

ON DIRECT SUM BANACH SPACES AND UNIFORM NON-SQUARENESS

千葉大学社会文化科学研究科 田村高幸 (Takayuki TAMURA)
九州工業大学工学部 加藤幹雄 (Mikio KATO)
新潟大学理学部 斎藤吉助 (Kichi-Suke SAITO)

Recently the strict convexity and the uniform convexity of the ψ -direct sum $X \oplus_\psi Y$ of Banach spaces X and Y were characterized in [15, 12]. We shall characterize the uniform non-squareness of $X \oplus_\psi Y$.

Let N_a denote the family of all absolute normalized norms on $\mathbb{C}^2$, that is,

$$\|(z, w)\| = \||z|, |w|\| \quad \text{and} \quad \|(1, 0)\| = \|(0, 1)\| = 1,$$

and let Ψ denote the family of all continuous convex functions ψ on $[0, 1]$ with $\psi(0) = \psi(1) = 1$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ ($0 \leq t \leq 1$). According to [3], the norms in N_a and the convex functions in Ψ correspond in a one-to-one way under the equation $\psi(t) = \|(1-t, t)\|$. Namely, for every element $\|\cdot\| \in N_a$ the function $\psi(t)$ defined by $\psi(t) = \|(1-t, t)\|$ belongs to Ψ ; and conversely for every element $\psi \in \Psi$, define

$$(1) \|(z, w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0). \end{cases}$$

Then $\|(\cdot, \cdot)\|_\psi$ is a norm in N_a and satisfies $\psi(t) = \|(1-t, t)\|_\psi$.

In [15], the ψ -direct sum $X \oplus_\psi Y$ of two Banach spaces X and Y was introduced as the direct sum $X \oplus Y$ with the norm $\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$ ($x \in X, y \in Y$). Recently the strict

convexity and the uniform convexity of $X \oplus_\psi Y$ were characterized in [15, 12]. In this note we characterize the uniform non-squareness of $X \oplus_\psi Y$. As an application we give an example of Banach spaces which are not uniformly convex but uniformly non-square.

Now recall that a Banach space X is called *uniformly non-square* ([6]; cf. [2, 10]) provided there exists a δ ($0 < \delta < 1$) such that, whenever $\|(x - y)/2\| > 1 - \delta$, $\|x\| = \|y\| = 1$, one has $\|(x+y)/2\| \leq 1 - \delta$. X is called *strictly convex* provided, if $\|x\| = \|y\| = 1$, $x \neq y$, then $\|\frac{x+y}{2}\| < 1$. X is called *uniformly convex* if any $\epsilon > 0$ there is a δ ($0 < \delta < 1$) such that, whenever $\|x - y\| \geq \epsilon$, $\|x\| \leq 1$, $\|y\| \leq 1$, one has $\|\frac{x+y}{2}\| < 1 - \delta$. As is well known, the notion of uniform non-squareness lies between uniform convexity and super-reflexivity. Also, it is well known that there exists a Banach space which is neither uniformly convex nor uniformly non-square but super-reflexive. (cf. [7], [1].) A function ψ on $[0, 1]$ is called *strictly convex* if, for any $s, t \in [0, 1]$, $s \neq t$, and for any c ($0 < c < 1$), one has $\psi((1-c)s + ct) < (1-c)\psi(s) + c\psi(t)$.

THEOREM A ([15, 12]). *Let X and Y be Banach spaces and let $\psi \in \Psi$. Then*

- (i) *$X \oplus_\psi Y$ is strictly convex if and only if X and Y are strictly convex, and ψ is strictly convex ([15, Theorem 1]).*
- (ii) *$X \oplus_\psi Y$ is uniformly convex if and only if X and Y are uniformly convex, and ψ is strictly convex ([12, Theorem 1]).*

Saito-Kato-Takahashi [13] gave the following characterization of the absolute norms on $\mathbb{C}^2$ which are uniformly non-square.

Proposition 1 ([13]). *Let $\psi \in \Psi$. Then the following are equivalent.*

- (i) $(\mathbb{C}^2, \|\cdot\|_\psi)$ is uniformly non-square.
- (ii) $\psi \neq \psi_1$ and $\psi \neq \psi_\infty$.

1. Monotonicity Property of Absolute Norms

We discuss the monotonicity property of absolute norms on $\mathbb{C}^2$ for later use. Recall the following fundamental facts. Proposition 2 played an essential role in the proof of Theorem A.

Lemma 1 ([2, p.36, Lemma 2]). *Let $\|\cdot\| \in N_a$.*

- (i) *If $|p| \leq |r|$ and $|q| \leq |s|$, then $\|(p, q)\| \leq \|(r, s)\|$.*
- (ii) *If $|p| < |r|$ and $|q| < |s|$, then $\|(p, q)\| < \|(r, s)\|$.*

Proposition 2 (Takahashi, Kato and Saito [15]). *Let $\psi \in \Psi$. Then the following assertions are equivalent:*

- (i) *If $|z| \leq |u|$ and $|w| < |v|$, or $|z| < |u|$ and $|w| \leq |v|$, then $\|(z, w)\|_\psi < \|(u, v)\|_\psi$.*
- (ii) $\psi(t) > \psi_\infty(t)$ for all $t \in (0, 1)$.

A more precise (component-wise) result is given in [15]. Next we present a condition on (z, w) and (u, v) for which the above assertion (i) is valid (component-wise) for a general $\psi \in \Psi$.

