

Non Reversible Knots Exist

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$S^3 \supset K$, tame knot

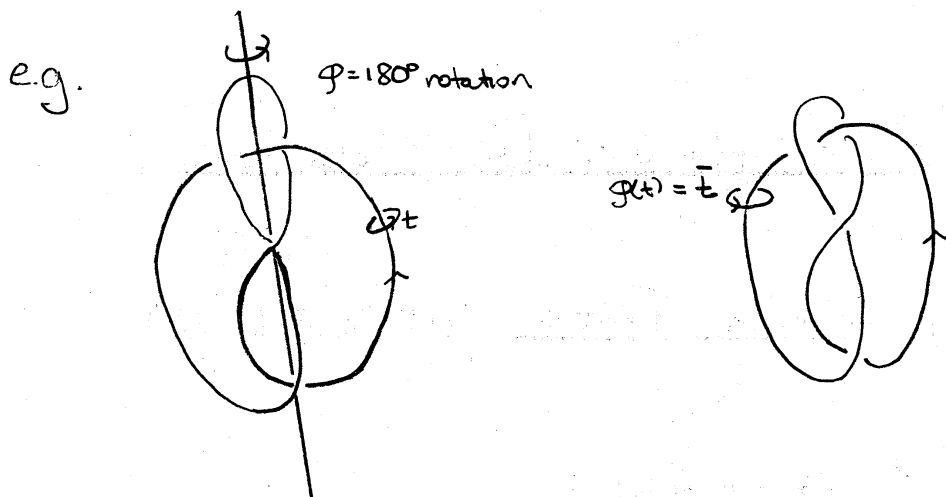
$$G = \pi_1(S^3 - K)$$

Definition. K is reversible $\Leftrightarrow \exists \rho: G \xrightarrow{\cong} G$ automorphism such that the induced $\bar{\rho}: G/G' \xrightarrow{\cong} G/G'$ takes a generator to its inverse.

Examples:

- (I) K invertible $\Rightarrow K$ reversible
- (II) K positive amphichiral $\Rightarrow K$ reversible.

Proof. (I) K invertible. So $\exists \varphi: S^3 \xrightarrow{\cong} S^3$ orientation preserving homeomorphism s.t. $\varphi(\vec{K}) = -\vec{K}$.
Take $\rho = \varphi_*$ (We may assume φ fixes the base point).



(iii) K positive amphicheiral. So $\exists \psi: S^3 \xrightarrow{\cong} S^3$ orientation reversing homeomorphism s.t. $\psi(K) = K$.

Take $\rho = \psi_*$.

Question: Do non-reversible knots exist?

(Answer: Yes!)

Notation: $P = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$, presentation of K .

$F = \langle x_1, \dots, x_m \rangle$ free.

$F \xrightarrow{\varphi} P \xrightarrow{\psi} P/P'$, canonical projections.

$P/P' \cong \langle t \rangle$; choose $y \in P/P'$, a generator.

$\left. \begin{array}{l} y \xrightarrow{\alpha_y} t \\ \bar{y} \xrightarrow{\alpha_{\bar{y}}} t \end{array} \right\}$ defines two isomorphisms.

$$\left. \begin{aligned} A &= \left(\frac{\partial r_i}{\partial x_j} \right) \tilde{x} \tilde{a}_j \\ \bar{A} &= \left(\frac{\partial r_i}{\partial x_j} \right) \tilde{x} \tilde{a}_j \end{aligned} \right\} \text{Laurent polynomial matrices.}$$

$\bar{A}$ is conjugate to A . (Here $\sim$ denotes extension to group rings) In future, we ignore $\tilde{a}_j, \tilde{a}_j$.

Definition: $A, \bar{A}$ are the Alexander matrices of P .

Definition: (matrix equivalence).

(a) Proper definition (see [P1]) A, A_* matrices over $J\langle t \rangle$.

$A \sim A_* \iff \exists (A, X), (A_*, X_*)$ satisfactory s.t.

$(A, X) \sim (A_*, X_*)$.

(b) Rough outline:

The usual equivalence relation allows:

(0) Interchange of rows, Interchange of columns.

