

GLOBAL DYNAMICS OF SOLUTIONS WITH GROUP INVARIANCE FOR THE NONLINEAR SCHRÖDINGER EQUATION

TAKAHISA INUI

ABSTRACT. We consider the focusing mass-supercritical and energy-subcritical nonlinear Schrödinger equation (NLS). We are interested in the global behavior of the solutions to (NLS) with group invariance. By the group invariance, we can determine the global behavior of the solutions above the ground state standing waves.

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1. INTRODUCTION

1.1. Background. We consider the focusing mass-supercritical and energy-subcritical nonlinear Schrödinger equation:

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $d \in \mathbb{N}$ and $1 + 4/d < p < 1 + 4/(d - 2)$. Note that we regard $1 + 4/(d - 2)$ as ∞ if $d = 1, 2$. It is known (see [19] and the standard texts [7, 47, 35]) that this

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equation (NLS) is locally well-posed in $H^1(\mathbb{R}^d)$ and the energy, the mass, and the momentum, which are defined as follows, are conserved.

$$\text{(Energy)} \quad E(u) := \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}}^{p+1},$$

$$\text{(Mass)} \quad M(u) := \|u\|_{L^2}^2,$$

$$\text{(Momentum)} \quad P(u) := \text{Im} \int_{\mathbb{R}^d} \overline{u(x)} \nabla u(x) dx.$$

Since a pioneer work by Kenig and Merle [25], many researchers have studied the global dynamics for (NLS). For the 3d cubic Schrödinger equation, Holmer and Roudenko [21] proved that the following two statements hold if the initial data $u_0 \in H^1$ is radially symmetric and satisfies the mass-energy condition $M(u_0)E(u_0) < M(Q)E(Q)$ where Q is the ground state solution.

- $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2} \Rightarrow$ the solution scatters.
- $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} > \|Q\|_{L^2} \|\nabla Q\|_{L^2} \Rightarrow$ the solution blows up in finite time.

For the non-radial solutions, Duyckaerts, Holmer, and Roudenko [12] obtained the scattering result and Holmer and Roudenko [22] proved that the solutions in the above blow-up region blow up in finite time or grow up at infinite time. Fang, Xie, and Cazenave [18] extended the scattering result and Akahori and Nawa [1] extended both the scattering and the blow-up result to (NLS). To explain their result in detail, we introduce some notations. Let ω be a positive number. We define the action S_ω by

$$S_\omega(\varphi) := E(\varphi) + \frac{\omega}{2} M(\varphi).$$

We also call this action mass-energy. Moreover, let K denote the functional which appears in the virial identity (see (1.1)), that is,

$$K(\varphi) := \partial_\lambda (S_\omega(\varphi^\lambda))|_{\lambda=0} = \frac{2}{d} \|\nabla \varphi\|_{L^2}^2 - \frac{p-1}{p+1} \|\varphi\|_{L^{p+1}}^{p+1},$$

where $\varphi^\lambda(x) := e^{\lambda} \varphi(e^{\frac{2}{d}\lambda} x)$. We consider the minimizing problem

$$l_\omega := \inf \{ S_\omega(\varphi) : \varphi \in H^1(\mathbb{R}^d) \setminus \{0\}, K(\varphi) = 0 \}.$$

It is known that there exists a unique radial positive solution Q_ω of the elliptic equation

$$-\Delta \varphi + \omega \varphi - |\varphi|^{p-1} \varphi = 0,$$

and it attains the minimizing problem l_ω , that is, $l_\omega = S_\omega(Q_\omega)$ and $K(Q_\omega) = 0$ hold (see [44, 4, 5, 32]). Fang, Xie, and Cazenave [18] proved (1) and Akahori and Nawa [1] proved both (1) and (2) in the following theorem. See also (1.26) in [1] for the setting by the frequency ω .

Theorem 1.1. *Let $\omega > 0$, $u_0 \in H^1(\mathbb{R}^d)$ satisfy $S_\omega(u_0) < l_\omega$, and u be the solution of (NLS) with the initial data u_0 . Then, the following statements hold.*

- (1) *If $K(u_0) \geq 0$, then the solution u scatters.*
- (2) *If $K(u_0) < 0$, then the solution u blows up in finite time or grows up at infinite time. More precisely, one of the following four cases occurs.*
 - (a) *u blows up in finite time in both time directions.*

- (b) u blows up in positive finite time and u is global in the negative time direction and $\limsup_{t \rightarrow -\infty} \|\nabla u(t)\|_{L^2} = \infty$.
- (c) u blows up in negative finite time and u is global in the positive time direction and $\limsup_{t \rightarrow \infty} \|\nabla u(t)\|_{L^2} = \infty$.
- (d) u is global in both time directions and $\limsup_{t \rightarrow \pm\infty} \|\nabla u(t)\|_{L^2} = \infty$.

For other studies of global dynamics of dispersive equations, see [16, 41, 17] (other global dynamics of (NLS)), [25, 15, 33, 30, 9, 40] (energy-critical NLS), [31, 29, 10] (mass-critical NLS), [38, 39, 27] (mass-subcritical NLS), [26, 14, 13] (wave equations), [23, 28, 42, 24] (nonlinear Klein-Gordon equations), and references therein.

We apply Theorem 1.1 to the even and odd solutions of (NLS) in the one dimensional case, *i.e.* $d = 1$. For even solutions, Theorem 1.1 is optimal since we have

$$\inf\{S_\omega(\varphi) : \varphi \in H_{\text{even}}^1(\mathbb{R}) \setminus \{0\}, K(\varphi) = 0\} = l_\omega,$$

where $H_{\text{even}}^1(\mathbb{R}) = \{\varphi \in H^1(\mathbb{R}) : \varphi(x) = \varphi(-x)\}$. On the other hand, for odd solutions, Theorem 1.1 is not optimal. Indeed, by an easy calculation, we can show

$$\inf\{S_\omega(\varphi) : \varphi \in H_{\text{odd}}^1(\mathbb{R}) \setminus \{0\}, K(\varphi) = 0\} = 2l_\omega,$$

where $H_{\text{odd}}^1(\mathbb{R}) = \{\varphi \in H^1(\mathbb{R}) : \varphi(x) = -\varphi(-x)\}$. Thus, we expect that we can determine the global behavior of the odd solutions such that $S_\omega(u) < 2l_\omega$ by the functional K .

