE across different geometries.
- **Algebraic localization and dimensional transmutation:** Our current framework assumes exponential localization of skin modes. Extending the Amoeba formulation to cases with algebraic localization remains an open challenge. Furthermore, when dimensional transmutation occurs, and the effective dimension of momentum space shrinks, it is of great interest to determine how the corresponding Szegő limit theorem should be modified to describe the resulting spectral measure.

These directions promise to reveal new connections between topology, geometry, and localization phenomena in non-Hermitian systems, with potential applications ranging from photonics to active matter.

Appendix A

Mathematical foundations of the Amoeba theory

In this appendix, we establish the mathematical foundations underlying the Amoeba formulation. First, we summarize the asymptotic properties of Toeplitz matrices and provide a heuristic proof of the Szegő limit theorem based on the trace-log formula. These results justify the spectral potential correspondence introduced in Chapter 3. Furthermore, we review the properties of plurisubharmonic functions and prove a key theorem: the average of a plurisubharmonic function over a torus yields a convex function. This theorem guarantees the convexity of the Ronkin function for holomorphic symbols, a property that is crucial for the optimization problem in the generalized Szegő limit theorem.

A.1 Asymptotic properties of Toeplitz matrices

In this section, we summarize the asymptotic properties of Toeplitz matrices that justify the approximations used in the subsequent section. Detailed proofs and error evaluations can be found in Ref. [173].

Let σ_1 and σ_2 be symbols, and $\mathcal{T}_N[\sigma_1]$ and $\mathcal{T}_N[\sigma_2]$ are corresponding Toeplitz matrices defined on the d -dimensional domain Ω , where N is the linear scale of Ω .

We first evaluate the error associated with the product of Toeplitz matrices,

$$\begin{aligned}\Delta\mathcal{T}_N[\sigma_1, \sigma_2]_{x,y} &= \mathcal{T}_N[\sigma_1\sigma_2]_{x,y} - \sum_{z \in \Omega} \mathcal{T}_N[\sigma_1]_{x,z} \mathcal{T}_N[\sigma_2]_{z,y} \\ &= \sum_{z \notin \Omega} \mathcal{T}_N[\sigma_1]_{x,z} \mathcal{T}_N[\sigma_2]_{z,y}.\end{aligned}\tag{A.1.1}$$

Intuitively, by explicitly writing out the matrix elements, one can observe that the components of $\Delta\mathcal{T}_N$ are non-zero only in the vicinity of the boundaries. Consequently, only $O(N^{d-1})$ elements contribute to the error, making it negligible compared to the total N^d sites in the

thermodynamic limit. Therefore, we can approximate $\mathcal{T}_N[\sigma_1]\mathcal{T}_N[\sigma_2]$ with $\mathcal{T}_N[\sigma_1\sigma_2]$ in the thermodynamic limit.

More rigorously, the norm of this error can be bounded by the singular values of the matrices (specifically the Schatten p -norms), and it has been shown that the error is asymptotically small in the thermodynamic limit [173].

While setting $\sigma_1 = \sigma$ and $\sigma_2 = \sigma^{-1}$ suggests a possible evaluation of the inverse matrix $\mathcal{T}_N[\sigma]^{-1} \approx \mathcal{T}_N[\sigma^{-1}]$, this approximation requires careful justification. Consider a Hermitian symbol σ that is gapped on the unit circle ($|\beta| = 1$). In this case, the Toeplitz matrix $\mathcal{T}_N[\sigma^{-1}]$ is well-defined and remains gapped. However, it does not necessarily coincide with $\mathcal{T}_N[\sigma]^{-1}$ in the thermodynamic limit. The failure of this approximation is fundamentally rooted in the presence of topological zero modes. Crucially, in non-Hermitian systems, such failure can occur even when all eigenvalues are far from zero. In such cases, the singular values of $\mathcal{T}_N[\sigma]$ approach zero (or become exponentially small with respect to N), signaling that the operator becomes effectively non-invertible in the thermodynamic limit. Consequently, the validity of the approximation $\mathcal{T}_N[\sigma]^{-1} \approx \mathcal{T}_N[\sigma^{-1}]$ can be assessed by examining the singular values of $\mathcal{T}_N[\sigma]$. This approach is consistent with the stability analysis using the Schatten norm of the operators.

Conversely, if the symbol σ is topologically trivial (e.g., the winding number vanishes), it can be adiabatically connected to the identity matrix without closing the spectral gap. In this regime, the singular values remain bounded away from zero, and the correspondence between the Toeplitz inverse and the inverse symbol is rigorously justified. This stability ensures that the trace-log expansion and the subsequent Szegő limit theorem can be safely applied.

A.2 Potential correspondence via Szegő limit theorem

As an alternative approach, we can understand the correspondence between the OBC and PBC spectral potentials heuristically using the trace-log formula [173]. We assume that the Toeplitz matrix $\mathcal{T}_N[\sigma]$ can be adiabatically connected to the identity matrix while maintaining its invertibility. This assumption is satisfied when the symbol σ belongs to the topologically trivial class (i.e., $W = 0$), ensuring that the spectral gap remains open during the deformation.

Using the identity $\ln \det A = \text{tr} \ln A$, the spectral potential is rewritten as

$$\frac{1}{N^d} \ln \det \mathcal{T}_N[\sigma] = \frac{1}{N^d} \text{tr} \ln \mathcal{T}_N[\sigma]. \quad (\text{A.2.1})$$

To evaluate the right-hand side, we introduce a parameter $\lambda \in [0, 1]$ to adiabatically connect the identity matrix to σ (e.g., $\sigma_\lambda = (1 - \lambda)\mathbb{I} + \lambda\sigma$). Using the cyclic property of the trace

and the derivative with respect to λ , we obtain

$$\begin{aligned}\frac{1}{N^d} \operatorname{tr} \ln \mathcal{T}_N[\sigma] &= \frac{1}{N^d} \int_0^1 d\lambda \frac{d}{d\lambda} \operatorname{tr} \ln \mathcal{T}_N[\sigma_\lambda] \\ &= \frac{1}{N^d} \int_0^1 d\lambda \operatorname{tr} \left(\mathcal{T}_N[\sigma_\lambda]^{-1} \partial_\lambda \mathcal{T}_N[\sigma_\lambda] \right).\end{aligned}\tag{A.2.2}$$

In the thermodynamic limit $N \rightarrow \infty$, the algebra of Toeplitz matrices becomes closed. Specifically, we can approximate the inverse and the product of Toeplitz matrices as $\mathcal{T}_N[A]^{-1} \approx \mathcal{T}_N[A^{-1}]$ and $\mathcal{T}_N[A]\mathcal{T}_N[B] \approx \mathcal{T}_N[AB]$, provided that the symbol is well-behaved. Applying these approximations to Eq. (A.2.2), we get

$$\begin{aligned}\frac{1}{N^d} \operatorname{tr} \ln \mathcal{T}_N[\sigma] &\approx \frac{1}{N^d} \int_0^1 d\lambda \operatorname{tr} \left(\mathcal{T}_N[\sigma_\lambda^{-1}] \mathcal{T}_N[\partial_\lambda \sigma_\lambda] \right) \\ &\approx \frac{1}{N^d} \int_0^1 d\lambda \operatorname{tr} \mathcal{T}_N[\sigma_\lambda^{-1} \partial_\lambda \sigma_\lambda].\end{aligned}\tag{A.2.3}$$

Finally, by representing the Toeplitz matrix in the basis of plane waves via the Fourier transform, the trace per unit cell can be evaluated in momentum space. In the thermodynamic limit, this trace converges to the integral of the symbol over the Brillouin zone,

$$\lim_{N \rightarrow \infty} \frac{1}{N^d} \operatorname{tr} \mathcal{T}_N[f] = \int \frac{d^d k}{(2\pi)^d} \operatorname{tr} f(k),\tag{A.2.4}$$

which yields the Szegő limit theorem:

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{1}{N^d} \operatorname{tr} \ln \mathcal{T}_N[\sigma] &= \int_0^1 d\lambda \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \operatorname{tr} \left(\sigma_\lambda(e^{ik})^{-1} \partial_\lambda \sigma_\lambda(e^{ik}) \right) \\ &= \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \operatorname{tr} \ln \sigma(e^{ik}) \\ &= \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln \det \sigma(e^{ik}).\end{aligned}\tag{A.2.5}$$

Taking the real part recovers the PBC spectral potential.

It is important to note that the approximation $\mathcal{T}_N[\sigma]^{-1} \approx \mathcal{T}_N[\sigma^{-1}]$ fails when the winding number $W[\sigma]$ is non-zero, as the inverse of the Toeplitz matrix deviates from the bulk symbol due to the presence of vanishing singular values.

A.3 Subharmonic and plurisubharmonic functions

In this section, we explain the mathematical properties of plurisubharmonic functions, which are a generalization of the subharmonic functions to several complex variables. The discussion in this section is based on Ref. [188].

A real-valued function u of a complex variable z is subharmonic, if, for any point z and for all $r > 0$, it satisfies the inequality:

$$u(z) \leq \int_0^{2\pi} \frac{d\theta}{2\pi} u(z + re^{i\theta}). \quad (\text{A.3.1})$$

This definition implies that the average value of the function on a circle is always greater than or equal to the value at its center. Obviously, the sum of subharmonic functions is also subharmonic.