(1) $A \leftrightarrow \begin{pmatrix} A \\ * \end{pmatrix}$, $*$ = linear combination of the rows of A .

$$(II) A \leftrightarrow \begin{pmatrix} A & \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \\ * & 1 \end{pmatrix}, \quad * \text{ arbitrary.}$$

We allow:

(0) As above.

(I) As above.

$$(II) A \leftrightarrow \begin{pmatrix} A & \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \\ * & 1 \end{pmatrix}, \quad \text{where if } * = (a_{m+1,1}, a_{m+1,2}, \dots, a_{m+1,n}),$$

$$\sum_{j=1}^n a_{m+1,j} \tilde{x}(x_j - 1) = -t\delta + 1, \quad \exists \delta \text{ integral.}$$

Difference: in (II), $*$ no longer arbitrary. (The reason is to ensure $\exists x'$ s.t. $((\begin{smallmatrix} A & \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \\ * & 1 \end{smallmatrix}), x')$ is satisfactory)

Theorem 1. K reversible. Then $A \sim \bar{A} \quad \forall P$.

Proof. Consider first the special case

$P = \langle x_1, \dots, x_m \mid r_1, \dots, r_{m-1} \rangle$ is Wirtinger. Then $r_i = x_a x_b \bar{x}_c \bar{x}_c$, $\exists a, b, c, \forall i$.

Define P 's reversed presentation $P_R = \langle x_1, \dots, x_m \mid s_1, \dots, s_{m-1} \rangle$ by if $r_i = x_a x_b \bar{x}_c \bar{x}_c$ then $s_i \in \bar{x}_a \bar{x}_b x_c x_c$, $\forall i$.
 $\exists$ an isomorphism $\lambda: P \xrightarrow{\sim} P_R$ induced by $x_j \mapsto \bar{x}_j, \forall j$.

λ induces $\mathcal{B} : \langle t \rangle \xrightarrow{\cong} \mathbb{R}/\mathbb{P}\mathbb{R}'$. Identify $\mathbb{R}/\mathbb{P}\mathbb{R}'$ with $\langle t \rangle$ so that $\mathcal{B}(t) = \bar{t}$. Then

$$\begin{aligned} X(x_j) &= \mathcal{B}^{-1}(\psi_R \lambda g(x_j)) \\ &= \mathcal{B}^{-1} \psi_R \varphi_R(\bar{x}_j) \\ &= \psi_R \varphi_R(x_j) \quad (\because \mathcal{B}(t) = \bar{t}) \end{aligned}$$

So

$$\begin{array}{ccc} F & & F \\ \varphi \swarrow & \lambda \rightarrow & \varphi_R \swarrow \\ & P & \mathbb{P}\mathbb{R}' \\ \chi \swarrow & & \chi \swarrow \\ & \downarrow \psi & \downarrow \psi_R \\ \langle t \rangle & \xrightarrow{\mathcal{B}} & \langle t \rangle \end{array}$$

K reversible $\Rightarrow \exists p$ s.t.

$$\begin{array}{ccccccc} F & & & & F & & \\ \varphi \swarrow & & & & \varphi_R \swarrow & & \\ & P & \xrightarrow{p} & P & \xrightarrow{\lambda} & \mathbb{P}\mathbb{R}' & \\ \chi \swarrow & & & & \chi \swarrow & & \\ & \downarrow \psi & & \downarrow & \downarrow \psi_R & & \\ \langle t \rangle & \xrightarrow{\mathcal{B}} & \langle t \rangle & \xrightarrow{\mathcal{B}} & \langle t \rangle & & \end{array}, \quad \mathcal{B} \circ \mathcal{B} = \text{id}.$$

$$t \mapsto \bar{t} \mapsto t$$

Hence $A = \left(\frac{\partial x_i}{\partial x_j} \right) \tilde{x} \sim \left(\frac{\partial \bar{x}_i}{\partial x_j} \right) \tilde{x}$.

(λp gives sequence of Tietze transformations)

Suppose $r_i = x_a x_b \bar{x}_a \bar{x}_c$. Then $s_i = \bar{x}_a \bar{x}_b x_a x_c$.

