

A summary of the works “A class of C^* -algebras generalizing both graph algebras and homeomorphism C^* -algebras”

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0 Introduction

In this note, we summarize the definitions and the results in [Ka1, Ka2, Ka3] where structures of C^* -algebras associated with topological graphs are examined. Topological graphs generalize (discrete) graphs as well as topological dynamical systems, and our construction of C^* -algebras from topological graphs is a common generalization of those of graph algebras [KPRR, KPR, FLR] and homeomorphism C^* -algebras.

Sections 1 and 2 contain definitions of topological graphs and C^* -algebras associated with them. In Sections 3, we give the 6-term exact sequences of KK -groups and K -groups. Sections 4 is devoted to a summary of examples of C^* -algebras associated with topological graphs. For the detail, see [Ka2]. The rest of the sections contain results of [Ka3] where we generalize many notion such as minimality from dynamical systems to topological graphs.

Some of the results in this note are generalized to the non-commutative setting in [Ka5, Ka6]. We changed some notations from [Ka4].

1 Topological correspondences

In this section, we introduce a notion of topological correspondences between locally compact spaces, and construct C^* -correspondences from them.

Let A, B be C^* -algebras. A (right) Hilbert B -module X is a Banach space with a B -valued inner product $\langle \cdot, \cdot \rangle$ and a right action of B satisfying certain conditions (for the detail, see [L]). For a Hilbert B -module X , we denote by $\mathcal{L}(X)$ the set of adjointable operators on X , and by $\mathcal{K}(X)$ the ideal of $\mathcal{L}(X)$ spanned by the elements of the form $\theta_{\xi, \eta}$ for $\xi, \eta \in X$ where $\theta_{\xi, \eta} \in \mathcal{L}(X)$ is defined by $\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle$ for $\zeta \in X$. By a left action of the C^* -algebra A on the Hilbert B -module X , we mean a $*$ -homomorphism $\pi : A \rightarrow \mathcal{L}(X)$. A Hilbert B -module X together with a left action of A on X is called a C^* -correspondence from A to B . From a $*$ -homomorphism

$\varphi : A \rightarrow B$, we can define a C^* -correspondence from A to B by taking $X = B$. Thus we consider C^* -correspondences as a generalization of $*$ -homomorphisms.

Definition 1.1 Let E^0 and F^0 be locally compact (Hausdorff) spaces. A *topological correspondence* (E^1, d, r) from E^0 to F^0 consists of a locally compact space E^1 , a local homeomorphism $d : E^1 \rightarrow E^0$, and a continuous map $r : E^1 \rightarrow F^0$.

A continuous map $\sigma : E^0 \rightarrow F^0$ gives a topological correspondence (E^0, id, σ) . More generally, a set of continuous maps $\sigma_i : O_i \rightarrow F^0$ defined only on open subsets O_i of E^0 gives a topological correspondence (E^1, d, r) by setting $E^1 = \coprod_i O_i$ and defining d by the embedding and r by σ_i 's. We can say that the subset $r(d^{-1}(v)) \subset F^0$ is the “image” of a point $v \in E^0$ under a topological correspondence (E^1, d, r) , which can be empty or infinite. Thus topological correspondences are “multi-valued” generalizations of continuous maps.

Let us take a topological correspondence (E^1, d, r) from E^0 to F^0 . Denote by $C_d(E^1)$ the set of continuous functions ξ of E^1 such that $\langle \xi, \xi \rangle(v) = \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$ for any $v \in E^0$ and $\langle \xi, \xi \rangle \in C_0(E^0)$. For $\xi, \eta \in C_d(E^1)$ and $f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi, \eta \rangle \in C_0(E^0)$ by

$$\begin{aligned} (\xi f)(e) &= \xi(e)f(d(e)) \quad \text{for } e \in E^1, \\ \langle \xi, \eta \rangle(v) &= \sum_{e \in d^{-1}(v)} \overline{\xi(e)}\eta(e) \quad \text{for } v \in E^0. \end{aligned}$$

With these operations, $C_d(E^1)$ is a Hilbert $C_0(E^0)$ -module ([Ka1, Proposition 1.10]). We define a left action π_r of $C_0(F^0)$ on $C_d(E^1)$ by $(\pi_r(f)\xi)(e) = f(r(e))\xi(e)$ for $e \in E^1$, $\xi \in C_d(E^1)$ and $f \in C_0(F^0)$. Thus we get a C^* -correspondence $C_d(E^1)$ from $C_0(F^0)$ to $C_0(E^0)$. A composition of two topological correspondences can be defined naturally, and this relates to the internal tensor product of C^* -correspondences.

2 C^* -algebras arising from topological graphs

A topological dynamical system $\Sigma = (X, \sigma)$ consists of a locally compact space X and a homeomorphism σ on X . Since topological correspondences generalize continuous maps, a pair of a locally compact space E^0 and a topological correspondence (E^1, d, r) from E^0 to itself generalizes a topological dynamical system. Such pair is called a topological graph.

Definition 2.1 A *topological graph* $E = (E^0, E^1, d, r)$ consists of two locally compact spaces E^0, E^1 , a local homeomorphism $d : E^1 \rightarrow E^0$, and a continuous map $r : E^1 \rightarrow E^0$.