This integral condition implies that the complex Laplacian of u is non-negative. To analyze the local convexity, it is convenient to use the Wirtinger derivatives:

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (\text{A.3.2})$$

Using these operators, we consider the Taylor expansion of u with respect to z and z^* :

$$\begin{aligned} u(z + w) &= u(z) + \frac{\partial u}{\partial z} w + \frac{\partial u}{\partial z^*} w^* \\ &\quad + \frac{1}{2} \left(\frac{\partial^2 u}{\partial z^2} w^2 + 2 \frac{\partial^2 u}{\partial z \partial z^*} |w|^2 + \frac{\partial^2 u}{\partial z^{*2}} w^{*2} \right) + o(|w|^2). \end{aligned} \quad (\text{A.3.3})$$

Letting $w = re^{i\theta}$, we average this expansion over the circle $\theta \in [0, 2\pi)$. The only surviving second-order term is the one involving $|w|^2 = r^2$:

$$\int_0^{2\pi} \frac{d\theta}{2\pi} u(z + re^{i\theta}) - u(z) = \frac{\partial^2 u}{\partial z \partial z^*} r^2 + o(r^3). \quad (\text{A.3.4})$$

The subharmonicity condition requires the left-hand side to be non-negative for sufficiently small r , which implies

$$\frac{\partial^2 u}{\partial z \partial z^*} \geq 0. \quad (\text{A.3.5})$$

This is equivalent to the standard Laplacian condition since $\Delta u = 4 \frac{\partial^2 u}{\partial z \partial z^*} \geq 0$.

A real-valued function u of n -complex variables $\mathbf{z} = (z_1, \dots, z_n)$ is plurisubharmonic if it is upper semi-continuous and its restriction to any complex line is subharmonic. That is, for any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$, the function of a single complex variable ζ , defined by

$$u_{\mathbf{a}, \mathbf{b}}(\zeta) = u(a_1 \zeta + b_1, \dots, a_n \zeta + b_n). \quad (\text{A.3.6})$$

is subharmonic. Based on the above discussion, this is equivalent to the condition $\partial_{\zeta} \partial_{\zeta^*} u_{\mathbf{a}, \mathbf{b}} \geq 0$. By applying the chain rule for Wirtinger derivatives, we obtain

$$\frac{\partial^2 u_{\mathbf{a}, \mathbf{b}}}{\partial \zeta \partial \zeta^*} = \sum_{j, k=1}^n \frac{\partial^2 u}{\partial z_j \partial z_k^*} a_j a_k^* \geq 0. \quad (\text{A.3.7})$$

This inequality implies that the complex Hessian matrix of u is positive semi-definite. The sum of plurisubharmonic functions is also plurisubharmonic.

Plurisubharmonic functions are closely related to convex functions. The average of a plurisubharmonic function over a torus is convex. We define its average over a torus as a function of the radii w_j :

$$\Phi(w_1, \dots, w_n) = \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} u(z_1 + w_1 e^{i\theta_1}, \dots, z_n + w_n e^{i\theta_n}). \quad (\text{A.3.8})$$

By construction, Φ depends only on the magnitudes $r_j = |w_j|$. We claim that Φ is a convex function with respect to the variables $\mu_j = \ln r_j$.

To prove this, we examine its real Hessian matrix,

$$\frac{\partial^2 \Phi}{\partial \mu_j \partial \mu_k} = \sum_{j,k} \frac{\partial^2 \Phi}{\partial r_j \partial r_k} r_j r_k + \sum_j \frac{\partial \Phi}{\partial r_j} r_j \delta_{jk}. \quad (\text{A.3.9})$$

Since u is plurisubharmonic, the function Φ inherits this property. Consequently, its complex Hessian matrix is positive semi-definite for any complex vector $\mathbf{a} \in \mathbb{C}^n$,

$$\sum_{j,k} \frac{\partial^2 \Phi}{\partial w_j \partial w_k^*} a_j a_k^* \geq 0, \quad \text{where} \quad \frac{\partial^2 \Phi}{\partial w_j \partial w_k^*} = \frac{1}{4} \frac{\partial^2 \Phi}{\partial r_j \partial r_k} e^{i(\varphi_k - \varphi_j)} + \frac{1}{4r_j} \frac{\partial \Phi}{\partial r_j} \delta_{jk}, \quad (\text{A.3.10})$$

where $w_j = r_j e^{i\varphi_j}$.

To connect this to the real Hessian, we choose the specific vector $a_j = \alpha_j r_j e^{-i\varphi_j}$ for an arbitrary real vector $\alpha \in \mathbb{R}^n$. Substituting this into the complex Hessian form yields

$$\sum_{j,k} \frac{\partial^2 \Phi}{\partial w_j \partial w_k^*} a_j a_k^* = \frac{1}{4} \sum_{j,k} \frac{\partial^2 \Phi}{\partial \mu_j \partial \mu_k} \alpha_j \alpha_k \geq 0. \quad (\text{A.3.11})$$

This inequality demonstrates that the quadratic form of the real Hessian matrix is non-negative for any real vector α . Therefore, Φ is a convex function of μ .

A.4 Convexity of the Ronkin function

In this section, we examine the convexity of the Ronkin function. We first prove that the total Ronkin function is inherently convex due to the analytic properties of the characteristic polynomial. We then demonstrate why this property breaks down for naive band-resolved Ronkin functions, necessitating the symmetry-based decomposition proposed in the main text.

A.4.1 Proof of convexity for the Ronkin function

Let f be a Laurent polynomial with d complex variables $\beta = (\beta_1, \dots, \beta_d)$,

$$f(\beta) = \sum_{\mathbf{n}} f_{\mathbf{n}} \beta^{\mathbf{n}} = \sum_{n_1=-p_1}^{q_1} \cdots \sum_{n_d=-p_d}^{q_d} f_{n_1, \dots, n_d} \beta_1^{n_1} \cdots \beta_d^{n_d}. \quad (\text{A.4.1})$$

The Ronkin function $R_f(\boldsymbol{\mu})$ is defined as the average of $\ln |f(e^{\boldsymbol{\mu} + i\mathbf{k}})|$ on the torus,

$$R_f(\boldsymbol{\mu}) = \int_0^{2\pi} \frac{d^d k}{(2\pi)^d} \ln |f(e^{\boldsymbol{\mu} + i\mathbf{k}})|. \quad (\text{A.4.2})$$

While the symbol $f(z)$ is a Laurent polynomial, the pole term $\ln |\beta^{-\mathbf{p}}|$ contributes only a linear term $-\mathbf{p} \cdot \boldsymbol{\mu}$ to the Ronkin function, which preserves convexity. Therefore, we focus on the polynomial part $g(\beta) = \beta^{\mathbf{p}} f(\beta)$.

Since $g(\beta)$ is a multivariate polynomial, it is analytic everywhere in $\mathbf{C}^d$. Consequently, its restriction to any complex line, $g_{\mathbf{a}, \mathbf{b}}(\zeta) = g(\mathbf{a}\zeta + \mathbf{b})$, is a univariate holomorphic function with respect to ζ . A fundamental property of holomorphic functions is Jensen's formula, which relates the value at the center to the integral over the boundary:

$$\begin{aligned} \ln |g_{\mathbf{a}, \mathbf{b}}(\zeta)| &= \int_0^{2\pi} \frac{d\theta}{2\pi} \ln |g_{\mathbf{a}, \mathbf{b}}(\zeta + r e^{i\theta})| - \sum_k \ln \frac{r}{|\zeta_k|} \\ &\leq \int_0^{2\pi} \frac{d\theta}{2\pi} \ln |g_{\mathbf{a}, \mathbf{b}}(\zeta + r e^{i\theta})|, \end{aligned} \quad (\text{A.4.3})$$

where ζ_k are the zeros of $g_{\mathbf{a}, \mathbf{b}}$ within the disk of radius r (i.e., $|\zeta_k| < r$). This inequality is precisely the defining condition for subharmonicity.

Applying the theorem in the previous section, the torus average of the plurisubharmonic function $\ln |g(\beta)|$, which is the Ronkin function, is guaranteed to be convex.

A.4.2 Breakdown of convexity in band-resolved Ronkin functions

The previous subsection established the convexity of the Ronkin function for Laurent polynomials. The proof relied on the global analyticity of multivariate polynomials, which ensures plurisubharmonicity via Jensen's inequality. We now apply this framework to non-Bloch Hamiltonians to elucidate why convexity breaks down for band-resolved Ronkin functions. For simplicity, we focus on one-dimensional systems without loss of generality.