Noting that the odd solutions can be regarded as the invariant solutions by the action of a group (see the following section), we expect that we can determine the global behavior of the invariant solutions by the action of more general group whose mass-energy S_ω is larger than l_ω . In the present paper, we weaken the condition $S_\omega < l_\omega$ in Theorem 1.1 by group invariance.

1.2. Main Result. We are interested in the global behavior of the solutions with group invariance for (NLS). Let $O(d)$ denote the set of $d \times d$ orthogonal matrices, *i.e.*

$$O(d) := \{\mathcal{R} \in M_d(\mathbb{R}) : \mathcal{R}^T \mathcal{R} = \mathcal{I}_d\},$$

where the transpose of a matrix $\mathcal{A}$ is written by $\mathcal{A}^T$ and $\mathcal{I}_d$ denotes the $d \times d$ identity matrix. $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ is a group with the binary operation $(+, \cdot)$. Let a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ satisfy the following assumption throughout this paper.

(A). For $(\theta_1, \mathcal{G}_1), (\theta_2, \mathcal{G}_2) \in G$, if $\mathcal{G}_1 = \mathcal{G}_2$, then we have $\theta_1 = \theta_2$.

Due to the assumption (A), we can use the notation $\mathcal{G}$ without confusion to denote not only a matrix but also an element of G . For a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, we say that a function φ is G -invariant (or with G -invariance) if $\varphi = \mathcal{G}\varphi$ for all $\mathcal{G} \in G$, where $\mathcal{G}\varphi(x) := e^{-i\theta}(\varphi \circ \mathcal{G}^{-1})(x) = e^{-i\theta}\varphi(\mathcal{G}^{-1}x)$ for $\mathcal{G} = (\theta, \mathcal{G}) \in \mathbb{R}/2\pi\mathbb{Z} \times O(d)$. For example, the radial functions are the invariant functions by the action of the group $G = \{(0, \mathcal{G}) : \mathcal{G} \in O(d)\}$ and the odd functions are the invariant functions by the action of the group $G = \{(0, \mathcal{I}_d), (\pi, -\mathcal{I}_d)\}$. We define the Sobolev space with G -invariance by

$$H_G^1 := \{\varphi \in H^1(\mathbb{R}^d) : \varphi = \mathcal{G}\varphi, \forall \mathcal{G} \in G\}.$$

If the initial data u_0 belongs to H_G^1 , then the corresponding solution to (NLS) is also G -invariant since the Laplacian Δ is invariant for group actions by $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$.

and (NLS) is gauge invariant. We remark that if (A) does not hold, then the results in the present paper is trivial since we have $H_G^1 = \{0\}$.

For a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, we consider the restricted minimizing problem

$$l_\omega^G := \inf\{S_\omega(\varphi) : \varphi \in H_G^1 \setminus \{0\}, K(\varphi) = 0\},$$

and we define subsets $\mathcal{K}_{G,\omega}^\pm$ in $H^1(\mathbb{R}^d)$ by

$$\begin{aligned}\mathcal{K}_{G,\omega}^+ &:= \{\varphi \in H_G^1 : S_\omega(\varphi) < l_\omega^G, K(\varphi) \geq 0\}, \\ \mathcal{K}_{G,\omega}^- &:= \{\varphi \in H_G^1 : S_\omega(\varphi) < l_\omega^G, K(\varphi) < 0\}.\end{aligned}$$

We define a critical action for the data with G -invariance by

$$\begin{aligned}\mathbf{S}_\omega^G &:= \sup\{\mathbf{S} \in (-\infty, l_\omega^G] : \forall \varphi \in \mathcal{K}_{G,\omega}^+, S_\omega(\varphi) < \mathbf{S} \\ &\Rightarrow \text{the solution to (NLS) with the initial data } \varphi \text{ belongs to } L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))\}.\end{aligned}$$

See (3.1) below for the definition of α and r . We remark that $u \in L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$ implies that the solution u scatters (see Proposition 3.2). Here, we say that the solution u to (NLS) scatters if and only if there exist $\varphi_\pm \in H^1(\mathbb{R}^d)$ such that

$$\|u(t) - e^{it\Delta}\varphi_\pm\|_{H^1} \rightarrow 0 \text{ as } t \rightarrow \pm\infty,$$

where $e^{it\Delta}$ denotes the free propagator of the Schrödinger equation.

Our aim in the present paper is to prove the following theorem.

Theorem 1.2. *Let $\omega > 0$, G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, $u_0 \in H_G^1$ satisfy $S_\omega(u_0) < l_\omega^G$, and u be the solution of (NLS) with the initial data u_0 . Then, the following statements hold.*

- (1) *In addition, we assume either that (i) G is a finite group or that (ii) G is an infinite group such that the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact. If $K(u_0) \geq 0$, then the solution u scatters.*
- (2) *If $K(u_0) < 0$, then the solution u blows up in finite time or grows up at infinite time. More precisely, one of (a)–(d) in Theorem 1.1 occurs*

If $G = \{(0, \mathcal{I}_d)\}$, then Theorem 1.2 is nothing but Theorem 1.1 ([18, 1]). Theorem 1.2 means that we can classify the solution whose mass-energy is larger than that of the ground state standing waves into scattering and blow-up (grow-up) by group invariance. For example, in the one dimensional case, we can determine the global behavior of the odd solutions whose mass-energy is smaller than $2l_\omega$ by the functional K . See Appendix B for other applications of Theorem 1.2.