We think that E^0 is a set of vertices and E^1 is a set of edges and that an edge $e \in E^1$ is directed from its domain $d(e) \in E^0$ to its range $r(e) \in E^0$. We denote by $E_\Sigma = (X, X, \text{id}, \sigma)$ the topological graph defined by a topological dynamical system $\Sigma = (X, \sigma)$. As we saw in Section 1, a topological graph $E = (E^0, E^1, d, r)$ gives us a C^* -correspondence $C_d(E^1)$ from $C_0(E^0)$ to itself.

Definition 2.2 Let $E = (E^0, E^1, d, r)$ be a topological graph. A *Toeplitz E -pair* on a C^* -algebra A is a pair of maps $T = (T^0, T^1)$ where $T^0 : C_0(E^0) \rightarrow A$ is a $*$ -homomorphism and $T^1 : C_d(E^1) \rightarrow A$ is a linear map satisfying that

- (i) $T^1(\xi)^* T^1(\eta) = T^0(\langle \xi, \eta \rangle)$ for $\xi, \eta \in C_d(E^1)$,
- (ii) $T^0(f) T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

For $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$, the equation $T^1(\xi) T^0(f) = T^1(\xi f)$ holds automatically from the condition (i). For a Toeplitz E -pair $T = (T^0, T^1)$, we write $C^*(T)$ for denoting the C^* -algebra generated by the images of the maps T^0 and T^1 . We can define a $*$ -homomorphism $\Phi^1 : \mathcal{K}(C_d(E^1)) \rightarrow C^*(T)$ by $\Phi^1(\theta_{\xi, \eta}) = T^1(\xi) T^1(\eta)^*$ for $\xi, \eta \in C_d(E^1)$.

Definition 2.3 Let $E = (E^0, E^1, d, r)$ be a topological graph. We define an open subset E_{rg}^0 of E^0 by

$$E_{\text{rg}}^0 = \{v \in E^0 \mid \text{there exists a neighborhood } V \text{ of } v \\ \text{such that } r^{-1}(V) \subset E^1 \text{ is compact, and } r(r^{-1}(V)) = V\},$$

and set $E_{\text{sg}}^0 = E^0 \setminus E_{\text{rg}}^0$.

A vertex in E_{rg}^0 is called *regular*, and a vertex in E_{sg}^0 is called *singular*. We can show that the restriction of π_r to $C_0(E_{\text{rg}}^0)$ is an injection into $\mathcal{K}(C_d(E^1))$ [Ka1].

Definition 2.4 Let $E = (E^0, E^1, d, r)$ be a topological graph. A Toeplitz E -pair $T = (T^0, T^1)$ is called a *Cuntz-Krieger E -pair* if $T^0(f) = \Phi^1(\pi_r(f))$ for all $f \in C_0(E_{\text{rg}}^0)$. We denote by $\mathcal{O}(E)$ the universal C^* -algebra generated by a Cuntz-Krieger E -pair $t = (t^0, t^1)$.

When E is a discrete graph, $\mathcal{O}(E)$ is isomorphic to the graph algebra of the opposite graph of E . For a topological graph E_Σ defined by a topological dynamical system $\Sigma = (X, \sigma)$, the C^* -algebra $\mathcal{O}(E_\Sigma)$ is isomorphic to the homeomorphism C^* -algebra $A(\Sigma) = C_0(X) \rtimes_\sigma \mathbb{Z}$. The universal Cuntz-Krieger pair can be characterized by the following two conditions.

Definition 2.5 A Toeplitz pair $T = (T^0, T^1)$ is called *injective* when T^0 is injective, and said to *admit a gauge action* when for each complex number z with $|z| = 1$ there exists an automorphism β'_z on $C^*(T)$ such that $\beta'_z(T^0(f)) = T^0(f)$ and $\beta'_z(T^1(\xi)) = zT^1(\xi)$.

For an injective Toeplitz pair $T = (T^0, T^1)$, the linear map T^1 is isometric.

Theorem 2.6 ([Ka1, Theorem 4.5]) *A Cuntz-Krieger pair T is universal if and only if it is injective and admits a gauge action.*

We can define a C^* -algebra $\mathcal{O}(E)$ without using the open set E_{rg}^0 nor a notion of Cuntz-Krieger pairs.

Proposition 2.7 ([Ka2]) *Let E be a topological graph. For an injective Toeplitz E -pair T admitting a gauge action, there exists a (unique) surjection $\rho : C^*(T) \rightarrow \mathcal{O}(E)$ such that $t^i = \rho \circ T^i$ for $i = 0, 1$.*

Thus $\mathcal{O}(E)$ can be defined as the smallest C^* -algebra generated by an injective Toeplitz E -pair admitting a gauge action. Note that the existence of such smallest C^* -algebra is a non-trivial fact. Now Theorem 2.6 can be rephrased as follows.

Proposition 2.8 *Let E be a topological graph. For an injective Toeplitz E -pair T admitting a gauge action, the surjection $\rho : C^*(T) \rightarrow \mathcal{O}(E)$ in Proposition 2.7 is an isomorphism if and only if T is a Cuntz-Krieger E -pair.*

We can construct the C^* -algebra $\mathcal{O}(E)$ concretely using the Fock space. This construction gives us an isomorphism between the C^* -algebra $\mathcal{O}(E)$ and the relative Cuntz-Pimsner algebra of $C_d(E^1)$ with respect to the ideal $C_0(E_{\text{rg}}^0)$ of $C_0(E^0)$ defined in [MS].

