

Coanalytic sets with Borel sections

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Fact. (Fujita and Mátrai) *Let $B \subset \mathbf{R} \times \mathbf{R}$ be a Borel set such that the horizontal section B^y is Σ_α^0 for every $y \in \mathbf{R}$. Then there is a dense G_δ set $D \subset \mathbf{R}$ such that $B \cap (\mathbf{R} \times D)$ is $\Sigma_\alpha^0 \upharpoonright \mathbf{R} \times D$.*

This can be proved by a straightforward induction using A.Louveau's solution ([Lo]) of the section problem of Borel sets. This fact has been used in order to solve an old problem by M.Laczkovich about differences of Borel measurable function (See [FM].)

Theorem. *The following statements are equivalent:*

- (1) *If $A \subset \mathbf{R} \times \mathbf{R}$ is Π_1^1 and all horizontal sections A^y are Borel, then there is a dense G_δ set $D \subset \mathbf{R}$ such that $A \cap (\mathbf{R} \times D)$ is Borel;*
- (2) *similar, but A^y are Σ_α^0 and $A \cap (\mathbf{R} \times D)$ is $\Sigma_\alpha^0 \upharpoonright \mathbf{R} \times D$;*
- (3) *similar, but A^y are closed and $A \cap (\mathbf{R} \times D)$ is Borel;*
- (4) $\text{BP}(\Sigma_2^1)$, i.e., every Σ_2^1 set of reals has the property of Baire. ◀

PROOF. From (1) to (2): use FACT.

From (2) to (3): immediate from the case $\alpha = 1$ of (2).

From (3) to (4): given Σ_2^1 set $P \subset \mathbf{R}$, let $A \subset \mathbf{R} \times \mathbf{R}$ be Π_1^1 such that $y \in P \iff \exists x [\langle x, y \rangle \in A]$. Uniformize A by a function $f : P \rightarrow \mathbf{R}$ with Π_1^1 graph. Apply (3) to the graph of f . Then $P \cap D$ is Σ_1^1 and D is co-meager. So P has BP.

From (4) to (1): this is the main part of this note.

Let $\mathbb{C}$ be the Cohen poset. Given a transitive model M of set theory, let $\text{Co}(M)$ be the set of all $\mathbb{C}$ -generic reals over M .

Lemma A. (Solovay) $\text{BP}(\Sigma_2^1)$ holds if and only if $\text{Co}(L[r])$ is co-meager for every $r \in \mathbf{R}$. ◀

Let WO be set the of $w \in {}^\omega 2$ which codes a wellordering on ω . For each $q \in \text{WO}$ let $\|w\|$ be the order-type (i.e., countable ordinal) that w codes.

Definition. A set $X \subset \mathbf{R} \times \omega_1$ is Π_2^1 in the codes if the set

$$\left\{ \langle x, w \rangle \in \mathbf{R} \times {}^\omega 2 \mid w \in \text{WO}, \langle x, \|w\| \rangle \in X \right\}$$

is (lightface) Π_2^1 . ◀

Lemma B. Let $X \subset \mathbf{R} \times \omega_1$ be Π_2^1 in the codes. Suppose that for every $y \in \mathbf{R}$ there is $\xi < \omega_1$ such that $\langle y, \xi \rangle \in X$. Then there is a countable ordinal δ such that for every $c \in \text{Co}(L)$ there is $\xi < \delta$ such that $\langle c, \xi \rangle \in X$. ◀

PROOF OF (4) $\implies$ (1) [taking Lemmas for granted]. We put $\mathbf{R} = {}^\omega \omega$ and assume A is lightface Π_1^1 . Let $f : \mathbf{R} \times \mathbf{R}$ be a recursive function such that $A = f^{-1}[\text{WO}]$. Since A^y is Borel, the image $f[A^y \times \{y\}]$ is bounded in WO , that is to say,

$$\forall y \in \mathbf{R} \exists \xi < \omega_1 \forall x \left[\langle x, y \rangle \in A \implies \|f(x, y)\| < \xi \right].$$