Remark 1.1.

- (1) *For Theorem 1.2 (1):*
 - (i) *If G is finite, then the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is not compact.*
 - (ii) *If G satisfies that for all $x \in \mathbb{R}^d \setminus \{0\}$, $Gx := \{Gx : G \in G\}$ has infinitely many elements, then the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact (see [3]).*
 - (iii) *In the case that G is an infinite group such that the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is not compact, the scattering result for the solutions with G -invariance is an open problem.*

- (iv) *The solution near a scattering solution scatters. Thus, the solution near a large scattering solution with a group invariance also scatters even if it is not group invariant.*
- (2) *For Theorem 1.2 (2):*
 - (i) *If the solution blows up in finite time, then $\|\nabla u(t)\|_{L^2}$ diverges at the maximal existence time by the local well-posedness.*
 - (ii) *If $G = \{(0, \mathcal{G}) : \mathcal{G} \in O(d)\}$, that is, the solution is radially symmetric, then the solution blows up in finite time in both time directions. See [43] and [21, Theorem 1.1 (2)].*
 - (iii) *If the initial data u_0 satisfies $|x|u_0 \in L^2(\mathbb{R}^d)$, then the corresponding solution u blows up in finite time in both time directions. This statement follows from Glassey's argument (see [20]) and the virial identity*

$$(1.1) \quad \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 4dK(u(t)).$$

See [36] for other sufficient conditions to blow up.

- (iv) *For the cubic nonlinear Schrödinger equation in two dimensions, Martel and Raphaël [37] obtained a grow-up solution. However, it is not known whether a grow-up solution exists or not for (NLS).*

In the present paper, we mainly prove Theorem 1.2 (1) since (2) follows from the result of Du, Wu, and Zhang [11]. To prove Theorem 1.2 (1), we prove the following Proposition 1.3. To state it, we prepare some notations.

We say that a subgroup G' of G satisfies $(*)$ if there exists a sequence $\{x_n\} \subset \mathbb{R}^d$ such that

$$\begin{cases} \{x_n - \mathcal{G}'x_n\} \text{ is bounded for all } \mathcal{G}' \in G', \\ |x_n - \mathcal{G}x_n| \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for all } \mathcal{G} \in G \setminus G'. \end{cases}$$

A subgroup G' of G satisfying $(*)$ is the stabilizer subgroup with respect to some $x \in \mathbb{R}^d$. For a finite group G , we define

$$m_\omega^G := \min_{G' \subsetneq G \text{ satisfying } (*)} \frac{\#G}{\#G'} S_\omega^{G'},$$

where $\#X$ denotes the number of the elements in a set X .

Proposition 1.3. *Let $\omega > 0$, G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, $u_0 \in H_G^1$ satisfy $S_\omega(u_0) < l_\omega^G$, and u be the solution of (NLS) with the initial data u_0 . In addition, we assume one of the following.*

- (i) *G is a finite group and u_0 satisfies $S_\omega(u_0) < m_\omega^G$*
- (ii) *G is an infinite group such that the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact.*

Then, the solution u scatters if $K(u_0) \geq 0$.

For an infinite group G such that the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact, Theorem 1.2 (1) is nothing but Proposition 1.3. On the other hand, for a finite group G , we add to assume that $S_\omega(u_0) < m_\omega^G$ in Proposition 1.3. We will prove $m_\omega^G \geq l_\omega^G$ by an induction argument. See Section 4.3.

1.3. Idea of proof. Theorem 1.2 (1) (i) follows from Proposition 1.3 (i) and Noether's induction argument and Theorem 1.2 (1) (ii) is equivalent to Proposition 1.3 (ii). The proof of Proposition 1.3 is based on the contradiction argument by Kenig and Merle [25]. That is, we find a critical element, whose orbit is precompact in H_G^1 , by assuming that Proposition 1.3 fails and using a concentration compactness argument, and we eliminate it by a rigidity argument. In the argument, we use the non-admissible Strichartz estimate, which was also used in Fang, Xie, and Cazenave [18]. However, unlike [18] and [1], we use a linear profile decomposition lemma for functions with group invariance to extend the mass-energy condition. See Proposition 4.1 in the case that G is finite. That is why we need to modify the construction of a critical element and the rigidity argument. If G satisfies that the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact, we can eliminate the translation parameter in the linear profile decomposition by the compactness of the embedding. In this case, the same argument as in the radial case does work.

The blow-up part, Theorem 1.2 (2), follows directly from the result of Du, Wu, and Zhang [11].

The rest of the present paper is organized as follows. In Section 2, we reorganize variational argument for the data with G -invariance and show the blow-up result, Theorem 1.2 (2), by the method of [11]. Section 3 is devoted to preliminaries for the proof of the scattering result, Theorem 1.2 (1). In Section 4, we consider the case that G is finite. In Section 4.1, we show the linear profile decomposition lemma for functions with the finite group invariance, which is a key ingredient. In Section 4.2, we prove Proposition 1.3 (i) by constructing a critical element and the rigidity argument. In Section 4.3, we derive Theorem 1.2 (1) (i) from Proposition 1.3 (i) by Noether's induction argument. In Section 5, we prove Proposition 1.3 (ii) (or equivalently Theorem 1.2 (1) (ii)). The proof is similar to in the radial case. We collect useful lemmas in Appendix A. In Appendix B, we introduce some applications of Theorem 1.2.