3 Nuclearity and KK -groups

For all topological graph E , the C^* -algebra $\mathcal{O}(E)$ is nuclear ([Ka1, Proposition 6.1]), and it satisfies the Universal Coefficient Theorem (UCT) of [RoSc] when it is separable ([Ka1, Proposition 6.6]). The C^* -algebra $\mathcal{O}(E)$ is separable if and only if E is second countable, which means both E^0 and E^1 are second countable. We get 6-term exact sequences of KK -groups and K -groups which help to compute those of the C^* -algebra $\mathcal{O}(E)$.

Let us denote by $\iota_* \in KK(C_0(E_{\text{rg}}^0), C_0(E^0))$ the element defined by the inclusion $\iota : C_0(E_{\text{rg}}^0) \hookrightarrow C_0(E^0)$, and by $[\pi_r] \in KK(C_0(E_{\text{rg}}^0), C_0(E^0))$ the element defined by the triple $(C_d(E^1), \pi_r, 0)$. Note that the element $[\pi_r]$ is obtained from the map $\pi_r : C_0(E_{\text{rg}}^0) \rightarrow \mathcal{K}(C_d(E^1))$, the strong Morita equivalence between $\mathcal{K}(C_d(E^1))$ and $C_0(d(E^1))$ defined by the Hilbert module $C_d(E^1)$, and the inclusion $C_0(d(E^1)) \subset C_0(E^0)$.

Proposition 3.1 ([Ka1, Proposition 6.9]) *Let E be a second countable topological graph. For any separable C^* -algebra B we have the following two exact sequences:*

$$\begin{array}{ccccc} KK_0(B, C_0(E_{\text{rg}}^0)) & \xrightarrow{\iota_* - [\pi_r]} & KK_0(B, C_0(E^0)) & \xrightarrow{t_*^0} & KK_0(B, \mathcal{O}(E)) \\ \uparrow & & & & \downarrow \\ KK_1(B, \mathcal{O}(E)) & \xleftarrow{t_*^0} & KK_1(B, C_0(E^0)) & \xleftarrow{\iota_* - [\pi_r]} & KK_1(B, C_0(E_{\text{rg}}^0)) \end{array}$$

and

$$\begin{array}{ccccc} KK_0(C_0(E_{\text{rg}}^0), B) & \xleftarrow{\iota_* - [\pi_r]} & KK_0(C_0(E^0), B) & \xleftarrow{t_*^0} & KK_0(\mathcal{O}(E), B) \\ \downarrow & & & & \uparrow \\ KK_1(\mathcal{O}(E), B) & \xrightarrow{t_*^0} & KK_1(C_0(E^0), B) & \xrightarrow{\iota_* - [\pi_r]} & KK_1(C_0(E_{\text{rg}}^0), B). \end{array}$$

Corollary 3.2 ([Ka1, Corollary 6.10]) *For a second countable topological graph E , the following sequence of K -groups is exact:*

$$\begin{array}{ccccc}
 K_0(C_0(E_{\text{rg}}^0)) & \xrightarrow{\iota_* - [\pi_r]} & K_0(C_0(E^0)) & \xrightarrow{t_*^0} & K_0(\mathcal{O}(E)) \\
 \uparrow & & & & \downarrow \\
 K_1(\mathcal{O}(E)) & \xleftarrow{t_*^0} & K_1(C_0(E^0)) & \xleftarrow{\iota_* - [\pi_r]} & K_1(C_0(E_{\text{rg}}^0)).
 \end{array}$$

4 Examples

Topological graphs are generalizations of not only (discrete) graphs and topological dynamical systems but also other notions such as partial homeomorphisms [E], singly generated dynamical systems [Re] and so on. Moreover the construction of C^* -algebras from topological graphs generalizes those of

- homeomorphism C^* -algebras,
- graph algebras [KPRR, KPR, FLR],
- crossed products by partial homeomorphisms [E],
- C^* -algebras associated with branched coverings [DM],
- C^* -algebras associated with singly generated dynamical systems [Re],
- C^* -algebras associated with infinite matrices [EL],
- C^* -algebras associated with subshifts [M].

The class of C^* -algebras arising from topological graphs contains many examples of nuclear C^* -algebras such as;

- all AF-algebras
- all simple AT-algebras with real rank zero,
- many AH-algebras including all Goodearl algebras (see [RøSt, Example 3.1.7]) and purely infinite AH-algebras constructed in [Rø],
- all simple separable nuclear purely infinite C^* -algebras satisfying UCT,
- many simple stably projectionless C^* -algebras.

The class of C^* -algebras arising from topological graphs is closed under taking

- direct sums,
- unitizations,
- tensor products with M_n or $\mathbb{K}$,

- tensor products with commutative C^* -algebras,
- inductive limits by certain connecting maps,
- ideals which is invariant under the gauge action,
- quotients by ideals which is invariant under the gauge action.