Let $h(\beta)$ be a non-Bloch Hamiltonian and $\varepsilon_m(\beta)$ be its m -th Bloch spectrum. We define the total and band-resolved Ronkin functions, respectively, as

$$R_{\sigma}(\boldsymbol{\mu}) = \int_0^{2\pi} \frac{dk}{2\pi} \ln \left| \det [E - h(e^{\boldsymbol{\mu} + i\mathbf{k}})] \right|, \quad (\text{A.4.4})$$

$$R_{\sigma}^{[m]}(\boldsymbol{\mu}) = \int_0^{2\pi} \frac{dk}{2\pi} \ln |E - \varepsilon_m(e^{\boldsymbol{\mu} + i\mathbf{k}})| \quad (\text{A.4.5})$$

First, consider a single-band system. In this case, the energy dispersion $\varepsilon(\beta)$ is identical to the Hamiltonian symbol $h(\beta)$, which is a Laurent polynomial. Consequently, the term $E - \varepsilon(\beta)$ is analytic on the entire complex plane. Therefore, the Ronkin function for a single-band system, defined as the torus average of $\ln |E - \varepsilon(\beta)|$, naturally satisfies the condition for convexity.

In contrast, for multiband systems, the individual energy bands $\varepsilon_m(\beta)$ are determined as the roots of the characteristic polynomial $\det[E - h(\beta)]$. Mathematically, these are algebraic functions rather than polynomials. To define $\varepsilon_m(\beta)$ as a single-valued function, one must inevitably introduce branch cuts in the complex β -plane. Across these branch cuts, the functions $\varepsilon_m(\beta)$ are discontinuous or non-differentiable, losing their analyticity. Crucially, Jensen's inequality and thus subharmonicity rely on the function being holomorphic within the domain of integration. In the vicinity of branch cuts, the integrand $\ln |E - \varepsilon_m(\beta)|$ ceases to be plurisubharmonic. This violation renders the theorem in Appendix A.3 inapplicable. Consequently, the convexity of the band-resolved Ronkin functions is not guaranteed and often violated.

The breakdown of convexity in the band-resolved Ronkin function, as discussed here, is numerically demonstrated for a specific model in Sec. 6.2.1

Appendix B

Concrete Algorithm for Wiener-Hopf factorization

In this appendix, we provide the concrete algorithm for the Wiener-Hopf factorization (WHF) and its symmetrization. These algorithms are established in Ref. [182], but we provide an overview here to ensure a self-contained discussion.

B.1 Wiener-Hopf factorization

First, we outline the iterative procedure for the WHF. The goal is to decompose a matrix polynomial $A(\beta)$ into the form $A = A_+ D A_-$, where $A_\pm$ are analytic and invertible in their respective domains, and D is a diagonal matrix of partial indices.

Setup

Let $A(\beta)$ be an analytic and invertible matrix-valued polynomial defined as

$$A(\beta) = \sum_{j=0}^q a_j \beta^j, \quad (\text{B.1.1})$$

where a_j are $M \times M$ matrices.

We assume $p = 0$ without loss of generality; for a general case with $p > 0$, the WHF of the auxiliary polynomial $A_0(\beta) = \beta^p A(\beta)$ yields the factorization for $A(\beta)$.

$$A_0 = A_+ D_0 A_- \implies A = A_+ (\beta^{-p} D_0) A_-. \quad (\text{B.1.2})$$

Let $\beta_1, \dots, \beta_n$ be the roots of $\det A(\beta)$ lying inside the unit circle. The algorithm constructs a sequence of factorizations $A = A_+^{(i)} D^{(i)} A_-^{(i)}$, starting from $A_+^{(n)} = A$ and $D^{(n)} = A_-^{(n)} = \mathbb{I}_M$, where $\mathbb{I}_M$ is the identity matrix.

Summary of the algorithm

In the discussion below, we express the matrix in terms of its column vectors.

$$A_+^{(i)} = \begin{pmatrix} A_{+,1}^{(i)} & \cdots & A_{+,M}^{(i)} \end{pmatrix}. \quad (\text{B.1.3})$$

For each root β_i , perform the following three steps:

1. **Identify the dependent column:** Find the smallest index s such that the s -th column vector of $A_+^{(i)}(\beta)$ is linearly dependent on the preceding columns at $\beta = \beta_i$,

$$A_{+,s}^{(i)}(\beta_i) = \sum_{m=1}^{s-1} \alpha_m A_{+,m}^{(i)}(\beta_i) \implies \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{s-1})^\top. \quad (\text{B.1.4})$$

Based on this index s , we introduce a block structure to define the subsequent transformation matrices. Specifically, the $M \times M$ matrices are partitioned into blocks of sizes $(s-1)$, 1, and $(M-s)$.

2. **Construct the transformation matrices:** Using the coefficients $\boldsymbol{\alpha}$, define the transformation matrix $U_i(\beta)$,

$$U_i(\beta) = \begin{pmatrix} \mathbb{I}_{s-1} & \boldsymbol{\alpha} & \\ & \beta - \beta_i & \\ & & \mathbb{I}_{M-s} \end{pmatrix}, \quad U_i^{-1}(\beta) = \begin{pmatrix} \mathbb{I}_{s-1} & -(\beta - \beta_i)^{-1} \boldsymbol{\alpha} & \\ & (\beta - \beta_i)^{-1} & \\ & & \mathbb{I}_{M-s} \end{pmatrix}. \quad (\text{B.1.5})$$

Also, define the update matrix $V_i(\beta)$ for the negative factor,

$$V_i(\beta) = \begin{pmatrix} \mathbb{I}_{s-1} & \beta^{\kappa_s^{(i)}} [D_{s-1}^{(i)}]^{-1} \boldsymbol{\alpha} & \\ & 1 - \beta_i \beta^{-1} & \\ & & \mathbb{I}_{M-s} \end{pmatrix}, \quad (\text{B.1.6})$$

where $D_{s-1}^{(i)} = \text{diag}(\beta^{\kappa_1^{(i)}}, \dots, \beta^{\kappa_{s-1}^{(i)}})$ denotes the upper-left block of $D^{(i)}$.

3. **Update the factors:** Update A_+ , D , and A_- as follows:

$$\begin{cases} A_+^{(i-1)} &= A_+^{(i)} U_i^{-1}(\beta) \Pi_i, \\ D^{(i-1)} &= \Pi_i D^{(i)} \text{diag}(\mathbb{I}_{s-1}, \beta, \mathbb{I}_{M-s}) \Pi_i, \\ A_-^{(i-1)} &= \Pi_i V_i(\beta) A_-^{(i)}, \end{cases} \quad (\text{B.1.7})$$

where Π_i is the permutation matrix that sorts the partial indices in $D^{(i-1)}$ into descending order.

Proof of the algorithm

The algorithm iteratively moves the roots of $\det A_+$ into the diagonal factor D while preserving the product $A = A_+ D A_-$.

First, we show that $U_i(\beta)$ removes the root β_i from $A_+^{(i)}$. Applying U_i^{-1} to $A_+^{(i)}$ from the right replaces the s -th column of the product as

$$A_{+,s}^{(i)} \mapsto \frac{1}{\beta - \beta_i} \left(A_{+,s}^{(i)}(\beta) - \sum_{m=1}^{s-1} \alpha_m A_{+,m}^{(i)}(\beta) \right). \quad (\text{B.1.8})$$

By the definition of α in Step 1, the numerator vanishes at $\beta = \beta_i$. Since each entry is a polynomial, the s -th column contains the factor $(\beta - \beta_i)$, making the singularity removable. From a global perspective, since $\det U_i^{-1}(\beta) = (\beta - \beta_i)^{-1}$, this multiplication cancels one zero of $\det A_+^{(i)}(\beta)$ at $\beta = \beta_i$, ensuring that $A_+^{(i-1)}$ remains analytic and invertible inside the unit circle.

Correspondingly, the extraction of the factor $(\beta - \beta_i)$ from $A_+^{(i)}$ leads to an update of the diagonal factor. This process increments the s -th partial index by 1, which is equivalent to multiplying the s -th diagonal entry of $D^{(i)}$ by β .

Finally, we define the update matrix $V_i(\beta)$ for the negative factor to compensate for the transformations of $A_+^{(i)}$ and $D^{(i)}$. Let $\Delta_i(\beta) = \text{diag}(\mathbb{I}_{s-1}, \beta, \mathbb{I}_{M-s})$. To ensure the invariance of the product, we require $A_+^{(i)} D^{(i)} A_-^{(i)} = (A_+^{(i)} U_i^{-1})(D^{(i)} \Delta_i)(V_i A_-^{(i)})$, which leads to:

$$V_i(\beta) = \Delta_i^{-1}(\beta) [D^{(i)}(\beta)]^{-1} U_i(\beta) D^{(i)}(\beta). \quad (\text{B.1.9})$$

By substituting the block matrices, we obtain the explicit form of $V_i(\beta)$ presented in Step 2. Since $(\beta - \beta_i)/\beta = 1 - \beta_i \beta^{-1}$ is analytic and invertible outside the unit circle, the updated factor $A_-^{(i-1)}$ preserves the required properties of the WHF.

It is important to note that while the partial indices κ_j in D are uniquely determined as topological invariants of the matrix $A(\beta)$, the side factors $A_{\pm}$ are not unique. This non-uniqueness arises because the roots β_i can be processed in any order. Since the transformation matrices U_i and permutations Π_i do not generally commute, different orderings of the roots lead to different side factors, although they always yield the same set of indices $\{\kappa_j\}$.