Proposition 3. *Let $\psi \in \Psi$ and let $(z, w), (u, v) \in \mathbb{C}^2$.*

- (i) *Let $|z| < |u|$ and $|w| = |v|$. Then $\|(z, w)\|_\psi = \|(u, v)\|_\psi$ if and only if $\|(z, w)\|_\psi = |w|$.*
- (ii) *Let $|z| = |u|$ and $|w| < |v|$. Then $\|(z, w)\|_\psi = \|(z, v)\|_\psi$ if and only if $\|(z, w)\|_\psi = |z|$.*

Proposition 3 is important in the proof of the uniform non-squariness of $X \oplus_\psi Y$.

3. Uniform Non-squareness of $X \oplus_\psi Y$

We need the following lemma.

Lemma 2. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Banach space X whose norms are convergent to non-zero limits.*

- (i) $\lim_{n \rightarrow \infty} \|x_n + y_n\| = \lim_{n \rightarrow \infty} (\|x_n\| + \|y_n\|).$
- (ii) $\lim_{n \rightarrow \infty} \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| = 2.$

By Proposition 3 and Lemma 2, we obtain the following main theorem.

Theorem 1. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*

- (i) $X \oplus_\psi Y$ is uniformly non-square.
- (ii) X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.

Now consider the Lorentz $\ell_{p,q}$ -norm $\|\cdot\|_{p,q}$,
 $1 \leq q \leq p \leq \infty$, $q < \infty$:

$$\|(z_1, z_2)\|_{p,q} = \left\{ z_1^{*q} + 2^{(q/p)-1} z_2^{*q} \right\}^{1/q},$$

where $\{z_1^*, z_2^*\}$ is the non-increasing rearrangement of $\{|z_1|, |z_2|\}$. (Note that in case of $1 \leq p < q \leq \infty$, $\|\cdot\|_{p,q}$ is not a norm but a quasi-norm (cf. [8], [16, p.126]). Clearly $\|\cdot\|_{p,q}$ is an absolute normalized norm and the corresponding convex function $\psi_{p,q}$ is given by

$$(2) \quad \psi_{p,q}(t) = \begin{cases} \{(1-t)^q + 2^{q/p-1} t^q\}^{1/q} & \text{if } 0 \leq t \leq 1/2, \\ \{t^q + 2^{q/p-1} (1-t)^q\}^{1/q} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Then $\psi_{p,q}$ yields the $\ell_{p,q}$ -sum $X \oplus_{p,q} Y$:

$$(3) \quad \|(x, y)\|_{p,q} = \left\{ \max(\|x\|^q, \|y\|^q) + 2^{(q/p)-1} \min(\|x\|^q, \|y\|^q) \right\}^{1/q}$$

COROLLARY 1. *Let $1 \leq q \leq p \leq \infty$ and not $p = q = 1, \infty$. Then, $\ell_{p,q}$ -sum $X_1 \oplus_{p,q} X_2$ is uniformly non-square if and only if X_1 and X_2 are uniformly non-square.*

In particular, ℓ_p -sum $X_1 \oplus_p X_2$, $1 < p < \infty$, is uniformly non-square if and only if X_1 and X_2 are uniformly non-square.

Theorem A and Theorem 1 easily gives an example of Banach spaces which are not uniformly convex but uniformly non-square.

EXAMPLE 1 (cf. [12, 13]). Let X and Y be uniformly convex Banach space and let $1/2 < \alpha < 1$. Now we define $\psi_\alpha \in \Psi$ by

$$(4) \quad \psi_\alpha(t) = \begin{cases} \frac{\alpha-1}{\alpha}t + 1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases}$$

Then the norm of $X \oplus_{\psi_\alpha} Y$ is given by

$$(5) \quad \|(x, y)\|_{\psi_\alpha} = \max\{\|x\| + (2 - \frac{1}{\alpha})\|y\|, \|y\|\}.$$

$X \oplus_{\psi_\alpha} Y$ is an example of uniformly non-square Banach spaces without uniform convexity.

References

- [1] J.-B. Baillon and R. Schoneberg, Asymptotic normal structure and fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.*, **81** (1981), 257–264.
- [2] B. Beauzamy, *Introduction to Banach Spaces and their Geometry*, 2nd ed., North-Holland, 1985.
- [3] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10** (1973).
- [4] R. Bhatia, *Matrix Analysis*, Springer, 1997.
- [5] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1967.
- [6] C. James, Uniformly non-square Banach spaces, *Ann. of Math.* **80** (1964), 542–550.
- [7] L. A. Karlovitz, Existence of fixed points of nonexpansive mappings in a space without normal structure, *Pacific J. Math.*, **66** (1976), 153–159.
- [8] M. Kato, On Lorentz spaces $\ell_{p,q}\{E\}$, *Hiroshima Math. J.* **6** (1976), 73–93.
- [9] M. Kato, K.-S. Saito and T. Tamura, On the ψ -direct sums of Banach spaces and convexity, submitted.
- [10] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, 1998.
- [11] K. Mitani, K.-S. Saito and T. Suzuki, Smoothness of absolute norms on $\mathbb{C}^n$, to appear in *J. Convex Analysis*.
- [12] K.-S. Saito and M. Kato, Uniform convexity of ψ -direct sums of Banach spaces, to appear in *J. Math. Anal. Appl.*
- [13] K.-S. Saito, M. Kato and Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$, *J. Math. Anal. Appl.* **244** (2000), 515–532.
- [14] K.-S. Saito, M. Kato and Y. Takahashi, On absolute norms on $\mathbb{C}^n$, *J. Math. Anal. Appl.* **252** (2000), 879–905.
- [15] Y. Takahashi, M. Kato and K.-S. Saito, Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, *J. Inequal. Appl.* **7** (2002), 179–186.
- [16] H. Triebel, *Intepolation Theory, Function spaces, Differential Operators*, North-Holland, 1978.