5 Orbits and invariant sets

In this section, we introduce a notion of orbits and invariant sets of topological graphs. These are closely related ideal structures of the C^* -algebra $\mathcal{O}(E)$ of a topological graph E . The difference of definitions of positive orbit spaces and negative orbit spaces comes from the irreversible feature of topological correspondences.

Let us fix a topological graph $E = (E^0, E^1, d, r)$. We set $d^0 = r^0 = \text{id}_{E^0}$ and $d^1 = d, r^1 = r$. For $n = 2, 3, \dots$, we define a space E^n of *paths* with length n by

$$E^n = \{(e_1, e_2, \dots, e_n) \in E^1 \times \dots \times E^1 \times E^1 \mid d(e_k) = r(e_{k+1}) \text{ for } k = 1, 2, \dots, n-1\}.$$

We define domain and range maps $d^n, r^n : E^n \rightarrow E^0$ by $d^n(e) = d(e_n)$ and $r^n(e) = r(e_1)$ for $e = (e_1, e_2, \dots, e_n) \in E^n$. Note that (E^n, d^n, r^n) is the n -times composition of the topological correspondence (E^1, d, r) on E^0 . An *infinite path* $e = (e_1, e_2, \dots, e_n, \dots)$ means that $e_k \in E^1$ and $d(e_k) = r(e_{k+1})$ for each $k = 1, 2, \dots$. The set of all infinite paths is denoted by E^∞ . The range $r^\infty(e) \in E^0$ of an infinite path $e = (e_1, e_2, \dots, e_n, \dots) \in E^\infty$ is defined by $r(e_1)$.

Definition 5.1 We define the *positive orbit space* $\text{Orb}^+(v)$ of $v \in E^0$ by

$$\text{Orb}^+(v) = \{r^n(e) \in E^0 \mid e \in (d^n)^{-1}(v) \subset E^n, n \in \mathbb{N}\}.$$

Definition 5.2 For $n \in \mathbb{N} \cup \{\infty\}$, a path $e \in E^n$ is called a *negative orbit* of $v \in E^0$ if $r^n(e) = v$ and $d^n(e) \in E_{\text{sg}}^0$ when $n < \infty$.

Note that each $v \in E^0$ has at least one negative orbit, but may have many negative orbits in general.

Definition 5.3 For a negative orbit $e = (e_1, e_2, \dots, e_n) \in E^n$ of $v \in E^0$ with $n \in \mathbb{N}$, the *negative orbit space* $\text{Orb}^-(v, e)$ is defined by

$$\text{Orb}^-(v, e) = \{v, d(e_1), d(e_2), \dots, d(e_n)\} \subset E^0.$$

Similarly, for a negative orbit $e = (e_1, e_2, \dots, e_k, \dots) \in E^\infty$ of $v \in E^0$, the *negative orbit space* $\text{Orb}^-(v, e)$ is defined by

$$\text{Orb}^-(v, e) = \{v, d(e_1), d(e_2), \dots, d(e_k), \dots\} \subset E^0.$$

Definition 5.4 We define the *orbit space* $\text{Orb}(v, e)$ of $v \in E^0$ with respect to a negative orbit e of v by

$$\text{Orb}(v, e) = \bigcup_{v' \in \text{Orb}^-(v, e)} \text{Orb}^+(v').$$

Remark 5.5 A negative orbit e of $v \in E^0$ determines “the past” of v , and the negative orbit space $\text{Orb}^-(v, e)$ consists of the points in “the past”, while the positive orbit space $\text{Orb}^+(v)$ is the set of all points in “the future” of v . The orbit space $\text{Orb}(v, e)$ of v with respect to the negative orbit e consists of the all points which are reached from some point in “the past”. “The past” $e \in E^n$ may have an “origin” $d^n(e)$ which should be a singular point (when $n < \infty$), or may come from long, long time ago (when $n = \infty$). When “the past” e has an “origin” $v' \in E_{\text{sg}}^0$, the orbit space $\text{Orb}(v, e)$ coincides with the positive orbit space $\text{Orb}^+(v')$ of the “origin” v' .

Definition 5.6 A subset X of E^0 is said to be *positively invariant* if $d(e) \in X$ implies $r(e) \in X$ for each $e \in E^1$, and *negatively invariant* if for $v \in X \cap E_{\text{rg}}^0$, there exists $e \in E^1$ with $r(e) = v$ and $d(e) \in X$. A subset X of E^0 is said to be *invariant* if X is both positively and negatively invariant.

It is easy to see the following two lemmas.

Lemma 5.7 For each $v \in E^0$, the positive orbit space $\text{Orb}^+(v)$ is positively invariant. A subset X of E^0 is positively invariant if and only if $\text{Orb}^+(v) \subset X$ for all $v \in X$.

Lemma 5.8 For each $v \in E^0$ and each negative orbit e of v , the negative orbit space $\text{Orb}^-(v, e)$ is negatively invariant. A subset X of E^0 is negatively invariant if and only if for each $v \in X$, there exists a negative orbit e of v such that $\text{Orb}^-(v, e) \subset X$.

From these lemmas, we get the following.

Proposition 5.9 For each $v \in E^0$ and each negative orbit e of v , the orbit space $\text{Orb}(v, e)$ is invariant. A subset X of E^0 is invariant if and only if for each $v \in X$, there exists a negative orbit e of v such that $\text{Orb}(v, e) \subset X$.