B.2 Symmetric Wiener-Hopf factorization

Once we obtain a Wiener-Hopf factorization, we can compute the symmetric Wiener-Hopf factorization via symmetrization.

While the symmetric Wiener-Hopf factorization can be generalized to all ten symmetry classes in the AZ classification (see Ref. [182] for the complete discussion), in this appendix, we focus on class A and class DIII, which are directly relevant to the lattice models analyzed in the main text. The construction for other classes proceeds in an analogous manner by imposing the corresponding symmetry constraints.

B.2.1 Factorization of constant matrix

Before discussing the factorization of polynomial matrices, we briefly review the factorization of constant (scalar-valued) matrices. Depending on the symmetry of the matrix, we employ either the Hermitian factorization or the Youla decomposition.

Summary of the algorithm

1. **Hermitian matrix factorization:** For a complex-valued invertible Hermitian matrix $C \in \mathbb{C}^{M \times M}$:
 - (a) Diagonalize the matrix as $C = U\Lambda U^\dagger$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_M)$ and the unitary matrix U consists of the corresponding eigenvectors as its columns.
 - (b) Construct the signature matrix $S = \text{diag}(\text{sgn}(\lambda_1), \dots, \text{sgn}(\lambda_M))$.
 - (c) Construct the transformation matrix G as $G = U\sqrt{|\Lambda|}$, where $\sqrt{|\Lambda|} = \text{diag}(\sqrt{|\lambda_1|}, \dots, \sqrt{|\lambda_M|})$.
 - (d) This yields the factorization $C = GSG^\dagger$.
2. **Skew-symmetric matrix factorization (Youla decomposition):** For a complex-valued invertible skew-symmetric matrix $C \in \mathbb{C}^{2M \times 2M}$ ($C = -C^\top$):
 - (a) Construct the Hermitian matrix $\tilde{C} = CC^\dagger$ and find its eigenvalues and eigenvectors. The eigenvalues are positive and doubly degenerate, denoted as $\lambda_1^2, \dots, \lambda_M^2$ ($\lambda_j > 0$).
 - (b) For each pair λ_j^2 , pick only *one* normalized eigenvector v_j .
 - (c) For each v_j , calculate its partner vector $w_j = -Cv_j^*/\lambda_j$.
 - (d) Construct the transformation matrix G as $G = (v_M, \dots, v_1, w_1, \dots, w_M)\sqrt{\Lambda}$, where $\sqrt{\Lambda} = \text{diag}(\sqrt{\lambda_M}, \dots, \sqrt{\lambda_1}, \sqrt{\lambda_1}, \dots, \sqrt{\lambda_M})$.
 - (e) This yields the factorization $C = GJG^\top$, where J is the $2M \times 2M$ skew-symmetric matrix:

$$J = \begin{pmatrix} 0 & \mathbb{X}_M \\ -\mathbb{X}_M & 0 \end{pmatrix}. \quad (\text{B.2.1})$$

Here, $\mathbb{X}_M$ denotes the $M \times M$ anti-diagonal identity matrix, $(\mathbb{X}_M)_{ij} = \delta_{i, M-j+1}$.

Proof of the algorithm

Here, we provide the derivation for the factorizations presented above.

For the Hermitian case, the spectral theorem ensures that $C = U\Lambda U^\dagger$. By absorbing the magnitudes of the eigenvalues into the basis vectors, we have:

$$C = U\sqrt{|\Lambda|} \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_M \end{pmatrix} \sqrt{|\Lambda|}U^\dagger = GSG^\dagger, \quad (\text{B.2.2})$$

where $G = U\sqrt{|\Lambda|}$ is invertible.

For the skew-symmetric case, we utilize the Youla decomposition. Let v_j be a normalized eigenvector of the positive semi-definite matrix $CC^\dagger$ with eigenvalue $\lambda_j^2 > 0$, $CC^\dagger v_j = \lambda_j^2 v_j$. We define a partner vector w_j by the relation $C^\dagger v_j = -\lambda_j w_j^*$, which implies $Cw_j^* = -\lambda_j v_j$. The skew-symmetry $C^\top = -C$ ensures that v_j and w_j are orthogonal,

$$w_j^\dagger v_j = -\frac{1}{\lambda_j} (Cv_j^*)^\dagger v_j = -\frac{1}{\lambda_j} v_j^\top C^\dagger v_j = 0, \quad (\text{B.2.3})$$

since $v^\top Cv = 0$ for any skew-symmetric matrix and any vector.

To align with the symmetry-protected structure, we arrange the M pairs of orthonormal vectors in the following order:

$$U = (v_M, \dots, v_1, w_1, \dots, w_M), \quad (\text{B.2.4})$$

which yields the global relation,

$$CU^* = U\Lambda J, \quad J = \begin{pmatrix} 0 & \mathbb{X}_M \\ -\mathbb{X}_M & 0 \end{pmatrix}, \quad (\text{B.2.5})$$

where $\Lambda = \text{diag}(\lambda_M, \dots, \lambda_1, \lambda_1, \dots, \lambda_M)$.

Finally, by defining $G = U\sqrt{\Lambda}$, we arrive at the target factorization,

$$C = GJG^\top. \quad (\text{B.2.6})$$

B.2.2 Symmetrization for class A

Starting from the WHF of $A = A_+DA_-$, we can symmetrize it through the following procedure.

Summary of the algorithm

1. **Construct the auxiliary matrix B :** Let $\mathbb{X}_M$ be the permutation matrix defined by $[\mathbb{X}_M]_{m,n} = \delta_{m+n, M+1}$. We first define the auxiliary matrix B by the relation:

$$A_-^\dagger \mathbb{X}_M = A_+ B \iff B = A_+^{-1} A_-^\dagger \mathbb{X}_M. \quad (\text{B.2.7})$$

Substituting this into the Hermiticity condition of A , we obtain the intermediate factorization:

$$A = A_+(BD\mathbb{X}_M)A_+^\dagger. \quad (\text{B.2.8})$$

2. **Factorize the central term (Construct G):** Next, we factorize the Hermitian central term as $BD\mathbb{X}_M = GD_{\text{Herm}}G^\dagger$. Given the block structure of B based on $M_\pm, M_0$, the transformation matrix G can be explicitly constructed block-wise. First, isolate the central block corresponding to the zero partial indices: $B_{00}\mathbb{X}_0$, where B_{00} and $\mathbb{X}_0$ denote the corresponding blocks of B and $\mathbb{X}$, respectively. Perform the constant Hermitian matrix factorization (see Appendix B.2.1) to find G_0 and the signature matrix S_0 ,

$$B_{00}\mathbb{X}_0 = G_0S_0G_0^\dagger. \quad (\text{B.2.9})$$

Using these, construct the full transformation matrix G as

$$G = \begin{pmatrix} \mathbb{I} & 0 & \frac{1}{2}B_{+-} \\ & G_0 & B_{0-} \\ & & B_{--} \end{pmatrix}. \quad (\text{B.2.10})$$

3. **Symmetrization:** Finally, we define the symmetric positive factor as $\tilde{A}_+ = A_+G$. This yields the target SWHF:

$$A = \tilde{A}_+D_{\text{Herm}}\tilde{A}_+^\dagger. \quad (\text{B.2.11})$$

Proof of the algorithm

Here, we provide the derivation and justification for the algorithm presented above.

Due to the Hermiticity of $A(\beta)$, the relation $A_+DA_- = A_-^\dagger D^\dagger A_+^\dagger$ holds. Since the Hermitian conjugate operation reverses the signs of the partial indices, the uniqueness of the partial indices implies that every positive partial index κ must be paired with a corresponding negative index $-\kappa$. This implies that the Hermitian conjugate of the diagonal factor takes the form $D^\dagger = \mathbb{X}D\mathbb{X}$.

Consequently, we arrive at the alternative factorization:

$$A = A_+DA_- = A_+(BD\mathbb{X})A_+^\dagger, \quad (\text{B.2.12})$$

where the matrix polynomial B is defined by $A_-^\dagger\mathbb{X} = A_+B$. Since A_+ is invertible inside the unit circle, B is well-defined and analytic within the unit disk. Furthermore, the Hermiticity of A ensures that the central factor $BD\mathbb{X}$ is itself Hermitian. Our objective is to factorize this central term symmetrically.

First, we analyze the structure of B . Equation (B.2.12) implies

$$B = D[A_-(A_+^\dagger)^{-1}\mathbb{X}]D^{-1} \implies B_{mn} = \beta^{\kappa_m - \kappa_n} [A_-(A_+^\dagger)^{-1}\mathbb{X}]_{mn}. \quad (\text{B.2.13})$$

The term $A_-(A_+^\dagger)^{-1}\mathbb{X}$ is analytic outside the unit circle (including infinity), whereas B is analytic inside the unit disk. Therefore, due to the constraint on the degrees, the entry B_{mn} must vanish if $\kappa_m < \kappa_n$.