2. VARIATIONAL ARGUMENT AND BLOW-UP RESULT

Lemma 2.1. *If $K(\varphi) \geq 0$, then we have*

$$(2.1) \quad S_\omega(\varphi) \leq \frac{1}{2} \|\nabla \varphi\|_{L^2}^2 + \frac{\omega}{2} \|\varphi\|_{L^2}^2 \leq \frac{d(p-1)}{d(p-1)-4} S_\omega(\varphi).$$

Proof. The left inequality holds obviously. We prove the right inequality. We have

$$0 \leq K(\varphi) = \left(\frac{2}{d} - \frac{p-1}{2} \right) \|\nabla \varphi\|_{L^2}^2 + (p-1)E(\varphi).$$

Adding $\omega(p-1)M(\varphi)/2$, we obtain

$$(2.2) \quad \left(\frac{p-1}{2} - \frac{2}{d} \right) \|\nabla \varphi\|_{L^2}^2 + \frac{\omega}{2} (p-1)M(\varphi) \leq (p-1)S_\omega(\varphi).$$

Therefore,

$$(2.3) \quad \left(p-1 - \frac{4}{d} \right) \left\{ \frac{1}{2} \|\nabla \varphi\|_{L^2}^2 + \frac{\omega}{2} M(\varphi) \right\} \leq (p-1)S_\omega(\varphi).$$

This completes the proof. $\square$

Lemma 2.2. *If $u_0 \in \mathcal{K}_{G,\omega}^+$, then the corresponding solution $u(t)$ belongs to $\mathcal{K}_{G,\omega}^+$ for all existence time t . Moreover, if $u_0 \in \mathcal{K}_{G,\omega}^-$, then the corresponding solution $u(t)$ belongs to $\mathcal{K}_{G,\omega}^-$ for all existence time t .*

Proof. Let $u_0 \in \mathcal{K}_{G,\omega}^+$. Since the energy and the mass are conserved and the solution belongs to H_G^1 , we have $u(t) \in \mathcal{K}_{G,\omega}^+ \cup \mathcal{K}_{G,\omega}^-$ for all existence time t . We assume that there exists $t_1 > 0$ such that $u(t_1) \in \mathcal{K}_{G,\omega}^-$. By the continuity of the solution in $H^1(\mathbb{R}^d)$, there exists $t_0 \in (0, t_1)$ such that $K(u(t_0)) = 0$ and $K(u(t)) < 0$ for $t \in (t_0, t_1]$. By the definition of l_ω^G , if $u(t_0) \neq 0$, then we see that

$$l_\omega^G > E(u_0) + \frac{\omega}{2}M(u_0) = E(u(t_0)) + \frac{\omega}{2}M(u(t_0)) \geq l_\omega^G.$$

This is a contradiction. Thus, $u(t_0) = 0$. By the uniqueness of the solution, $u = 0$ for all time. However, this contradicts $u(t_1) \in \mathcal{K}_{G,\omega}^-$. Thus, we see that $u(t) \in \mathcal{K}_{G,\omega}^+$ for all t . The second statement follows from the same argument. $\square$

Remark 2.1. *By Lemmas 2.1 and 2.2, we obtain the global existence of the solution in $\mathcal{K}_{G,\omega}^+$.*

Lemma 2.3. *Let $\varphi \in H_G^1$ satisfy $S_\omega(\varphi) < l_\omega^G$. Then, one of the following holds.*

$$(2.4) \quad K(\varphi) \geq \min\{4(l_\omega^G - S_\omega(\varphi))/d, \delta \|\nabla \varphi\|_{L^2}^2\}, \text{ or } K(\varphi) \leq -4(l_\omega^G - S_\omega(\varphi))/d, \\ \text{for some } \delta > 0.$$

Proof. Since the statement holds if $\varphi = 0$, we may assume that $\varphi \neq 0$. Let $s(\lambda) := S_\omega(\varphi^\lambda)$, where $\varphi^\lambda(x) = e^\lambda \varphi(e^{\frac{2}{d}\lambda})$. Then, $s(0) = S_\omega(\varphi)$ and $s'(0) = K(\varphi)$. By direct calculations,

$$(2.5) \quad s(\lambda) = \frac{1}{2}e^{\frac{4}{d}\lambda} \|\nabla \varphi\|_{L^2}^2 + \frac{\omega}{2} \|\varphi\|_{L^2}^2 - \frac{1}{p+1}e^{(p-1)\lambda} \|\varphi\|_{L^{p+1}}^{p+1},$$

$$(2.6) \quad s'(\lambda) = \frac{2}{d}e^{\frac{4}{d}\lambda} \|\nabla \varphi\|_{L^2}^2 - \frac{p-1}{p+1}e^{(p-1)\lambda} \|\varphi\|_{L^{p+1}}^{p+1},$$

$$(2.7) \quad s''(\lambda) = \frac{8}{d^2}e^{\frac{4}{d}\lambda} \|\nabla \varphi\|_{L^2}^2 - \frac{(p-1)^2}{p+1}e^{(p-1)\lambda} \|\varphi\|_{L^{p+1}}^{p+1}.$$

Thus, we have $s'' \leq 4s'/d$. First, we consider the case of $K < 0$. Let λ_0 be defined by

$$(2.8) \quad \lambda_0 := \left(p - 1 - \frac{4}{d}\right)^{-1} \log \left(\frac{\frac{2}{d} \|\nabla \varphi\|_{L^2}^2}{\frac{p-1}{p+1} \|\varphi\|_{L^{p+1}}^{p+1}} \right).$$

Then, $s'(\lambda_0) = 0$. Moreover, $\lambda_0 < 0$ since $K < 0$. Integrating $s'' \leq 4s'/d$ on $[\lambda_0, 0]$, we obtain

$$(2.9) \quad s'(0) - s'(\lambda_0) \leq \frac{4}{d}(s(0) - s(\lambda_0)).$$

This completes the proof in the case of $K < 0$. Next, we consider the case of $K > 0$. We define

$$(2.10) \quad \lambda_1 := \left(p - 1 - \frac{4}{d}\right)^{-1} \log \left(\frac{\frac{16}{d^2} \|\nabla \varphi\|_{L^2}^2}{\frac{p-1}{p+1} \left(p - 1 + \frac{4}{d}\right) \|\varphi\|_{L^{p+1}}^{p+1}} \right).$$