We are interested in closed invariant subsets.

Lemma 5.10 If a subset X of E^0 is positively invariant or negatively invariant, then so is the closure $\overline{X}$. Hence $\overline{X}$ is invariant for an invariant set $X \subset E^0$.

By this lemma, $\overline{\text{Orb}(v, e)}$ is a closed invariant set for a negative orbit e of $v \in E^0$. Let X^0 be a closed subset of E^0 , and define $X^1 = d^{-1}(X^0) \subset E^1$. If X^0 is positively invariant, then we have $r(X^1) \subset X^0$ and so $X = (X^0, X^1, d, r)$ is a topological graph. We can state a condition for a closed positively invariant set X^0 to be negatively invariant (hence invariant) using the topological graph X .

Proposition 5.11 *A closed positively invariant subset X^0 of E^0 is invariant if and only if $X_{\text{sg}}^0 \subset E_{\text{sg}}^0$.*

By this proposition, for a closed invariant subset X^0 of E^0 , we have the inclusion $X_{\text{sg}}^0 \subset E_{\text{sg}}^0 \cap X^0$. In general, this inclusion is not equal. We will see this difference in the study of ideals of $\mathcal{O}(E)$ (see Definition 7.1). We finish this section by studying complements of invariant subsets.

Definition 5.12 A subset V of E^0 is said to be *hereditary* if V satisfies $d(r^{-1}(V)) \subset V$, and said to be *saturated* if we have $v \in V$ for $v \in E_{\text{rg}}^0$ satisfying $d(r^{-1}(v)) \subset V$.

Proposition 5.13 *A set X is positively invariant if and only if the complement V of X is hereditary, and negatively invariant if and only if V is saturated.*

Definition 5.14 For a subset V of E^0 , we define $H(V), S(V) \subset E^0$ by

$$H(V) = \bigcup_{n=0}^{\infty} d^n((r^n)^{-1}(V)).$$

and by $S(V) = \bigcup_{n=0}^{\infty} V_n$ where $V_0 = V$ and for $n = 1, 2, \dots$, V_n is defined inductively by

$$V_n = V_{n-1} \cup \{v \in E_{\text{rg}}^0 \mid d(r^{-1}(v)) \subset V_{n-1}\}.$$

Proposition 5.15 *For a subset V of E^0 , $H(V)$ is the smallest hereditary subset containing V and $S(V)$ is the smallest saturated subset containing V .*

It is not difficult to see that if a subset V is hereditary then so is $S(V)$. Hence we have the following.

Proposition 5.16 *For a subset V of E^0 , $S(H(V))$ is the smallest hereditary and saturated subset containing V .*

By noting that if V is open then so is both $H(V)$ and $S(V)$, we get the following.

Proposition 5.17 *For an open subset V of E^0 , the open set $S(H(V))$ is the smallest open set which contains V and whose complement is a closed invariant subset.*

6 The space of negative orbits, and the one-sided Markov shift

We denote by E_{∞}^0 the set of all negative orbits, and by E_{∞}^1 the subset of E_{∞}^0 consisting of the negative orbits whose length is greater than or equal to 1. We define topologies on E_{∞}^0 and E_{∞}^1 as follows.

Let $\tilde{E}^1 = E^1 \cup \{\infty\}$ be the one-point compactification of E^1 . We consider a negative orbit $e \in E^n$ with $n \geq 1$ as an element of the infinite direct product $\tilde{E}^1 \times \tilde{E}^1 \times \dots$ of the compact space $\tilde{E}^1$ by

$$\begin{aligned} E^n \ni (e_1, \dots, e_n) &\mapsto (e_1, \dots, e_n, \infty, \infty, \dots) \in \tilde{E}^1 \times \tilde{E}^1 \times \dots && \text{when } n < \infty, \\ E^\infty \ni (e_1, \dots, e_k, \dots) &\mapsto (e_1, \dots, e_k, \dots) \in \tilde{E}^1 \times \tilde{E}^1 \times \dots && \text{when } n = \infty. \end{aligned}$$

Thus we can consider E_∞^1 as a subset of the compact set $\tilde{E}^1 \times \tilde{E}^1 \times \dots$, and we define the relative topology on E_∞^1 .

The set E_∞^0 is a disjoint union of E_{sg}^0 and E_∞^1 . We consider E_∞^0 as a subset of $E^0 \times \tilde{E}^1 \times \tilde{E}^1 \times \dots$ by the embeddings

$$\begin{aligned} E_{\text{sg}}^0 \ni v &\mapsto (v, \infty, \infty, \dots) \in E^0 \times \tilde{E}^1 \times \tilde{E}^1 \times \dots, \\ E_\infty^1 \ni (e_1, e_2, \dots) &\mapsto (r(e_1), e_1, e_2, \dots) \in E^0 \times \tilde{E}^1 \times \tilde{E}^1 \times \dots, \end{aligned}$$

and define the relative topology on E_∞^0 . We denote by r_∞ the embedding $E_\infty^1 \rightarrow E_\infty^0$. Then we have the following.