So

$$\frac{\partial r_i}{\partial x_j} = \begin{array}{ccc} a & b & c \\ 1 - x_c & x_a & -1 \\ \downarrow \tilde{x} & & \\ 1 - t & t & -1 \quad (i: P \text{ Wirtinger}) \end{array}$$

$$\text{and } \frac{\partial s_i}{\partial x_j} = \begin{array}{ccc} a & b & c \\ -\bar{x}_a + \bar{x}_a \bar{x}_b & -\bar{x}_a \bar{x}_b & \bar{x}_c \\ \downarrow \tilde{x} & & \\ -\bar{t} + \bar{t}^2 & -\bar{t}^2 & \bar{t} \\ \uparrow (x \mapsto t) & & \\ 1 - \bar{t} & \bar{t} & -1 \end{array}$$

Hence $(\frac{\partial s_i}{\partial x_j})^{\tilde{x}} \sim \bar{A}$, and $A \sim \bar{A}$.

Generally: $Q = \langle y_1, \dots, y_n \mid S_1, \dots, S_r \rangle$, arbitrary presentation of K . Alexander matrices $B, \bar{B}$.

$\exists \lambda: P \xrightarrow{\sim} Q$ isomorphism s.t.

$$\begin{array}{ccccc} F & \rightarrow & P & \xrightarrow{\lambda} & Q \leftarrow \langle y_1, \dots, y_n \rangle \\ & \searrow x & \downarrow & & \downarrow \\ & & \langle t \rangle & \xrightarrow{B} & \langle t \rangle \end{array}$$

We may suppose $(\frac{\partial}{\partial x_j})^{\tilde{x}} = A$, $(\frac{\partial}{\partial x_j})^{\tilde{x}^*} = B$.

Then $S = \text{identity} \Rightarrow A \sim B$; $S(t) = \bar{t} \Rightarrow A \sim \bar{B}$.

But $A \sim \bar{A}$, and $A \sim B \Leftrightarrow \bar{A} \sim \bar{B}$,

$\Rightarrow$ EITHER $B \sim A \sim \bar{A} \sim \bar{B}$

OR $B \sim \bar{A} \sim A \sim \bar{B}$

$\Rightarrow B \sim \bar{B}$. Q.E.D.

Theorem 2 (Partial converse to Thm. 1)

$A \sim \bar{A} \Leftrightarrow \exists \bar{P}: G/G'' \xrightarrow{\cong} G/G''$ s.t. the induced $S: G/G' \xrightarrow{\cong} G/G'$ takes a generator to its inverse.

Proof. See [P2], (1-5).

Definition. K weakly reversible $\Leftrightarrow \exists \bar{P}: G/G'' \xrightarrow{\cong} G/G''$ s.t. the induced $S: G/G' \xrightarrow{\cong} G/G'$ takes a generator to its inverse.

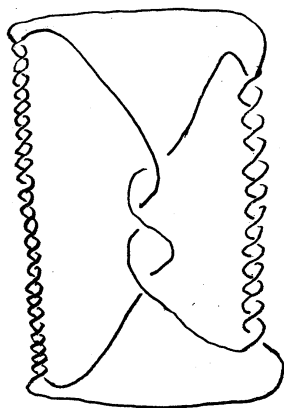
Theorem 1 + Theorem 2 $\Rightarrow$

Summary Reversible $\Rightarrow$ Weakly reversible

Weakly reversible $\Leftrightarrow A \sim \bar{A}$, $\forall P$.

Proposition The pretzel knot $(25, -3, 13)$ is not weakly reversible. Hence it is not reversible.

$(25, -3, 13) =$



Proof. $A = \begin{bmatrix} 5(1-t) & 2-t & 0 \\ 1-2t & 11(1-t) & 0 \end{bmatrix}$ is an Alexander matrix.

$A \sim \bar{A} \Rightarrow$ Row class $A \sim$ column class A (See [FS] for definitions etc.)

$R =$ ring of integers of $\mathbb{Q}(e)$, $e = \sqrt{-211}$.