Then, $s''(\lambda_1) + 4s'(\lambda_1)/d = 0$. If $\lambda_1 \geq 0$, then, by the definition of λ_1 , we obtain

$$(2.11) \quad K(\varphi) \geq \left(p - 1 + \frac{4}{d}\right)^{-1} \frac{2}{d} \left(p - 1 - \frac{4}{d}\right) \|\nabla \varphi\|_{L^2}^2.$$

Letting $\delta := 2(p - 1 - 4/d)/\{d(p - 1 + 4/d)\}$, we obtain $K(\varphi) \geq \delta \|\nabla \varphi\|_{L^2}^2$. If $\lambda_1 < 0$, then $s''(\lambda) < -4s'(\lambda)/d$ for $\lambda \in [0, \lambda_0]$, where we note that $\lambda_0 > 0$ since $K \geq 0$ and that $s''(\lambda) + 4s'(\lambda)/d < 0$ for all $\lambda > \lambda_1$. Integrating the inequality $s''(\lambda) < -4s'(\lambda)/d$ on $[0, \lambda_0]$, this completes the proof. $\square$

Proof of Theorem 1.2 (2). By Lemmas 2.2 and 2.3, if $u_0 \in \mathcal{H}_{G,\omega}^-$, then the solution u satisfies $K(u(t)) < -4(l_\omega^G - S_\omega(u_0))/d < 0$ for all existence time t . Therefore, Theorem 1.2 (2) follows directly from Theorem 2.1 in [11]. $\square$

3. PRELIMINARIES FOR THE PROOF OF THE SCATTERING RESULT

In this subsection, we introduce some basic facts used to prove the scattering result. Their proofs also can be found in [18]. Let

$$(3.1) \quad \begin{aligned} \alpha &:= \frac{2(p-1)(p+1)}{4-(d-2)(p-1)}, \quad \beta := \frac{2(p-1)(p+1)}{d(p-1)^2+(d-2)(p-1)-4}, \quad \gamma := \frac{2(d+2)}{d} \\ q &:= \frac{4(p+1)}{d(p-1)}, \quad r := p + 1, \quad s := \frac{d}{2} - \frac{2}{p-1}. \end{aligned}$$

Moreover, let β' and r' denote the Hölder exponents of the exponent β and r , respectively.

Lemma 3.1 (Strichartz estimates). *The following estimates are valid.*

$$(3.2) \quad \|e^{it\Delta}\varphi\|_{L^q(\mathbb{R};L^r)} + \|e^{it\Delta}\varphi\|_{L^\gamma(\mathbb{R};L^\gamma)} \lesssim \|\varphi\|_{L^2},$$

$$(3.3) \quad \|e^{it\Delta}\varphi\|_{L^\alpha(\mathbb{R};L^r)} \lesssim \|\varphi\|_{\dot{H}^s},$$

$$(3.4) \quad \left\| \int_0^t e^{i(t-t')\Delta} f(t') dt' \right\|_{L^\alpha(I;L^r)} \lesssim \|f\|_{L^{\beta'}(I;L^{r'})},$$

where I is a time interval and the implicit constant is independent of I .

Proof. The first estimate is a standard Strichartz estimate. The second one is obtained by the Sobolev inequality and the Strichartz estimate. The third one is a non-admissible Strichartz estimate (see [8, Lemma 2.1] or [7, Proposition 2.4.1]). $\square$

Proposition 3.2. *Let $u_0 \in H^1(\mathbb{R}^d)$ and u be the solution to (NLS) with the initial data u_0 . If the solution u is positively global and $u \in L^\alpha((0, \infty) : L^r(\mathbb{R}^d))$, then the solution scatters in the positive time direction. Moreover, the same statement holds in the negative case.*

Proof. Proposition 2.3 in [8] implies u belongs to $L^\eta((0, \infty) : L^\rho(\mathbb{R}^d))$ for any admissible pair (η, ρ) , and then a standard argument gives us the fact that u scatters in the positive time directions (see the argument in Theorem 7.8.1 in [7]). $\square$

Proposition 3.3. *There exists $\varepsilon_{sd} > 0$ such that if $u_0 \in H^1(\mathbb{R}^d)$ and $\|e^{it\Delta}u_0\|_{L^\alpha((0,\infty);L^r)} \leq \varepsilon_{sd}$, then the solution u of (NLS) with the initial data u_0 is positively global and we have*

$$(3.5) \quad \|u\|_{L^\alpha((0,\infty);L^r)} \lesssim \varepsilon_{sd}.$$

In particular, if $\|u_0\|_{H^1} \leq \varepsilon_{sd}$, then the solution u is global and we have

$$(3.6) \quad \|u\|_{L^7(\mathbb{R};L^7)} + \|u\|_{L^\alpha(\mathbb{R};L^r)} + \|u\|_{L^\infty(\mathbb{R};H^1)} \lesssim \|u_0\|_{H^1}.$$

Proof. The first statement follows from Proposition 2.4 in [8]. By (3.3), we obtain $\|e^{it\Delta}u_0\|_{L^\alpha(\mathbb{R};L^r)} \lesssim \|u_0\|_{\dot{H}^s} \leq \varepsilon_{sd}$. Applying Proposition 2.4 in [8] and the Strichartz estimates, we obtain the second statement. See also Proposition 4.3 in [18]. $\square$

Lemma 3.4. *If $\psi \in H_G^1$ satisfies $\|\nabla\psi\|_{L^2}^2/2 + \omega M(\psi)/2 < l_\omega^G$, then there exists a global solution U_+ to (NLS) such that $U_+(0) \in \mathcal{H}_{G,\omega}^+$ and $\|U_+(t) - e^{it\Delta}\psi\|_{H^1} \rightarrow 0$ as $t \rightarrow \infty$. Moreover, the same statement holds in the negative case.*

