

On Some Doubly Infinite, Finite and Mixed Sums derived from The N- Fractional Calculus of A Power Function

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Abstract

In a previous paper, some doubly infinite, finite and mixed sums are reported using the N- fractional calculus $((z-c)^{\alpha+\beta})_\gamma$ by the author and his colleagues.

In this article the same doubly infinite sums in a previous paper are discussed again using $((z-c)^\beta \cdot (z-c)^\alpha)_\gamma$, the N-fractional calculus of products of power functions.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}^+), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in \mathbb{C}$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) Lemma. We have [1]

$$(i) \quad ((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii). (Γ ; Gamma function),

$$(iv) \quad (u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \begin{pmatrix} u = u(z), \\ v = v(z) \end{pmatrix}.$$

§ 1. Doubly Infinite, Finite and Mixed Infinite Sums

In the following $\alpha, \beta, \gamma \in \mathbb{R}$.

Theorem 1. *Let*

$$L(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(k-\alpha+m)\Gamma(\gamma-\beta-m)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(k-\alpha)\Gamma(-\beta)}. \quad (1)$$

(i) When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$, we have the following doubly infinite sums;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma; k, m) \left(\frac{-c}{z}\right)^k \left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\alpha}, \quad (2)$$

where

$$Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi \beta \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \beta)} \quad (|Q| = M < \infty), \quad (3)$$

$$|-c/z| < 1, \quad |(z-c)/z| < 1,$$

and

$$\left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right|, \quad \left| \frac{\Gamma(\gamma - \beta - m)}{\Gamma(-\beta)} \right| < \infty.$$

(ii) When $\alpha, \beta \notin \mathbb{Z}^+$, we have the following mixed infinite sums;

$$\sum_{k=0}^{\infty} \sum_{m=0}^s L(\alpha, \beta, s; k, m) \left(\frac{-c}{z}\right)^k \left(\frac{z-c}{z}\right)^m = Q(\alpha, \beta; s) \frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z}\right)^{\alpha}, \quad (4)$$

for $s \in \mathbb{Z}^+$ where

$$|-c/z| < 1, \quad |(z-c)/z| < \infty,$$

and

$$\left| \frac{\Gamma(s-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right|, \quad \left| \frac{\Gamma(s-m-\beta)}{\Gamma(-\beta)} \right| < \infty.$$

Proof of (i). We have

$$(z-c)^{\alpha} = z^{\alpha} \left(1 - \frac{c}{z}\right)^{\alpha} \quad (5)$$

$$= z^{\alpha} \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} z^{-k} \quad (|z| > |c|) \quad (6)$$

$$= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} z^{\alpha-k}. \quad (7)$$

Next make $(z-c)^\beta \times (7)$, then operate N^γ to its both sides, we obtain

$$((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} ((z-c)^\beta \cdot z^{\alpha-k})_\gamma \quad (8)$$

$$= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+1)}{m! \Gamma(\gamma+1-m)} ((z-c)^\beta)_{\gamma-m} (z^{\alpha-k})_m. \quad (9)$$

Now we have

$$((z-c)^\beta)_{\gamma-m} = e^{-i\pi(\gamma-m)} \frac{\Gamma(\gamma-m-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\gamma+m}, \quad (10)$$

$$\left(\left| \frac{\Gamma(\gamma-m-\beta)}{\Gamma(-\beta)} \right| < \infty \right)$$

and

$$(z^{\alpha-k})_m = e^{-i\pi m} \frac{\Gamma(m+k-\alpha)}{\Gamma(k-\alpha)} z^{\alpha-k-m}. \quad (11)$$

respectively.