Proposition 6.1 *The topological spaces E_∞^0 and E_∞^1 are locally compact, and the map $r_\infty : E_\infty^1 \rightarrow E_\infty^0$ is a homeomorphism onto an open subset of E_∞^0 .*

We define a map $d_\infty : E_\infty^1 \rightarrow E_\infty^0$ by $d_\infty(e_1, e_2, \dots) = (d(e_1), e_2, e_3, \dots)$. Then d_∞ is a local homeomorphism, and so we get a topological graph $E_\infty = (E_\infty^0, E_\infty^1, d_\infty, r_\infty)$. In the case that a topological graph E has finitely many vertices and edges, and has no sinks or sources (which means that d and r are surjective), the topological graph E_∞ is nothing but the one-sided Markov shift considered in [CK].

We define two maps $m^0 : E_\infty^0 \rightarrow E^0$ and $m^1 : E_\infty^1 \rightarrow E^1$ by

$$m^0(v, e_1, e_2, \dots) = v, \quad m^1(e_1, e_2, \dots) = e_1.$$

Then both m^0 and m^1 are surjective proper continuous maps and we have $m^0 \circ d_\infty = d \circ m^1$ and $m^0 \circ r_\infty = r \circ m^1$. The pair $m = (m^0, m^1)$ satisfying these conditions (and one more condition) is called a *factor map* from E_∞ to E [Ka2]. Let us define a $*$ -homomorphism $\mu^0 : C_0(E^0) \ni f \mapsto f \circ m^0 \in C_0(E_\infty^0)$ and a linear map $\mu^1 : C_d(E^1) \ni \xi \mapsto \xi \circ m^1 \in C_{d_\infty}(E_\infty^1)$. Since the factor map $m = (m^0, m^1)$ satisfies the condition called regularity, we get a $*$ -homomorphism $\mu : \mathcal{O}(E) \rightarrow \mathcal{O}(E_\infty)$ such that $\mu \circ t^i = t_\infty^i \circ \mu^i$ for $i = 0, 1$ where $t = (t^0, t^1)$ is the universal Cuntz-Krieger E -pair on $\mathcal{O}(E)$ and $t_\infty = (t_\infty^0, t_\infty^1)$ is the universal Cuntz-Krieger E_∞ -pair on $\mathcal{O}(E_\infty)$. The following is one of the main theorems of [Ka7].

Theorem 6.2 *The $*$ -homomorphism $\mu : \mathcal{O}(E) \rightarrow \mathcal{O}(E_\infty)$ is an isomorphism.*

By this theorem, the C^* -algebra $\mathcal{O}(E)$ is shown to be related to the dynamical system $E_\infty = (E_\infty^0, E_\infty^1, d_\infty, r_\infty)$ which can be considered as a generalization of one-sided Markov shifts. Recall that this observation was important in the work of [CK]. We also see from Theorem 6.2 that the C^* -algebra $\mathcal{O}(E)$ is obtained from a topological groupoid whose unit space is E_∞^0 .

7 Gauge invariant ideals

The set of all gauge invariant ideals is parameterized by pairs of two closed subsets of E^0 called admissible pairs.

Definition 7.1 A pair $\rho = (X^0, Z)$ of closed subsets of E^0 satisfying the following two conditions is called an *admissible pair*;

- (i) X^0 is invariant,
- (ii) $X_{\text{sg}}^0 \subset Z \subset E_{\text{sg}}^0 \cap X^0$.

Define a C^* -subalgebra $\mathcal{F}^1 \subset \mathcal{O}(E)$ and a $*$ -homomorphism $\pi_0^1 : \mathcal{F}^1 \rightarrow C_0(E_{\text{sg}}^0)$ by

$$\mathcal{F}^1 = \{t^0(f) + \varphi^1(x) \mid f \in C_0(E^0), x \in \mathcal{K}(C_d(E^1))\},$$

and $\pi_0^1(t^0(f) + \varphi^1(x)) = f|_{E_{\text{sg}}^0}$. For an ideal I of $\mathcal{O}(E)$, we define closed subsets X_I^0 and Z_I of E^0 by

$$\begin{aligned} X_I^0 &= \{v \in E^0 \mid f(v) = 0 \text{ for all } f \in C_0(E^0) \text{ with } t^0(f) \in I\}, \\ Z_I &= \{v \in E_{\text{sg}}^0 \mid f(v) = 0 \text{ for all } f \in \pi_0^1(I \cap \mathcal{F}^1)\}. \end{aligned}$$

Proposition 7.2 For an ideal I of $\mathcal{O}(E)$, the pair $\rho_I = (X_I^0, Z_I)$ is an admissible pair.

Definition 7.3 For an admissible pair $\rho = (X^0, Z)$, we define a topological graph $E_\rho = (E_\rho^0, E_\rho^1, d_\rho, r_\rho)$ as follows. Set $Y_\rho = Z \setminus X_{\text{sg}}^0$, $\partial Y_\rho = \overline{Y_\rho} \setminus Y_\rho$, and define

$$E_\rho^0 = X^0 \amalg_{\partial Y_\rho} \overline{Y_\rho}, \quad E_\rho^1 = X^1 \amalg_{d^{-1}(\partial Y_\rho)} d^{-1}(\overline{Y_\rho}).$$

The domain map $d_\rho : E_\rho^1 \rightarrow E_\rho^0$ is defined from $d : X^1 \rightarrow X^0$ and $d : d^{-1}(\overline{Y_\rho}) \rightarrow \overline{Y_\rho}$. The range map $r_\rho : E_\rho^1 \rightarrow E_\rho^0$ is defined from $r : X^1 \rightarrow X^0$ and $r : d^{-1}(\overline{Y_\rho}) \rightarrow X^0$.