Proof. We may assume that $\psi \neq 0$ since the statement is true if $\psi = 0$. It is known in [46, Theorem 17] (see also [45, Theorem 8]) that there exist $T \in \mathbb{R}$ and a unique solution $U_+ \in C((T, \infty) : H^1(\mathbb{R}^d))$ of (NLS) such that

$$(3.7) \quad \|U_+(t) - e^{it\Delta}\psi\|_{H^1} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

The uniqueness and the assumption that ψ is G -invariant imply that the solution U_+ is also G -invariant. By the triangle inequality, the Sobolev embedding, (3.7), and $\|e^{it\Delta}\psi\|_{L^{p+1}} \rightarrow 0$ as $t \rightarrow \infty$ (see [7, Corollary 2.3.7]), we have

$$\begin{aligned} \|U_+(t)\|_{L^{p+1}} &\leq \|U_+(t) - e^{it\Delta}\psi\|_{L^{p+1}} + \|e^{it\Delta}\psi\|_{L^{p+1}} \\ &\lesssim \|U_+(t) - e^{it\Delta}\psi\|_{H^1} + \|e^{it\Delta}\psi\|_{L^{p+1}} \rightarrow 0, \end{aligned}$$

as $t \rightarrow \infty$. Therefore, by the conservation laws and the assumption, we obtain

$$S_\omega(U_+) = \lim_{t \rightarrow \infty} S_\omega(U_+(t)) = \frac{1}{2} \|\nabla\psi\|_{L^2}^2 + \frac{\omega}{2} M(\psi) < l_\omega^G$$

and

$$\lim_{t \rightarrow \infty} K(U_+(t)) = \frac{2}{d} \|\nabla\psi\|_{L^2}^2 > 0.$$

Thus, $U_+(t)$ belongs to $\mathcal{H}_{G,\omega}^+$ for large $t > T$. This statement, Lemmas 2.1, and 2.2, imply that U_+ is global in both time directions and $U_+(0) \in \mathcal{H}_{G,\omega}^+$. $\square$

Lemma 3.5 (Perturbation Lemma). *Given $A \geq 0$, there exist $\varepsilon(A) > 0$ and $C(A) > 0$ with the following property. If $u \in C([0, \infty) : H^1(\mathbb{R}^d))$ is a solution of (NLS), if $\tilde{u} \in C([0, \infty) : H^1(\mathbb{R}^d))$ and $e \in L_{loc}^1([0, \infty) : H^{-1}(\mathbb{R}^d))$ satisfy $i\partial_t \tilde{u} + \Delta \tilde{u} + |\tilde{u}|^{p-1} \tilde{u} = e$, for a.e. $t > 0$, and if*

$$(3.8) \quad \|\tilde{u}\|_{L^\alpha([0,\infty);L^r)} \leq A,$$

$$(3.9) \quad \|e\|_{L^{\beta'}([0,\infty);L^{r'})} \leq \varepsilon(A),$$

$$(3.10) \quad \|e^{it\Delta}(u(0) - \tilde{u}(0))\|_{L^\alpha([0,\infty);L^r)} \leq \varepsilon \leq \varepsilon(A),$$

then $u \in L^\alpha((0, \infty) : L^r(\mathbb{R}^d))$ and $\|u - \tilde{u}\|_{L^\alpha([0,\infty);L^r)} \leq C\varepsilon$.

See [18, Proposition 4.7] for the proof.

4. PROOF OF THE SCATTERING RESULT FOR THE FINITE GROUP INVARIANT SOLUTIONS

In this section, we prove Theorem 1.2 (1) (i). Let a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(\mathbb{R}^d)$ be finite throughout this section.

4.1. Linear Profile Decomposition with finite group invariance. We prove a linear profile decomposition for functions with G -invariance. Let $\tau_y\varphi(x) = \varphi(x - y)$ throughout this paper.

Proposition 4.1 (Linear Profile Decomposition with finite group invariance). *Let $\{\varphi_n\} \subset H_G^1$ be a bounded sequence. Then, after replacing a subsequence, for $j \in \mathbb{N}$ there exist a subgroup G^j of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, $\psi^j \in H_{G^j}^1$, $\{W_n^j\} \subset H_{G^j}^1$, $\{t_n^j\} \subset \mathbb{R}$, and $\{x_n^j\} \subset \mathbb{R}^d$ such that*

$$(4.1) \quad \varphi_n = \sum_{j=1}^J e^{it_n^j \Delta} \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} + \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n^J}{\#G}$$

for every $J \in \mathbb{N}$, and the following statements hold.

- (1) For any fixed j , $\{t_n^j\}$ satisfies either $t_n^j = 0$ or $t_n^j \rightarrow \pm\infty$ as $n \rightarrow \infty$,
- (2) For any fixed j , $\{x_n^j\}$ satisfies $x_n^j = \mathcal{G}x_n^j$ for all $\mathcal{G} \in G^j$ and $|x_n^j - \mathcal{G}x_n^j| \rightarrow \infty$ for all $\mathcal{G} \in G \setminus G^j$.
- (3) We have the orthogonality of the parameters: for $j \neq h$,

$$\lim_{n \rightarrow \infty} |t_n^j - t_n^h| = \infty \text{ or } \lim_{n \rightarrow \infty} |\mathcal{G}x_n^j - \mathcal{G}'x_n^h| = \infty \text{ for all } \mathcal{G}, \mathcal{G}' \in G.$$
- (4) We have smallness of the remainder:

$$\limsup_{n \rightarrow \infty} \left\| e^{it_n^J \Delta} \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n^J}{\#G} \right\|_{L^\alpha(\mathbb{R}; L^r)} \rightarrow 0 \text{ as } J \rightarrow \infty.$$