On the other hand we have

$$((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k! \Gamma(\gamma+1-k)} ((z-c)^\beta)_{\gamma-k} ((z-c)^\alpha)_k \quad (12)$$

$$= e^{-i\pi\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1) \Gamma(\gamma-\beta-k) \Gamma(k-\alpha)}{k! \Gamma(\gamma+1-k) \Gamma(-\beta) \Gamma(-\alpha)} (z-c)^{\alpha+\beta-\gamma}. \quad (13)$$

$$= e^{-i\pi\gamma} \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)} (z-c)^{\alpha+\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [-\gamma]_k}{k! [1+\beta-\gamma]_k} \quad (14)$$

since

$$((z-c)^\beta)_{\gamma-k} = e^{-i\pi(\gamma-k)} \frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\gamma+k}, \quad (15)$$

$$\left(\left| \frac{\Gamma(\gamma-k-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

$$((z-c)^\alpha)_k = e^{-i\pi k} \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)} (z-c)^{\alpha-k}, \quad (16)$$

and

$$\Gamma(\lambda+1-k) = (-1)^{-k} \frac{\Gamma(\lambda+1) \Gamma(-\lambda)}{\Gamma(k-\lambda)} \quad (17)$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \text{ with } [\lambda]_0 = 1$$

(notation of Pochhammer).

Next we have the identity

$$\sum_{k=0}^{\infty} \frac{[a]_k [b]_k}{k! [c]_k} = {}_2F_1(a, b; c; 1) \quad (18)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \left(\begin{array}{l} \operatorname{Re}(c-a-b) > 0 \\ c \notin \mathbf{Z}_0^- \end{array} \right). \quad (19)$$

Therefore, we have

$$((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\beta)}{\Gamma(-\beta)} (z-c)^{\alpha+\beta-\gamma} {}_2F_1(-\alpha, -\gamma; 1+\beta-\gamma; 1) \quad (20)$$

$$= e^{-i\pi\gamma} \frac{\Gamma(\gamma-\beta)\Gamma(1+\alpha+\beta)\Gamma(1+\beta-\gamma)}{\Gamma(-\beta)\Gamma(1+\beta)\Gamma(1+\alpha+\beta-\gamma)} (z-c)^{\alpha+\beta-\gamma} \quad (21)$$

$$\left(\begin{array}{l} \operatorname{Re}(\alpha+\beta+1) > 0 \\ (1+\beta-\gamma) \notin \mathbf{Z}_0^- \end{array} \right)$$

$$= e^{-i\pi\gamma} \frac{\sin \pi \beta \cdot \sin \pi(\gamma-\alpha-\beta)}{\sin \pi(\alpha+\beta) \cdot \sin \pi(\gamma-\beta)} \cdot \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} (z-c)^{\alpha+\beta-\gamma} \quad (22)$$

$$= e^{-i\pi\gamma} Q(\alpha, \beta, \gamma) \cdot \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} (z-c)^{\alpha+\beta-\gamma} \quad (23)$$

from (14), because we have the identity

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbf{Z}). \quad (24)$$

Therefore, substituting (23), (10) and (11) into (9) we obtain

$$\begin{aligned} & Q(\alpha, \beta, \gamma) \cdot \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} (z-c)^{\alpha+\beta-\gamma} \\ &= \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(\gamma-m-\beta)\Gamma(m+k-\alpha)}{m! \Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)} \\ & \quad \times (z-c)^{\beta-\gamma+m} z^{\alpha-k-m}, \end{aligned} \quad (25)$$

we have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L(\alpha, \beta, \gamma; k, m) \left(\frac{-c}{z} \right)^k \left(\frac{z-c}{z} \right)^m = Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z} \right)^\alpha, \quad (2)$$

from (25), using the notation (1), under the conditions.

Proof of (ii). Set $\gamma = s \in \mathbb{Z}^+$ in (2), we have then (4) clearly under the conditions.

Corollary 1. When $r, s \in \mathbb{Z}^+$ we have the following doubly finite sums;

$$\sum_{k=0}^r \sum_{m=0}^s L(r, \beta, s; k, m) \left(\frac{-c}{z} \right)^k \left(\frac{z-c}{z} \right)^m = Q(r, \beta, s) \frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)} \left(\frac{z-c}{z} \right)^r, \quad (26)$$

where

$$|-c/z|, |(z-c)/z| < \infty,$$

and

$$\left| \frac{\Gamma(s-r-\beta)}{\Gamma(-r-\beta)} \right|, \left| \frac{\Gamma(s-\beta-m)}{\Gamma(-\beta)} \right| < \infty.$$

Proof. Set $\alpha = r$ and $\gamma = s$ in (2) we have then this corollary clearly.

§ 2. Direct calculation of the doubly infinite sums

The direct calculation (without the use of N-fractional calculus) of the doubly infinite sum in the LHS of § 1. (2) is shown as follows.