Reflecting this upper-triangular property, we partition the matrices based on the index sets $\mathcal{M}_+, \mathcal{M}_0, \mathcal{M}_-$,

$$B = \begin{pmatrix} B_{++} & B_{+0} & B_{+-} \\ & B_{00} & B_{0-} \\ & & B_{--} \end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} & & \mathbb{X}_+ \\ & \mathbb{X}_0 & \\ \mathbb{X}_+ & & \end{pmatrix}, \quad D = \begin{pmatrix} D_+ & & \\ & \mathbb{I} & \\ & & D_- \end{pmatrix}. \quad (\text{B.2.14})$$

We seek a matrix G that factorizes the central term as $BD\mathbb{X} = GD_{\text{Herm}}G^\dagger$. Since $D^\dagger = \mathbb{X}D\mathbb{X}$, the target diagonal factor D_{Herm} in this block representation possesses an anti-diagonal structure,

$$D_{\text{Herm}} = \begin{pmatrix} & D_+\mathbb{X}_+ \\ & S_0 \\ D_-\mathbb{X}_+ & \end{pmatrix}, \quad (\text{B.2.15})$$

where S_0 is a constant signature matrix to be determined. We assume the ansatz for G as:

$$G = \begin{pmatrix} \mathbb{I} & 0 & G_{+-} \\ & G_{00} & G_{0-} \\ & & G_{--} \end{pmatrix}. \quad (\text{B.2.16})$$

First, we determine the central components G_{00} and S_0 . Looking at the $(0,0)$ block of the equation $BD\mathbb{X} = GD_{\text{Herm}}G^\dagger$, we have the equation,

$$B_{00}\mathbb{X}_0 = G_{00}S_0G_{00}^\dagger. \quad (\text{B.2.17})$$

This is a factorization problem for a constant Hermitian matrix (see Appendix B.2.1). Thus, G_{00} can be identified as the factor G_0 , and S_0 is determined as the corresponding signature matrix.

Next, to determine the off-diagonal block G_{+-} , we examine the $(+,+)$ block of the equation $BD\mathbb{X} = GD_{\text{Herm}}G^\dagger$. Calculating the product with the ansatz, the $(+,+)$ block of the right-hand side expands to:

$$(GD_{\text{Herm}}G^\dagger)_{++} = G_{+-}D_-\mathbb{X}_+ + D_+\mathbb{X}_+G_{+-}^\dagger = G_{+-}D_-\mathbb{X}_+ + [G_{+-}D_-\mathbb{X}_+]^\dagger. \quad (\text{B.2.18})$$

On the other hand, the $(+,+)$ block of the left-hand side is simply given by $(BD\mathbb{X})_{++} = B_{+-}D_-\mathbb{X}_+$. Since the entire matrix $BD\mathbb{X}$ is Hermitian, this block must be self-adjoint. Thus, we can symmetrize the term as:

$$B_{+-}D_-\mathbb{X}_+ = \frac{1}{2}B_{+-}D_-\mathbb{X}_+ + \frac{1}{2}(B_{+-}D_-\mathbb{X}_+)^\dagger. \quad (\text{B.2.19})$$

Comparing the terms in the expanded equation, we can uniquely identify the solution as:

$$G_{+-}D_-\mathbb{X}_+ = \frac{1}{2}B_{+-}D_-\mathbb{X}_+ \implies G_{+-} = \frac{1}{2}B_{+-}. \quad (\text{B.2.20})$$

The remaining blocks are determined similarly by solving the equations.

Combining these results, the components of G are uniquely determined as follows:

$$G = \begin{pmatrix} \mathbb{I} & 0 & \frac{1}{2}B_{+-} \\ & G_0 & B_{0-} \\ & & B_{--} \end{pmatrix}. \quad (\text{B.2.21})$$

The explicit summation formula for G presented in Ref. [182] is consistent with the result of this block-wise construction.

B.2.3 Symmetrization for class DIII

As discussed in Sec. 4.1.2, the SWHF of the class DIII Hamiltonian reduces to the factorization of the off-diagonal block $A(\beta)$, which satisfies the skew-symmetry condition $A^\top(\beta) = -A(\beta)$. Starting from a generic WHF, $A = A_+DA_-$, we construct a skew-symmetric factorization following a procedure analogous to the class A case.

Summary of the algorithm

1. Construct the auxiliary matrix B :

Utilizing the relation $A_+DA_- = -A_-^\top D^\top A_+^\top$ derived from the skew-symmetry, we define the auxiliary matrix B as

$$B = A_+^{-1}A_-^\top. \quad (\text{B.2.22})$$

This leads to the intermediate factorization:

$$A = -A_+(BD\mathbb{X})A_+^\top. \quad (\text{B.2.23})$$

2. Factorize the central term:

We factorize the skew-symmetric central term as $-BD\mathbb{X} = GD_{\text{skew}}G^\top$. First, for the central block $-B_{00}D_0\mathbb{X}_0$, we perform the Youla decomposition for constant skew-symmetric matrices (see Appendix B.2.1) to find G_0 and the canonical block J ,

$$-B_{00}D_0\mathbb{X}_0 = G_0(-J)G_0^\top. \quad (\text{B.2.24})$$

where J is the canonical skew-symmetric matrix with anti-diagonal entries,

$$J = \begin{pmatrix} & \mathbb{X}_M \\ -\mathbb{X}_M & \end{pmatrix}. \quad (\text{B.2.25})$$

Using this G_0 , the full transformation matrix G is constructed exactly as in the class A case:

$$G = \begin{pmatrix} \mathbb{I} & 0 & \frac{1}{2}B_{+-} \\ & G_0 & B_{0-} \\ & & B_{--} \end{pmatrix}. \quad (\text{B.2.26})$$

The resulting diagonal factor D_{skew} takes the following form,

$$D_{\text{skew}} = \begin{pmatrix} & & D_+\mathbb{X}_+ \\ & -J & \\ -D_-\mathbb{X}_+ & & \end{pmatrix}. \quad (\text{B.2.27})$$

3. Final Symmetrization:

Finally, defining $\tilde{A}_+ = A_+G$ yields the target skew-symmetric factorization,

$$A = \tilde{A}_+ D_{\text{skew}} \tilde{A}_+^\top. \quad (\text{B.2.28})$$

Proof of the algorithm

The derivation follows parallel to the class A case. Since the transpose operation reverses the signs of partial indices, we have $D^\top = \mathbb{X}D\mathbb{X}$. The skew-symmetry $A^\top = -A$ implies

$$A = -A_-^\top D^\top A_+^\top = -A_-^\top (\mathbb{X}D\mathbb{X}) A_+^\top. \quad (\text{B.2.29})$$

Defining $B = A_+^{-1}A_-^\top\mathbb{X}$, we obtain $A = -A_+(BD\mathbb{X})A_+^\top$. Since A is skew-symmetric, the central term $BD\mathbb{X}$ must also be skew-symmetric. Furthermore, the analyticity of B (inside the unit disk) and the degree constraints on D force the lower triangular blocks of B to vanish.

We factorize $-BD\mathbb{X}$ using the same ansatz for G as in the class A case. The central block reduces to the factorization of a constant skew-symmetric matrix, $-B_{00}\mathbb{X}_0$, which is achieved via the Youla decomposition. For the off-diagonal blocks, the condition $-BD\mathbb{X} = GD_{\text{skew}}G^\top$ is solved block-wise. Due to the structure of D_{skew} (specifically its anti-diagonal blocks $\pm D_\pm\mathbb{X}_\pm$), the equation for the $(+, +)$ block involves the skew-symmetrization of the term $B_{+-}D_-\mathbb{X}_+$. Comparing the terms, we consistently obtain the same solution form $G_{+-} = \frac{1}{2}B_{+-}$, leading to the identical structure for G as in the class A case.

Appendix C

Hidden Symmetry and Fragility of $\kappa = 2$ Phase

In this Appendix, we examine the fragility of the $\kappa = 2$ phase in class AII[†] and show that its apparent stability originates from an underlying unitary symmetry. This hidden symmetry enforces the modified GBZ condition $\mu_2 = \mu_3$, replacing the conventional relation $\mu_1 = \mu_2$.

The $\kappa = 2$ phase is inherently fragile. Within the framework of the Hermitian-doubled Hamiltonian, this phase corresponds to the presence of two Kramers pairs. As in the case of $\mathbb{Z}_2$ topological phases, an even number of Kramers pairs is not protected against generic TRS[†]-preserving perturbations: pairs can hybridize and gap out each other, rendering the phase topologically trivial.

In our model, this inherent fragility can explicitly be confirmed. The $\kappa = 2$ phase, which exists for $\Delta_1 = 0$, disappears upon introducing a TRS[†]-preserving perturbation $\Delta h(e^{ik}) = -2\Delta_1 X \sin k$, even for a small value $\Delta_1 = 0.01$, as shown in Fig. C.1. Under this perturbation, the $\kappa = 2$ in the system changes to a $\kappa = 0$ phase without appreciable change in the PBC spectrum. The winding in the PBC spectrum remains unchanged, indicating that the underlying bulk topology of the PBC bands is unaffected.