- (5) We have the orthogonality in norms: for all $\lambda \in [0, 1]$,

$$(4.2) \quad \|\varphi_n\|_{\dot{H}^\lambda}^2 = \sum_{j=1}^J \left\| \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right\|_{\dot{H}^\lambda}^2 + \left\| \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n^J}{\#G} \right\|_{\dot{H}^\lambda}^2 + o_n(1),$$

$$(4.3) \quad \|\varphi_n\|_{L^{p+1}}^{p+1} = \sum_{j=1}^J \left\| e^{it_n^j \Delta} \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right\|_{L^{p+1}}^{p+1} + \left\| \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n^J}{\#G} \right\|_{L^{p+1}}^{p+1} + o_n(1)$$

and, in particular,

$$(4.4) \quad S_\omega(\varphi_n) = \sum_{j=1}^J S_\omega \left(e^{it_n^j \Delta} \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right) + S_\omega \left(\sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n^J}{\#G} \right) + o_n(1),$$

$$(4.5) \quad K(\varphi_n) = \sum_{j=1}^J K \left(e^{it_n^j \Delta} \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right) + K \left(\sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n^J}{\#G} \right) + o_n(1).$$

Lemma 4.2. *Let $a > 0$ and $\{\varphi_n\} \subset H_G^1$ satisfy*

$$\limsup_{n \rightarrow \infty} \|\varphi_n\|_{H^1} \leq a < \infty.$$

If $\|e^{it\Delta}\varphi_n\|_{L^\infty(\mathbb{R};L^{p+1})} \rightarrow A$ as $n \rightarrow \infty$, then there exist a subsequence, which is still denoted by $\{\varphi_n\}_{n \in \mathbb{N}}$, a subgroup G' of G , $\psi \in H_{G'}^1$, sequences $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$, and $\{W_n\}_{n \in \mathbb{N}} \subset H_{G'}^1$ such that

$$(4.6) \quad \varphi_n = e^{it_n\Delta} \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n}\psi)}{\#G} + \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n}{\#G},$$

and the following hold.

- (1) $e^{-it_n\Delta}\tau_{-g_{x_n}}\varphi_n \rightharpoonup \mathcal{G}\psi/(\#G/\#G')$ in $H^1(\mathbb{R}^d)$ and $e^{-it_n\Delta}\tau_{-g_{x_n}}\tilde{W}_n \rightarrow 0$ in $H^1(\mathbb{R}^d)$ for all $\mathcal{G} \in G$, where $\tilde{W}_n := \sum_{\mathcal{G} \in G} \mathcal{G}W_n/\#G$.
- (2) The sequence $\{t_n\}$ satisfies either $t_n = 0$ or $t_n \rightarrow \pm\infty$ as $n \rightarrow \infty$.
- (3) The sequence $\{x_n\}$ satisfies $\mathcal{G}'x_n = x_n$ for all $\mathcal{G}' \in G'$ and $|x_n - g_{x_n}| \rightarrow \infty$ for all $\mathcal{G} \in G \setminus G'$.
- (4) We have the orthogonality in norms:

$$\|\varphi_n\|_{\dot{H}^\lambda}^2 - \left\| \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n}\psi)}{\#G} \right\|_{\dot{H}^\lambda}^2 - \left\| \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n}{\#G} \right\|_{\dot{H}^\lambda}^2 \rightarrow 0 \text{ as } n \rightarrow \infty,$$

for all $0 \leq \lambda \leq 1$.

$$\|\varphi_n\|_{L^{p+1}}^{p+1} - \left\| e^{it_n\Delta} \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n}\psi)}{\#G} \right\|_{L^{p+1}}^{p+1} - \left\| \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n}{\#G} \right\|_{L^{p+1}}^{p+1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

- (5) We have

$$\|\psi\|_{H^1} \geq \nu A^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} a^{-\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}},$$

where $\Lambda := d(p-1)/\{2(p+1)\} \in (0, \min\{1, d/2\})$ and the constant $\nu > 0$ is independent of a , A , and $\{\varphi_n\}_{n \in \mathbb{N}}$.

- (6) If $A = 0$, then for every sequences $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$, and $\{W_n\}_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^d)$ satisfying (4.6) and (1), we must have $\psi = 0$.

Proof. Let $\widehat{\chi} \in C_0^\infty(\mathbb{R}^d)$ satisfy $0 \leq \widehat{\chi} \leq 1$ and

$$\widehat{\chi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2. \end{cases}$$

Given $\rho > 0$, we set $\widehat{\chi}_\rho(\xi) := \widehat{\chi}(\xi/\rho)$. For any $u \in H^1(\mathbb{R}^d)$ and $\lambda \in [0, 1]$, we observe that

$$\begin{aligned} \|u - \chi_\rho * u\|_{\dot{H}^\lambda}^2 &= \| |\xi|^\lambda (\widehat{u} - \widehat{\chi}_\rho \widehat{u}) \|_{L^2}^2 = \int_{\mathbb{R}^d} |\xi|^{2\lambda} (1 - \widehat{\chi}_\rho)^2 |\widehat{u}|^2 d\xi \leq \int_{|\xi| \geq \rho} |\xi|^{2\lambda} |\widehat{u}|^2 d\xi \\ &= \int_{|\xi| \geq \rho} |\xi|^{-2(1-\lambda)} |\xi|^2 |\widehat{u}|^2 d\xi \leq \rho^{-2(1-\lambda)} \int_{|\xi| \geq \rho} |\xi|^2 |\widehat{u}|^2 d\xi \\ &\leq \rho^{-2(1-\lambda)} \int_{\mathbb{R}^d} |\xi|^2 |\widehat{u}|^2 d\xi = \rho^{-2(1-\lambda)} \|\nabla u\|_{L^2}^2. \end{aligned}$$

Therefore, we have

$$(4.7) \quad \|u - \chi_\rho * u\|_{\dot{H}^\lambda} \leq \rho^{-(1-\lambda)} \|\nabla u\|_{L^2}.$$

Since $\Lambda < d/2$, by the Hölder inequality, we see that, for any $u \in H^1(\mathbb{R}^d)$,

$$(4.8) \quad |\chi_\rho * u(x)| \leq \int_{\mathbb{R}^d} |\widehat{\chi_\rho}(\xi)| |\widehat{u}(\xi)| d\xi \leq \int_{|\xi| \leq 2\rho} |\widehat{u}(\xi)| d\xi = \int_{|\xi| \leq 2\rho} |\xi|^{-\Lambda} |\xi|^\Lambda |\widehat{u}(\xi)| d\xi \\ \leq \left(\int_{|\xi| \leq 2\rho} |\xi|^{-2\Lambda} d\xi \right)^{1/2} \|u\|_{\dot{H}^\Lambda} \leq \kappa \rho^{\frac{d-2\Lambda}{2}} \|u\|_{\dot{H}^\Lambda},$$

where κ is a constant independent of ρ and u .