For each $\xi < \omega_1$ set

$$\text{WO}_\xi = \left\{ w \in \text{WO} \mid \|w\| < \xi \right\}$$

and let

$$X = \left\{ \langle y, \xi \rangle \mid f[A^y \times \{y\}] \subset \text{WO}_\xi \right\}.$$

Observe that X is Π_2^1 in the codes. Applying LEMMA B we find a countable ordinal δ such that

$$\forall c \in \text{Co}(L) \exists \xi < \delta \left[\langle c, \xi \rangle \in X \right].$$

Then we have

$$A \cap (\mathbf{R} \times \text{Co}(L)) = f^{-1}[\text{WO}_\delta] \cap (\mathbf{R} \times \text{Co}(L)).$$

By LEMMA A there is a dense G_δ set $D \subset \text{Co}(L)$. ◀

PROOF OF LEMMA B. Let $\varphi(y, w)$ be a Π_2^1 formula such that

$$\begin{aligned} \langle y, \xi \rangle \in X &\iff \exists w \in \text{WO} \left[\xi = \|w\| \wedge \varphi(y, w) \right] \\ &\iff \forall w \in \text{WO} \left[\xi = \|w\| \implies \varphi(y, w) \right]. \end{aligned}$$

Then we have, by the assumption of the lemma,

$$(*) \quad \forall y \exists \xi < \omega_1 \forall w \left[w \in \text{WO} \wedge \|w\| = \xi \implies \varphi(y, w) \right].$$

Let $\varphi^*(y, \xi)$ stand for “ $\forall w \dots$ ” part of $(*)$. Then $\varphi^*(y, \xi)$ is absolute for every proper class model in which ξ is countable.

Let $c \in \text{Co}(L)$ and suppose that $\langle c, \xi \rangle \in X$. Let $g : \omega \rightarrow \xi$ be $\text{Coll}(\xi)$ -generic over $L[c]$. Then

$$L[c, g] \models \varphi^*(c, \xi)$$

so that there are forcing conditions $p \in \mathbb{C}$ and $q \in \text{Coll}(\xi)$ such that c meets p , g meets q and

$$\langle p, q \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} L[\dot{c}, \dot{g}] \models \varphi^*(\dot{c}, \check{\xi}).$$

Then by absoluteness of forcing relations,

$$L \models \left[\langle p, q \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} \varphi^*(\dot{c}, \check{\xi}) \right].$$

By homogeneity of the poset $\text{Coll}(\xi)$,

$$L \models \left[\langle p, \emptyset \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} \varphi^*(\dot{c}, \check{\xi}) \right].$$

where $\emptyset$ is the weakest member of $\text{Co}(\xi)$.

For each $\xi < \omega_1$ let

$$Y_\xi = \left\{ p \in \mathbb{C} \mid L \models \left[\langle p, \emptyset \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} \varphi^*(\dot{c}, \check{\xi}) \right] \right\}.$$

Then $\bigcup_{\xi < \omega_1} Y_\xi$ is pre-dense in $\mathbb{C}$. By ccc, there is $\delta < \omega_1$ such that $\bigcup_{\xi < \delta} Y_\xi$ is already pre-dense in $\mathbb{C}$. $\blacktriangleleft$

Daisuke Ikegami observed that $\mathbb{C}$ in LEMMA B can be replaced by other alboreal Suslin ccc forcing notions. Daisuke also pointed out that Sacks forcing does not satisfy LEMMA B nor clause (3) of THEOREM.

By Montgomery's result on the category quantifier, we obtain

Corollary. *Assume $\text{BP}(\Sigma_2^1)$. Let $A \subseteq \mathbf{R} \times \mathbf{R}$ be Π_1^1 such that A^y is Σ_α^0 for every $y \in \mathbf{R}$. Then the set*

$$\exists^* A = \left\{ x \in \mathbf{R} \mid A_x \text{ is non-meager} \right\}$$

is Σ_α^0 . ◀

Question. Does the last statement imply $\text{BP}(\Sigma_2^1)$?

References

[FM] H.Fujita and T.Mátrai, *On the difference property of Borel measurable functions*, Fund. Math. to appear.

[Lo] A.Louveau, *A separation theorem for Σ_1^1* , Trans. Amer. Math. Soc. **260** (1980), 363–378.