$e^2 \equiv 1 \pmod{4} \Rightarrow R = \mathbb{Z} + \mathbb{Z}(\frac{1}{2}(1+e))$. ([ZS])

$R' = R[\frac{1}{53}]$.

Row class $A = ((52+e)/53, 11)_{R'}$

Column class $A = ((52+e)/53, 5)_{R'}$.

Equivalent $\stackrel{[P2]}{\Rightarrow} I = (5\alpha, \alpha^2, 25)_{R'}$ principle R ideal, $\alpha = (-3+e)/2$.

Write $I = R_\beta$, $\beta = \frac{1}{2}(q_0 + q_1 e) \in R$.

Now $I \subset (2S, \alpha)R$, and $|2S| = 5^4$, $|\alpha| = \alpha\bar{\alpha} = 55$.

$$\Rightarrow 5 \mid |r|, \forall r \in I$$

$$\Rightarrow 1 \notin I.$$

Further $| \beta | \neq 5$, $\because | \beta | = \frac{1}{4}(q_0^2 + 211q_1^2) \neq 5 \quad \forall q_0, q_1 \in \mathbb{Z}$.

But $2S \in I \Rightarrow \exists r \in R$ s.t. $2S = r\beta$.

$$\text{Hence } |2S| = 5^4 = |r||\beta|.$$

$$\text{But } 1 \notin I \Rightarrow | \beta | \neq 1.$$

And $| \beta | \neq 5$, and similarly $|r| \neq 5$. Hence

$$\text{EITHER } | \beta | = 2S \Rightarrow \beta = 5,$$

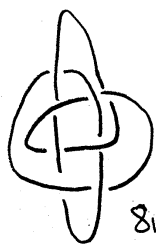
$$\text{OR } | \beta | = (2S)^2 \Rightarrow |r| = 1 \Rightarrow \beta = 2S.$$

$$\text{But } I \ni \alpha^2 = \frac{1}{4}(-3+e)(-3+e) = (-101-3e)/2 \neq r \cdot S \quad \forall r \in R \text{ or } r = 2S$$

a contradiction to the principality of I .

Hence $A \not\sim \bar{A}$. Q.E.D.

Remark 1.



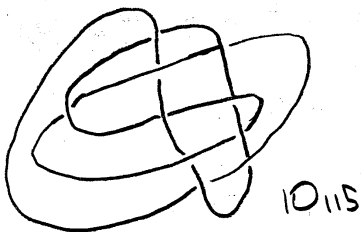
817 is non positive amphicheiral, and non-invertible, but negative amphicheiral ([K2]). Moreover A. Kawachi has shown (see Appendix) 817 is non-reversible. Further

8_{17} has a cyclic module; thus

$$A_{8_{17}} = [\Delta] = [\bar{\Delta}] = \bar{A} \quad (\Delta = \text{Alexander polynomial})$$

$\Rightarrow 8_{17}$ is weakly reversible.

Remark 2.



10_{115} is non +ve amphicheiral, and non-invertible.

$$A_{115} = \begin{bmatrix} t^2 - t + 1 & 2t & 0 \\ 2(t-1) & t^4 - 8t^3 + 17t^2 - 12t + 1 & 0 \end{bmatrix}$$

Hence row class = Column class, and the method of proof used in the proposition is inapplicable.

Conjecture. 10_{115} is not weakly reversible.

Finally, we construct an infinity of non reversible knots.

$$\text{Set } A = \begin{bmatrix} 5(1-t) & 2-t \\ 1-2t & 11(1-t) \end{bmatrix},$$

and $A(p, q) = \begin{bmatrix} A & & 0 \\ \vdots & \ddots & \vdots \\ \underbrace{A}_p & \underbrace{\bar{A}}_q & \vdots \\ & & \ddots \\ & & \underbrace{\bar{A}}_q & 0 \end{bmatrix}$ $2(p+q) \times 2(p+q)+1$ matrix.