Note that for an admissible pair $\rho = (X^0, Z)$ with $Z = X_{\text{sg}}^0$, we have $E_\rho = X$. By using Theorem 2.6, we can show the following.

Proposition 7.4 For a gauge-invariant ideal I of $\mathcal{O}(E)$, there exists a natural isomorphism $\mathcal{O}(E)/I \cong \mathcal{O}(E_{\rho_I})$.

From this proposition and some computation, we get the next theorem.

Theorem 7.5 The map $I \mapsto \rho_I$ gives us an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of all admissible pairs.

This theorem is a continuous counterpart of [BHRS, Theorem 3.6].

8 Freeness and topological freeness

A path $e \in E^n$ with $n \geq 1$ is called a *loop* if $r^n(e) = d^n(e)$. The vertex $r^n(e) = d^n(e)$ is called the *base point* of the loop e . A loop $e = (e_1, \dots, e_n)$ is said to be *simple* if $r(e_i) \neq r(e_j)$ for $i \neq j$, and *without entrances* if $r^{-1}(r(e_i)) = \{e_i\}$ for $i = 1, \dots, n$.

Definition 8.1 A topological graph E is said to be *topologically free* if the set of base points of loops without entrances has an empty interior.

This generalizes topological freeness of ordinary dynamical systems and Condition L of graph algebras (see, for example, [T] and [KPR]).

Theorem 8.2 ([Ka1, Theorem 5.12]) *If a topological graph $E = (E^0, E^1, d, r)$ is topologically free, then the natural surjection $\mathcal{O}(E) \rightarrow C^*(T)$ is an isomorphism for all injective Cuntz-Krieger E -pair $T = (T^0, T^1)$.*

The necessity of topological freeness in Theorem 8.2 is proved in [Ka3]. By Theorem 8.2, we have the following (cf. Proposition 7.4).

Proposition 8.3 *Let I be an ideal of $\mathcal{O}(E)$. If the topological graph E_{ρ_I} is topologically free, then I is gauge-invariant.*

Definition 8.4 For a positive integer n , we denote by $\text{Per}_n(E)$ the set of vertices v satisfying the following three conditions;

- (i) there exists a simple loop $(e_1, \dots, e_n) \in E^n$ whose base point is v ,
- (ii) for each $i = 1, 2, \dots, n$, there exist no $e \in E^1$ satisfying $r(e) = r(e_i)$ and $d(e) \in \text{Orb}^+(v)$ other than e_i ,
- (iii) v is isolated in $\text{Orb}^+(v)$.

We set $\text{Per}(E) = \bigcup_{n=1}^{\infty} \text{Per}_n(E)$ and $\text{Aper}(E) = E^0 \setminus \text{Per}(E)$.

An element in $\text{Per}(E)$ is called a *periodic point* while an element in $\text{Aper}(E)$ is called an *aperiodic point*. The conditions (i) and (ii) above mean that $v \in E^0$ is a base point of exactly one simple loop, and the condition (iii) says that there exist no “approximate loops” whose “base points” are v .

Definition 8.5 A topological graph E is said to be *free* if $\text{Aper}(E) = E^0$.

This is a generalization of freeness of ordinary dynamical systems and Condition K of graph algebras (see, for example, [KPRR]).

Proposition 8.6 *A topological graph E is free if and only if E_{ρ} is topologically free for every admissible pair ρ .*

In particular, free topological graphs are topologically free. From Theorem 7.5, Proposition 8.3 and Proposition 8.6, we have the following.

Theorem 8.7 *If a topological graph E is free, then every ideal is gauge-invariant. Hence the set of all ideals corresponds bijectively to the set of all admissible pairs by the map $I \mapsto \rho_I$.*

9 Minimality and topological transitivity

In [Ka3], we generalize minimality and topological transitivity from topological dynamical systems to topological graphs.

Definition 9.1 A topological graph E is said to be *minimal* if there exists no closed invariant sets other than $\emptyset$ or E^0 .

The following characterization of minimality is naturally expected.

Proposition 9.2 *For a topological graph E , the following conditions are equivalent.*

- (i) E is minimal.
- (ii) An orbit space $\text{Orb}(v, e)$ is dense in E^0 for all $v \in E^0$ and all negative orbit e of v .
- (iii) For every non-empty open set $V \subset E^0$, we have $S(H(V)) = E^0$.

The condition (ii) in Proposition 9.2 is related to cofinality of (discrete) graphs [KPRR]. By Theorem 7.5, E is minimal if and only if $\mathcal{O}(E)$ has no non-trivial gauge invariant ideals. We can prove the following.

Theorem 9.3 *For a topological graph E , the following conditions are equivalent.*

- (i) The C^* -algebra $\mathcal{O}(E)$ is simple.
- (ii) E is minimal and topologically free.
- (iii) E is minimal and free.