This behavior can be understood by analyzing the symmetries of the unperturbed Hamiltonian $h(k)$ (i.e., for $\Delta_1 = 0$):

$$h(e^{ik}) = 2t_1 \cos k + 2t_2 \cos 2k - 2(\Delta_2 X + ig_2 Z) \sin 2k. \quad (\text{C.0.1})$$

The Hamiltonian $h(k)$ can be diagonalized by a k -independent unitary matrix V ,

$$V^{-1}hV = \begin{pmatrix} h_+ & \\ & h_- \end{pmatrix}, \quad (\text{C.0.2})$$

where each block is given by

$$h_{\pm}(e^{ik}) = 2t_1 \cos k + 2t_2 \cos 2k \pm 2iZ\sqrt{g_2^2 - \Delta_2^2} \sin 2k. \quad (\text{C.0.3})$$

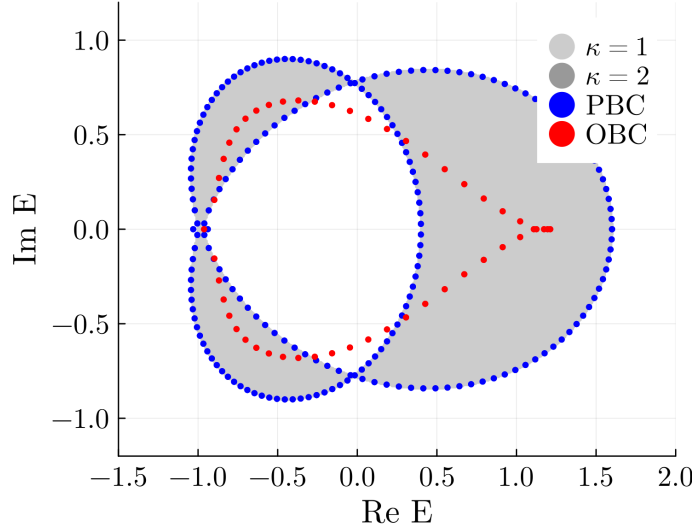


Figure C.1: Test of the stability of the $\kappa = 2$ phase under a $\text{TRS}^\dagger$ -preserving perturbation. The parameters are identical to those in Fig. 6.12 except for $\Delta_1 = 0.01$. Introducing the perturbation $\Delta h(e^{ik}) = -2\Delta_1 X \sin k$ removes the $\kappa = 2$ phase, confirming its fragility. The original $\kappa = 2$ phase changes to a $\kappa = 0$ phase without any qualitative change in the PBC spectrum, indicating that the bulk winding of the PBC Hamiltonian remains unchanged.

This diagonal form reveals an additional hidden unitary symmetry of the unperturbed Hamiltonian,

$$[h, VZV^{-1}] = 0. \quad (\text{C.0.4})$$

Owing to this symmetry, the system decouples into two independent blocks h_+ and h_- , corresponding respectively to Kramers partners with partial indices $\pm\kappa = W[h_\pm]$. As a result, hybridization between the Kramers pairs is prohibited, which stabilizes the $\kappa = 2$ phase for these specific parameters.

The perturbation $\Delta h(e^{ik}) = -2\Delta_1 X \sin k$, however, breaks this hidden symmetry, since the perturbed Hamiltonian $h(e^{ik}) + \Delta h(e^{ik})$ can no longer be diagonalized by a k -independent matrix. As a consequence, the two Kramers pairs hybridize and gap each other out, leading to the disappearance of the $\kappa = 2$ phase. This demonstrates that the $\kappa = 2$ phase is not protected by $\text{TRS}^\dagger$ alone but instead relies on the additional hidden unitary symmetry VZV^{-1} .

This hidden symmetry also drives the shift of the GBZ condition from the conventional $\mu_1 = \mu_2$ [147] to $\mu_2 = \mu_3$. The intrinsic instability of the $\kappa = 2$ phase underlies the emergence of the green region in Fig. 6.12, the parameter domain where $\mathcal{K}^* = (1, -1)$ and whose boundary satisfies the conventional relation $\mu_1 = \mu_2$. In our specific model, both the $\kappa = 2$ phase and this $\mathcal{K}^* \neq (0, 0)$ region are stabilized by a hidden unitary symmetry. When a small

TRS[†]-preserving perturbation breaks this symmetry, the $\kappa = 2$ phase disappears, and the OBC spectrum expands to the boundary of the green region, thereby restoring the standard GBZ condition. Thus, the nonvanishing partial indices and the accompanying “abnormal” GBZ relation ($\mu_2 = \mu_3$) are fragile artifacts of this hidden symmetry rather than a robust modification of the GBZ itself.

Figure C.2 corroborates this discussion by showing the OBC DOS of the perturbed Hamiltonian, scaled by the surface element $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 2 \times 10^{-4}$. Because the perturbation breaks the hidden symmetry, the conventional GBZ condition $\mu_1 = \mu_2$ is restored. Due to finite-size effects, the OBC spectrum obtained by direct diagonalization ($N = 50$, red dotted line) exhibits a small shift relative to the WHF-based results, which represent the $N \rightarrow \infty$ limit. Nevertheless, both results are consistent within the thermodynamic limit.

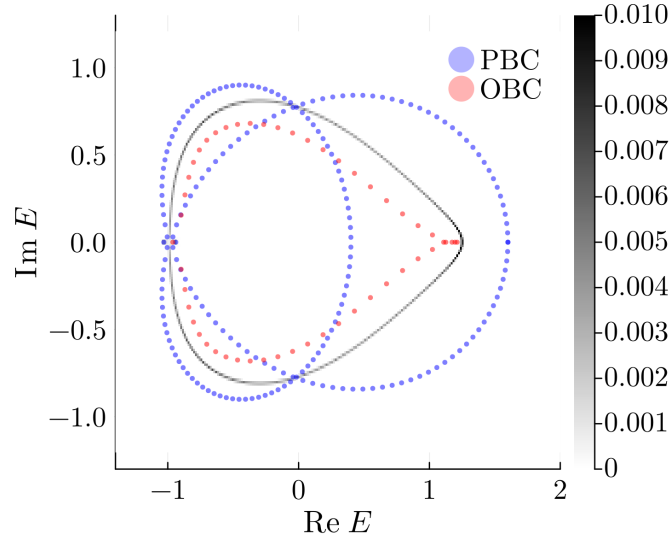


Figure C.2: DOS multiplies by surface element $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 2 \times 10^{-4}$. The Hamiltonian and parameters are identical to those in Fig. C.1. Due to the finite-size effect, the OBC spectrum shifts from the Amoeba results. However, these results coincide in the thermodynamic limit.

Appendix D

Amoeba formulation for geometry-dependent skin effect

In this appendix, we outline the phenomenological extension of the Amoeba formulation to two-dimensional systems exhibiting the geometry-dependent skin effect (GDSE).

D.1 Potential for GDSE

The standard Amoeba formulation assumes a global optimization of the Ronkin function over all spatial dimensions simultaneously. However, systems with GDSE exhibit localization properties that depend on the specific geometry of the boundaries, often behaving as quasi-one-dimensional chains coupled in the transverse direction. To describe this physics, a phenomenological modification based on hierarchical optimization has been proposed [109].

Consider a d -dimensional Hamiltonian h on a hypercubic lattice. The real-space representation on a finite lattice of size N can be expressed using multi-level Toeplitz operators:

$$H_{\text{OBC}} = \mathcal{T}_{N_1}^{(1)} \dots \mathcal{T}_{N_d}^{(d)}[h]. \quad (\text{D.1.1})$$

The core idea of this formulation is to treat the thermodynamic limit sequentially. We assume that the asymptotic behavior of the determinant along a specific direction j follows the one-dimensional Szegő limit theorem, treating the momenta in the other directions as external parameters:

$$\lim_{N_j \rightarrow \infty} \frac{1}{N_j} \ln |\det \mathcal{T}_{N_j}[\sigma]| \leftrightarrow \min_{\mu_j} \int_0^{2\pi} \frac{dk_j}{2\pi} \ln |\det \sigma_{\mu_j}|, \quad (\text{D.1.2})$$

where σ_{μ_j} is the symbol analytically continued only in the j -th component.

For a two-dimensional system ($d = 2$), this leads to two distinct candidates for the spectral potential, depending on the order in which the thermodynamic limits are taken (or

equivalently, which direction is treated as the primary "spatial" dimension). We define $\phi_1(E)$ as the potential obtained by first calculating the bulk potential along y (integrating k_y with optimized μ_y) and then treating the result as a one-dimensional system along x ,

$$\phi_1(E) = \min_{\mu_x} \int_0^{2\pi} \frac{dk_x}{2\pi} \left[\min_{\mu_y} \int_0^{2\pi} \frac{dk_y}{2\pi} \ln |\det \sigma(e^{\mu+ik})| \right], \quad (\text{D.1.3})$$

where $e^{\mu+ik} = (e^{\mu_1+ik_1}, e^{\mu_2+ik_2})$. Similarly, $\phi_2(E)$ is defined by reversing the hierarchy.

$$\phi_2(E) = \min_{\mu_y} \int_0^{2\pi} \frac{dk_y}{2\pi} \left[\min_{\mu_x} \int_0^{2\pi} \frac{dk_x}{2\pi} \ln |\det \sigma(e^{\mu+ik})| \right]. \quad (\text{D.1.4})$$

Physically, ϕ_1 and ϕ_2 correspond to the skin effect localized primarily along the x - and y -boundaries, respectively. The effective spectral potential for the full open boundary system is conjectured to be the minimum of these two values, representing the most favorable localization channel:

$$\phi(E) = \min\{\phi_1(E), \phi_2(E)\}. \quad (\text{D.1.5})$$

This potential $\phi(E)$ correctly describes the spectral area where the skin effect is dominated by the geometry of the lattice termination.