Theorem 2. Let

$$L = L(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\gamma-\beta-m)\Gamma(k-\alpha+m)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}, \quad (1)$$

and

$$Q = Q(\alpha, \beta, \gamma) := \frac{\sin \pi \beta \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \beta)} \quad (|Q(\alpha, \beta, \gamma)| = M < \infty). \quad (2)$$

We have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \cdot \left(\frac{z-c}{z} \right)^m \left(\frac{-c}{z} \right)^k = Q \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left(\frac{z-c}{z} \right)^{\alpha}, \quad (3)$$

where

$$|c/z| < 1,$$

and

$$(\alpha + \beta), (\gamma - \beta), (\gamma - \alpha - \beta) \notin \mathbb{Z}.$$

Proof. Now we have

$$L \cdot \left(\frac{z-c}{z} \right)^m \left(\frac{-c}{z} \right)^k = \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \cdot \frac{[-\alpha]_{k+m} [-\gamma]_m}{k! \cdot m! [1 + \beta - \gamma]_m} \left(\frac{c}{z} \right)^k \left(\frac{z-c}{z} \right)^m \quad (4)$$

using the identity

$$\Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1)\Gamma(-\lambda)}{\Gamma(k - \lambda)} \quad (5)$$

and

$$[-\alpha]_{k+m} = [-\alpha]_m [-\alpha + m]_k. \quad (6)$$

We have then

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \cdot \left(\frac{-c}{z}\right)^k \left(\frac{z-c}{z}\right)^m &= \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \\ &\times \sum_{m=0}^{\infty} \frac{[-\alpha]_m [-\gamma]_m}{m! [1 + \beta - \gamma]_m} \left(\frac{z-c}{z}\right)^m \sum_{k=0}^{\infty} \frac{[-\alpha + m]_k}{k!} \left(\frac{c}{z}\right)^k \end{aligned} \quad (7)$$

$$= \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \left(\frac{z-c}{z}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{[-\alpha]_m [-\gamma]_m}{m! [1 + \beta - \gamma]_m} \quad (8)$$

$$= \frac{\Gamma(\gamma - \beta)}{\Gamma(-\beta)} \left(\frac{z-c}{z}\right)^{\alpha} {}_2F_1(-\alpha, -\gamma; 1 + \beta - \gamma; 1) \quad (9)$$

$$= \frac{\Gamma(\gamma - \beta)\Gamma(1 + \beta - \gamma)\Gamma(1 + \alpha + \beta)}{\Gamma(-\beta)\Gamma(1 + \beta)\Gamma(1 + \alpha + \beta - \gamma)} \left(\frac{z-c}{z}\right)^{\alpha}, \quad (10)$$

where

$$\left|\frac{-c}{z}\right| < 1, \quad \left|\frac{z-c}{z}\right| < 1, \quad \operatorname{Re}(\alpha + \beta) > -1.$$

Because we have

$$\sum_{k=0}^{\infty} \frac{[-\alpha + m]_k}{k!} \left(\frac{c}{z}\right)^k = \left(\frac{z-c}{z}\right)^{\alpha-m} \quad (11)$$

since

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}, \quad (12)$$

and

$${}_2F_1(a, b; c; 1) = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m}{m! [c]_m} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \left(\begin{array}{l} \operatorname{Re}(c-a-b) > 0 \\ c \notin \mathbb{Z}_0^- \end{array} \right). \quad (13)$$

Moreover we have the identity

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbb{Z}), \quad (14)$$

then applying (14) to (10) we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} L \cdot \left(\frac{z-c}{z}\right)^m \left(\frac{-c}{z}\right)^k = Q \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left(\frac{z-c}{z}\right)^{\alpha}. \quad (15)$$

§ 3. Commentary

[I] In a previous paper, the results obtained by the author are derived by the use of $\left((z-c)^{\alpha+\beta}\right)_\gamma$, however the results shown in this article, the N- fractional calculus $\left((z-c)^\beta \cdot (z-c)^\alpha\right)_\gamma$ is used

{ II } When $Q = Q(\alpha, \beta, \gamma) = 1$, § 1. (2) overlaps Theorem 2 obtained in a previous paper .[11]

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