For topological dynamical systems $\Sigma = (X, \sigma)$, minimality implies topological freeness when X is infinite. This is not the case for topological graphs (or even discrete graphs).

Definition 9.4 A topological graph E is called *topologically transitive* if we have $H(V_1) \cap H(V_2) \neq \emptyset$ for two non-empty open sets $V_1, V_2 \subset E^0$.

Proposition 9.5 *If there exist $v \in E^0$ and a negative orbit e of v such that the orbit space $\text{Orb}(v, e)$ is dense in E^0 , then E is topologically transitive.*

The converse of Proposition 9.5 is true when E^0 is second countable, but in general it is false even for topological dynamical systems.

Proposition 9.6 *For a topological graph E , the following are equivalent.*

- (i) E is topologically transitive.
- (ii) For two non-empty open sets $V_1, V_2 \subset E^0$, we have $S(H(V_1)) \cap S(H(V_2)) \neq \emptyset$.
- (iii) If two closed invariant subsets X_1^0, X_2^0 satisfies $X_1^0 \cup X_2^0 = E^0$, then either $X_1^0 = E^0$ or $X_2^0 = E^0$ holds.

Theorem 9.7 *A C^* -algebra $\mathcal{O}(E)$ is primitive if and only if E is topologically free and topologically transitive.*

References

- [BHRS] Bates, T.; Hong, J.; Raeburn, I.; Szymański, W. *The ideal structure of the C^* -algebras of infinite graphs*. Preprint.
- [CK] Cuntz, J.; Krieger, W. *A class of C^* -algebras and topological Markov chains*. Invent. Math. **56** (1980), no. 3, 251–268.
- [DM] Deaconu, V.; Muhly, P. S. *C^* -algebras associated with branched coverings*. Proc. Amer. Math. Soc. **129** (2001), no. 4, 1077–1086.
- [E] Exel, R. *Circle actions on C^* -algebras, partial automorphisms, and a generalized Pimsner-Voiculescu exact sequence*. J. Funct. Anal. **122** (1994), no. 2, 361–401.
- [EL] Exel, R.; Laca, M. *Cuntz-Krieger algebras for infinite matrices*. J. Reine Angew. Math. **512** (1999), 119–172.
- [FLR] Fowler, N. J.; Laca, M.; Raeburn, I. *The C^* -algebras of infinite graphs*. Proc. Amer. Math. Soc. **128** (2000), no. 8, 2319–2327.
- [Ka1] Katsura, T. *A class of C^* -algebras generalizing both graph algebras and homeomorphism C^* -algebras I, fundamental results*. Preprint.
- [Ka2] Katsura, T. *A class of C^* -algebras generalizing both graph algebras and homeomorphism C^* -algebras II, examples*. Preprint.
- [Ka3] Katsura, T. *A class of C^* -algebras generalizing both graph algebras and homeomorphism C^* -algebras III, ideal structures*. Preprint.
- [Ka4] Katsura, T. *Continuous graphs and crossed products of Cuntz algebras*. To appear in Sūrikaiseikikenkyūsho Kōkyūroku.
- [Ka5] Katsura, T. *A construction of C^* -algebras from C^* -correspondences*. Preprint.
- [Ka6] Katsura, T. *Ideal structures of C^* -algebras associated with C^* -correspondences*. In preparation.
- [Ka7] Katsura, T. In preparation.
- [KPR] Kumjian, A.; Pask, D.; Raeburn, I. *Cuntz-Krieger algebras of directed graphs*. Pacific J. Math. **184** (1998), no. 1, 161–174.
- [KPRR] Kumjian, A.; Pask, D.; Raeburn, I.; Renault, J. *Graphs, groupoids, and Cuntz-Krieger algebras*. J. Funct. Anal. **144** (1997), no. 2, 505–541.
- [L] Lance, E. C. *Hilbert C^* -modules. A toolkit for operator algebraists*. London Mathematical Society Lecture Note Series, **210**. Cambridge University Press, Cambridge, 1995.

- [M] Matsumoto, K. *On C^* -algebras associated with subshifts*. Internat. J. Math. **8** (1997), no. 3, 357–374.
- [MS] Muhly, P. S.; Solel, B. *Tensor algebras over C^* -correspondences: representations, dilations, and C^* -envelopes*. J. Funct. Anal. **158** (1998), no. 2, 389–457.
- [P] Pimsner, M. V. *A class of C^* -algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$* . Free probability theory, 189–212, Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.
- [Re] Renault, J. *Cuntz-like algebras*. Operator theoretical methods, 371–386, Theta Found., Bucharest, 2000.
- [Rø] Rørdam, M. *A purely infinite AH -algebra and an application to AF -embeddability*. Preprint.
- [RøSt] Rørdam, M.; Stormer, E. *Classification of nuclear C^* -algebras. Entropy in operator algebras*. Encyclopaedia of Mathematical Sciences, **126**. Operator Algebras and Non-commutative Geometry, **7**. Springer-Verlag, Berlin, 2002. x+198 pp.
- [RoSc] Rosenberg, J.; Schochet, C. *The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized K -functor*. Duke Math. J. **55** (1987), no. 2, 431–474.
- [T] Tomiyama, J. *The interplay between topological dynamics and theory of C^* -algebras*. Lecture Notes Series, 2. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1992.