D.2 Application to two-dimensional systems with TRS[†]

In this section, we generalize the spectral potential for two-dimensional class AII[†] systems that possess reflection symmetry along the diagonal $x = y$. In the presence of this symmetry, the system behaves as a quasi-one-dimensional chain along the high-symmetry line, effectively decoupling the dynamics in the perpendicular direction.

We define TRS[†] and the reflection symmetry acting on the Bloch Hamiltonian $h(\beta_x, \beta_y)$ as,

$$U_T h(\beta_x, \beta_y)^\top U_T^{-1} = h(\beta_x^{-1}, \beta_y^{-1}), \quad (\text{D.2.1})$$

$$R h(\beta_x, \beta_y) R^{-1} = h(\beta_y, \beta_x), \quad (\text{D.2.2})$$

where U_T is a unitary operator satisfying $U_T U_T^* = -1$, and R is a unitary reflection operator. Note that the transpose operation acts on the internal degrees of freedom and does not invert the Laurent polynomial exponents β .

To exploit the reflection symmetry, we rotate the spatial basis by 45°. The coordinate transformation for the spatial vectors is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_\perp \\ x_\parallel \end{pmatrix}. \quad (\text{D.2.3})$$

Correspondingly, the complex-valued wavenumbers $(\ln \beta_x, \ln \beta_y)$ transform inversely to $(\ln \beta_\perp, \ln \beta_\parallel)$ as

$$\begin{pmatrix} \ln \beta_x & \ln \beta_y \end{pmatrix} = \begin{pmatrix} \ln \beta_\perp & \ln \beta_\parallel \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (\text{D.2.4})$$

Here, the $\parallel$ component corresponds to the high-symmetry axis ($x = y$), while the $\perp$ component corresponds to the direction perpendicular to it.

In these rotated coordinates, the reflection symmetry R swaps $\beta_x \leftrightarrow \beta_y$, which leaves $\beta_\parallel$ invariant but inverts $\beta_\perp \rightarrow \beta_\perp^{-1}$. By combining R and U_T , we identify a hidden symmetry for the effective quasi-one-dimensional system:

$$(RU_T)h(\beta_\perp, \beta_\parallel)^\top (RU_T)^{-1} = h(\beta_\perp, \beta_\parallel^{-1}). \quad (\text{D.2.5})$$

This relation implies that for a fixed $\beta_\perp$, the Hamiltonian respects an effective $\text{TRS}^\dagger$ with respect to $\beta_\parallel$. Since the reflection symmetry maps $x_\perp \rightarrow -x_\perp$, the NHSE is prohibited along the perpendicular direction. Consequently, the states are delocalized along $x_\perp$, allowing us to treat the perpendicular wavenumber as a conventional Bloch parameter by setting $\beta_\perp = e^{ik_\perp}$ with $k_\perp \in \mathbb{R}$.

The problem thus reduces to determining the localization along the parallel direction. We define the total Ronkin function for this rotated system as

$$R_\tau(\mu_\parallel) = \int_0^{2\pi} \frac{dk_\perp}{2\pi} R_\tau(\mu_\parallel; k_\perp), \quad (\text{D.2.6})$$

$$R_\tau(\mu_\parallel; k_\perp) = \int_0^{2\pi} \frac{dk_\parallel}{2\pi} \ln \left| \det \tau(e^{ik_\perp}, e^{\mu_\parallel + ik_\parallel}) \right|, \quad (\text{D.2.7})$$

where $\tau = (E - h)U_T$. The spectral potential is then obtained by optimizing the Ronkin function for each slice of $k_\perp$,

$$\phi_\tau(E) = \int_0^{2\pi} \frac{dk_\perp}{2\pi} \phi_\tau(k_\perp), \quad (\text{D.2.8})$$

$$\phi_\tau(k_\perp) = 2 \min_{\mu_\parallel} R_\tau^{(+)}(\mu_\parallel; k_\perp), \quad (\text{D.2.9})$$

where $R_\tau^{(+)}$ is the symmetry-decomposed Ronkin function introduced in Chapter 5, , and the factor of 2 reflects the Kramers degeneracy protected by the effective $\text{TRS}^\dagger$ for each fixed $k_\perp$.

This formulation signifies that the reflection symmetry effectively resolves the ambiguity in the hierarchy of limits. In two-dimensional non-Hermitian systems, the spectral distribution can depend on the order in which the thermodynamic limits $N_x, N_y \rightarrow \infty$ are taken. However, the presence of reflection symmetry identifies a preferred high-symmetry axis, rendering the order of limits irrelevant for the spectral density and providing a robust definition of the two-dimensional non-Bloch spectral potential.

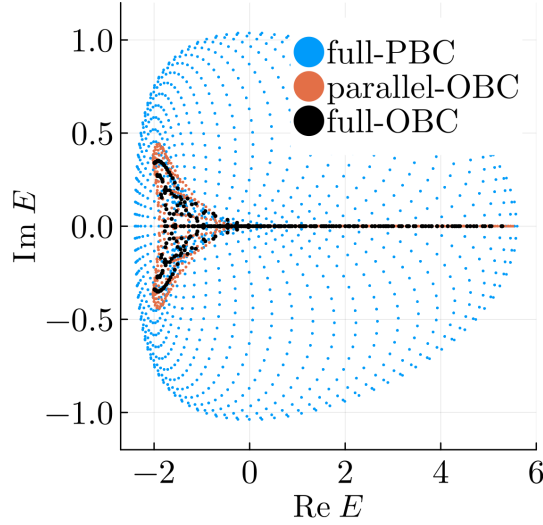


Figure D.1: Comparison of the two-dimensional spectra under various boundary conditions. The complex energy spectra are calculated for the Hamiltonian in Eq. (D.3.1) with $N_{\parallel} = 50$ and $N_{\perp} = 50$ sites. The colors represent different boundary conditions: ($\parallel$ -PBC, $\perp$ -PBC) (blue), ($\parallel$ -OBC, $\perp$ -PBC) (orange), and ($\parallel$ -OBC, $\perp$ -OBC) (black). Parameters are set to $t = 1.0$, $t_2 = 0.4$, $\Delta = 0.15$, and $g = 0.3$. Due to the reflection symmetry, localization along the $x_{\perp}$ direction is strictly prohibited, which eliminates spectral sensitivity to boundary conditions in that direction. Consequently, the choice of boundary conditions along $x_{\perp}$ becomes irrelevant to the spectral distribution, as evidenced by the perfect overlap of the orange and black data points.

D.3 Numerical verification

In this section, we validate the preceding sections numerically. To end this, we investigate the two-dimensional symplectic Hatano-Nelson model with next-nearest hopping. The non-Bloch Hamiltonian is defined as

$$h(\beta_x, \beta_y) = t(\beta_x + \beta_x^{-1}) + t(\beta_y + \beta_y^{-1}) + t_2(\beta_x + \beta_x^{-1})(\beta_y + \beta_y^{-1}) + (i\Delta X + gZ)(\beta_x - \beta_x^{-1}) + (i\Delta X + gZ)(\beta_y - \beta_y^{-1}), \quad (\text{D.3.1})$$

where X, Y and Z are Pauli matrices.

This model satisfies the reflection symmetry $Rh(\beta_x, \beta_y)R^{-1} = h(\beta_y, \beta_x)$ with $R = \mathbb{I}$, and respects $\text{TRS}^{\dagger}$ defined by $U_T = Y$. The t_2 term introduces a nontrivial coupling between the x and y directions, ensuring that the system is truly two-dimensional while preserving the diagonal reflection symmetry.

Figure D.1 compares the complex energy spectra under various boundary conditions with parameters set as $t = 1.0$, $t_2 = 0.4$, $\Delta = 0.15$, and $g = 0.3$. We consider a two-dimensional

lattice with $N_{\parallel} = 50$ and $N_{\perp} = 50$ sites, governed by the Hamiltonian in Eq. (D.3.1). The figure displays three types of spectra: the full periodic boundary conditions ($\parallel$ -PBC, $\perp$ -PBC; blue), the partial open boundary conditions ($\parallel$ -OBC, $\perp$ -PBC; orange), and the full open boundary conditions ($\parallel$ -OBC, $\perp$ -OBC; black). A striking feature of this result is the perfect overlap between the orange and black data points. This numerical consistency provides direct evidence for the role of reflection symmetry in two-dimensional non-Hermitian systems. As discussed in the previous section, the reflection symmetry R effectively prohibits the formation of the NHSE along the $x_{\perp}$ direction. Consequently, the system exhibits no spectral sensitivity to the boundary conditions imposed along $x_{\perp}$, rendering the choice between PBC and OBC in this direction irrelevant to the final spectral distribution. More importantly, this result signifies that the reflection symmetry effectively resolves the ambiguity in the hierarchy of thermodynamic limits. However, our formulation demonstrates that the presence of a high-symmetry axis (the reflection line) dictates a preferred direction for the non-Bloch generalization, ensuring a robust and unique definition of the two-dimensional non-Bloch spectrum. Thus, the spectral potential obtained via the WHF-based optimization [Eq. (5.5.16)] accurately captures the thermodynamic limit of the system regardless of the specific boundary configuration along the transverse direction.