Theorem 3. $A(p, q) \sim A(p', q') \Leftrightarrow p=p', q=q'.$

Proof. $\Leftarrow$ clear

$$\Rightarrow \Delta_{A(p, q)} = \Delta_{A(p', q')} \Rightarrow p+q = p'+q'.$$

Column class $A(p, q) = [(\alpha, 5)_{R'}]^p [(\alpha, 11)_{R'}]^q$

Row class $A(p', q') = [(\alpha, 5)_{R'}]^{p'} [(\alpha, 11)_{R'}]^{q'}.$

Equivalent $\Rightarrow I^n$ principle, $I = (5\alpha, \alpha^2, 25)_{R'}$, $p-p' = n = q'-q.$

Lemma: R Dedekind domain, I^n principle $\Rightarrow I$ principle.

Proof. (i) Without valuations; see [P2].

(ii) With valuations; see [D], p. 47.

By the proposition, I is not principle $\Rightarrow n=0$, or $p=p', q=q'.$

Q.E.D.

Set $K_1 = (25, -3, 13)$ pretzel knot.

Define $K(p, q) = \#_p K_1 \#_q (-K_1)$.

Then $A_{K(p, q)} = A(p, q)$. Theorem 3 $\Rightarrow$

Theorem 4. $\{K(p, q)\}_{p > q}$ is an infinite collection of non reversible knots with mutually inequivalent Alexander matrices.

Appendix

Claim 8_7 is non reversible.

Proof. (Kawachi) Let $K = 8_7$, $E = S^3 - \dot{N}(K)$, $G = \pi_1(E)$, and suppose $\exists p: G \xrightarrow{\cong} G$
 $t \mapsto \bar{t}$.

As K is hyperbolic, $\exists p \Rightarrow \exists h: E \rightarrow E$ of finite order such that $h_*(m) = -m + a$, $h_*(l) = \pm l$ in $H_1(\partial E)$. (Mostow rigidity). There are two cases:

(i) h orientation preserving $\Rightarrow h_*(l) = -l \Rightarrow a = 0$
 (ii) h has finite order $\Rightarrow h$ can be extended to a homeomorphism of S^3 , contradicting the non invertibility of K .

This in fact shows: K hyperbolic, non-invertible, signature $\sigma(K) \neq 0 \Rightarrow K$ non reversible, a result first pointed out to me by M. Sakuma.

(ii) h is orientation reversing. So $h_*(1) = -1$.

a) $\text{Fix}(h) \neq \emptyset$.

Proof. $H_*(E) = H_*(S^1)$. Lefschetz number

$\lambda(h) = 1 - (-1) = 2 \neq 0$. Lefschetz fixed point theorem $\Rightarrow \text{Fix}(h) \neq \emptyset$.

b) We can assume $h^{2^m} = \text{identity}$ ($m \geq 1$).

Proof. If the order of $h = 2^m n$, n odd, replace h by h^n .

c) Proof that $m \neq 2$

$h' = h^{2^{m-1}} : E \rightarrow E$ is an orientation preserving involution.

Hence $\exists \bar{h}' : S^3 \rightarrow S^3$ orientation preserving involution such that $\bar{h}'(K) = K$. $\bar{h}'$ preserves the orientation of K , and $\emptyset \neq \text{Fix}(h) \subset \text{Fix}(\bar{h}')$. Hence by Smith theory,

(note $H_2(E) \cong H_2(S')$) $\text{Fix}(h') = S'$, but $S' \cap K = \emptyset$.
 It now follows from [M] that the Alexander polynomial of K must be of the form $A_{817} = F(t)F_0(t)$, where $F_0(t) \equiv F(t)P_2(t)$. Here 2 is some odd number, $p_2(t) = (t^2 - 1)/(t - 1)$, and $\equiv$ means equal considered as $\mathbb{Z}_2\langle t \rangle$ polynomials. The Alexander polynomial of 817 is not of this form.

(d) Prove that $m \neq 1$.

$h^2 = \text{identity}$, h orientation reversing, $\text{Fix}(h) \neq \emptyset$ ^[KH] $\Rightarrow$
 $A_{817} = F(t)^2$, for some $F(t) \in \mathbb{Z}\langle t \rangle$. The Alexander polynomial of 817 is not of this form. Q.E.D.

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