To numerically verify Eq. (D.2.8), we apply the WHF-based Amoeba method to the two-dimensional symplectic model defined in Eq. (D.3.1). We first examine the behavior of the Ronkin functions for specific complex energies E to see how the WHF framework identifies the topological properties of each $k_{\perp}$ slice.

Figure D.2 compares the optimized values of the conventional total Ronkin function and the symmetry-decomposed Ronkin function for two representative reference energies. Blue and light-blue curves show the total Ronkin function $\min_{\mu} R_{\tau}(\mu; k_{\perp})$, while red and light-red curves represent the symmetry-decomposed Ronkin function $2 \min_{\mu} R_{\tau}^{(+)}(\mu; k_{\perp})$. The color intensity distinguishes the $\mathbb{Z}_2$ topological character of each effective one-dimensional Hamiltonian: solid red and light-blue segments correspond to the $\mathbb{Z}_2$ nontrivial sector ($\nu = 1$), whereas light-red and blue segments indicate the trivial sector ($\nu = 0$). The black dashed lines indicate the resulting OBC spectral potential $\phi_{\tau}(k_{\perp})$.

For $E = 1.6 - 0.2i$ [Fig. D.2(a)], the $\mathbb{Z}_2$ invariant $\nu(k_{\perp})$ of the effective one-dimensional Hamiltonian depends on $k_{\perp}$ and switches between the trivial ($\nu = 0$) and nontrivial ($\nu = 1$) sectors. Accordingly, the OBC spectral potential $\phi_{\tau}(k_{\perp})$ is obtained from different optimized Ronkin functions: in the trivial sector around $k_{\perp} \simeq \pi/2$ ($\nu = 0$), $\phi_{\tau}(k_{\perp})$ coincides with the conventional total Ronkin function $\min_{\mu} R_{\tau}(\mu; k_{\perp})$, whereas in the nontrivial sector ($\nu = 1$) it is dictated by the symmetry-decomposed Ronkin function $2 \min_{\mu} R_{\tau}^{(+)}(\mu; k_{\perp})$. This is directly confirmed in Fig. D.2(a), where the black dashed curve follows the blue branch near $k_{\perp} \simeq \pi/2$ and the red branch elsewhere.

In contrast, Fig. D.2(b) shows the case for $E = 2.61.7i$. In this case, the effective one-dimensional Hamiltonian remains in the $\mathbb{Z}_2$ trivial phase ($\nu = 0$) for all values of $k_{\perp}$. Ac-

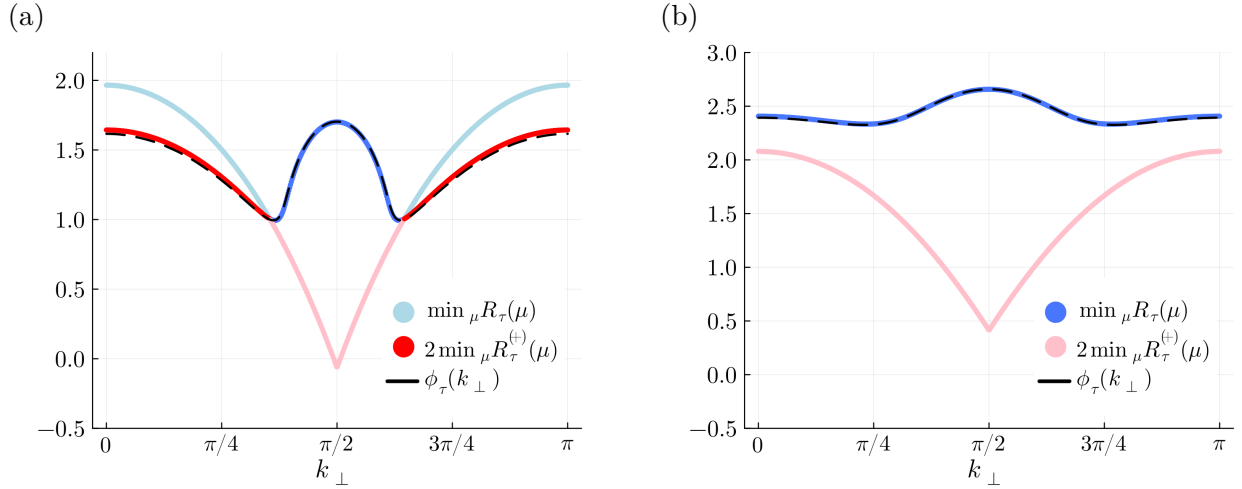


Figure D.2: Numerical validation of the WHF-based decomposition across $k_{\perp}$. The curves show the conventional total Ronkin function $\min_{\mu} R_{\tau}(\mu; k_{\perp})$ (blue/light blue) and the symmetry-decomposed Ronkin function $2 \min_{\mu} R_{\tau}^{(+)}(\mu; k_{\perp})$ (red/light red). The black dashed curve indicates the optimized OBC spectral potential $\phi_{\tau}(k_{\perp})$. (a) Evaluation at $E = 1.60.2i$. At this energy, the $\mathbb{Z}_2$ invariant of the effective one-dimensional Hamiltonian depends on $k_{\perp}$ and is nontrivial ($\nu = 1$) over most of the $k_{\perp}$ range. Accordingly, the OBC spectral potential is determined by the symmetry-decomposed Ronkin function in the nontrivial sector, while it coincides with the total Ronkin function in the trivial sector around $k_{\perp} \simeq \pi/2$. (b) Evaluation at $E = 2.61.7i$. In this case, the system remains in the $\mathbb{Z}_2$ trivial phase ($\nu = 0$) for all $k_{\perp}$, and the spectral potential is entirely captured by the conventional total Ronkin function.

cordingly, the OBC spectral potential $\phi_{\tau}(k_{\perp})$ is entirely determined by the conventional total Ronkin function $\min_{\mu} R_{\tau}(\mu; k_{\perp})$. This is clearly confirmed by the black dashed curve, which coincides with the blue branch over the entire $k_{\perp}$ range, while the symmetry-decomposed Ronkin function $2 \min_{\mu} R_{\tau}^{(+)}(\mu)$ (light red) does not contribute to the OBC physics.

The switching between the total and symmetry-decomposed Ronkin functions, equivalently the change of the $\mathbb{Z}_2$ invariant $\nu(k_{\perp})$, is governed by the behavior of the root of $\det \tau = 0$ with the smallest modulus. Specifically, the transition occurs when the innermost root $|\beta_1| = e^{\mu_1}$ crosses the unit circle, accompanied by a sign change of $\mu_1 = \ln |\beta_1|$.

By integrating these optimized potentials $\phi_{\tau}(k_{\perp})$ over the entire $k_{\perp}$ Brillouin zone, we reconstruct the full spectral density in the complex energy plane. Figure D.3 presents the resulting OBC spectrum and the corresponding DOS. The DOS, evaluated from the Laplacian of the spectral potential $\phi(E)$, shows an exact correspondence with the numerical OBC spectrum obtained via direct diagonalization.

These results demonstrate that the WHF-based Amoeba formulation correctly captures

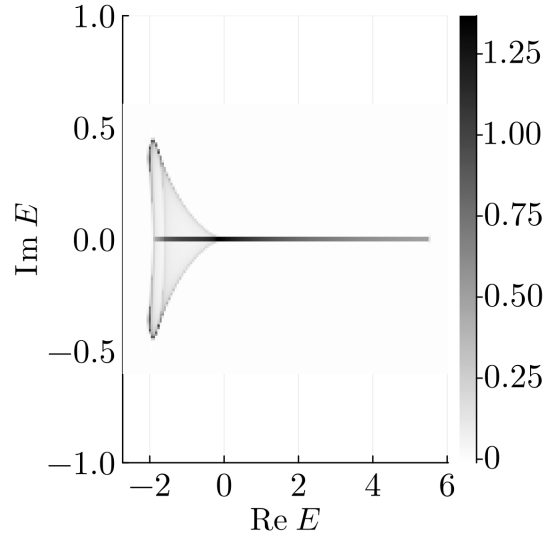


Figure D.3: DOS multiplied by a surface element $\Delta \text{Re } E \times \Delta \text{Im } E \simeq 6 \times 10^{-4}$. The Hamiltonian and parameters are identical to those in Fig. D.1. The OBC potential is evaluated by integrating the potentials for each slice of $k_{\perp}$.

the thermodynamic OBC spectral properties of the present two-dimensional system. A crucial ingredient in this construction is the presence of reflection symmetry, which allows the problem to be reduced to a family of effective one-dimensional Hamiltonians labeled by $k_{\perp}$. Extending this formulation to more general higher-dimensional geometries without such symmetry, where no natural dimensional reduction is available, remains an important open problem for future work.

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