First, we consider the case of $A > 0$. It follows from the Sobolev embedding $\|u\|_{L^{p+1}} \leq C \|u\|_{\dot{H}^\Lambda}$, the isometry of $e^{it\Delta}$ on $\dot{H}^\Lambda(\mathbb{R}^d)$, (4.7), $0 \leq \Lambda \leq 1$, and the assumption of $\limsup_{n \rightarrow \infty} \|\varphi_n\|_{H^1} \leq a < \infty$ that

$$\|e^{it\Delta} \varphi_n - e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^{p+1}} \leq C \|e^{it\Delta} \varphi_n - e^{it\Delta}(\chi_\rho * \varphi_n)\|_{\dot{H}^\Lambda} = C \|\varphi_n - (\chi_\rho * \varphi_n)\|_{\dot{H}^\Lambda} \\ \leq C \rho^{-(1-\Lambda)} \|\nabla \varphi_n\|_{L^2} \leq 2C \rho^{-(1-\Lambda)} a,$$

for large $n \in \mathbb{N}$. Choosing $\rho = (4Ca/A)^{\frac{1}{1-\Lambda}}$, we obtain

$$\|e^{it\Delta} \varphi_n - e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^{p+1}} \leq 2C \rho^{-(1-\Lambda)} a \leq \frac{A}{2}.$$

Thus, we have

$$(4.9) \quad \|e^{it\Delta} \varphi_n - e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^{p+1})} \leq \frac{A}{2}.$$

By the triangle inequality, $\|e^{it\Delta} \varphi_n\|_{L^\infty(\mathbb{R}; L^{p+1})} \rightarrow A$ as $n \rightarrow \infty$, and (4.9), we get

$$(4.10) \quad \|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^{p+1})} \\ \geq \|e^{it\Delta} \varphi_n\|_{L^\infty(\mathbb{R}; L^{p+1})} - \|e^{it\Delta} \varphi_n - e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^{p+1})} \\ \geq \frac{3A}{4} - \frac{A}{2} = \frac{A}{4},$$

for large n . On the other hand, for large n , we have

$$(4.11) \quad \|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^{p+1})} \leq \|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^2)}^{\frac{d-2\Lambda}{d}} \|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^\infty)}^{\frac{2\Lambda}{d}} \\ \leq \|\varphi_n\|_{L^\infty(\mathbb{R}; L^2)}^{\frac{d-2\Lambda}{d}} \|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^\infty)}^{\frac{2\Lambda}{d}} \\ \leq (2a)^{\frac{d-2\Lambda}{d}} \|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^\infty)}^{\frac{2\Lambda}{d}},$$

where we have used the Hölder inequality, the isometry of $e^{it\Delta}$ on $L^2(\mathbb{R}^d)$, $\|\chi_\rho * u\|_{L^2} \leq \|u\|_{L^2}$ for all $u \in H^1(\mathbb{R}^d)$, and the assumption of $\limsup_{n \rightarrow \infty} \|\varphi_n\|_{H^1} \leq a < \infty$. Combining (4.10) and (4.11), we get, for large n ,

$$\|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^\infty)}^{\frac{2\Lambda}{d}} \geq (2a)^{-\frac{d-2\Lambda}{d}} \frac{A}{4},$$

or equivalently,

$$\|e^{it\Delta}(\chi_\rho * \varphi_n)\|_{L^\infty(\mathbb{R}; L^\infty)} \geq (2a)^{-\frac{d-2\Lambda}{2\Lambda}} \left(\frac{A}{4}\right)^{\frac{d}{2\Lambda}}.$$

Therefore, there exist $\{\tilde{t}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ and $\{\tilde{x}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$ such that

$$(4.12) \quad \left| e^{-i\tilde{t}_n \Delta} (\chi_\rho * \varphi_n)(\tilde{x}_n) \right| \geq (4a)^{-\frac{d-2\Lambda}{2\Lambda}} \left(\frac{A}{4} \right)^{\frac{d}{2\Lambda}},$$

for large n . We consider the following two cases.

Case1: $\{\tilde{t}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is unbounded.

Case2: $\{\tilde{t}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is bounded.

Case1: Since $\{\tilde{t}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is unbounded, we may assume $\tilde{t}_n \rightarrow \pm\infty$ as $n \rightarrow \infty$ taking a subsequence. Let $t_n := \tilde{t}_n$. Taking a subsequence and using Lemma A.1, we obtain a subgroup of G' of G such that a subsequence, which is still denoted by $\{\tilde{x}_n\}$, satisfies that

$$\begin{cases} \tilde{x}_n - \mathcal{G}' \tilde{x}_n \rightarrow \bar{x}_{\mathcal{G}'} \text{ as } n \rightarrow \infty, & \forall \mathcal{G}' \in G', \\ |\tilde{x}_n - \mathcal{G} \tilde{x}_n| \rightarrow \infty \text{ as } n \rightarrow \infty, & \forall \mathcal{G} \in G \setminus G', \end{cases}$$

for some $\bar{x}_{\mathcal{G}'} \in \mathbb{R}^d$. Using Lemma A.2, we obtain a sequence $\{x_n\}$ such that

$$\begin{cases} x_n - \mathcal{G}' x_n = 0, & \forall \mathcal{G}' \in G', \\ |x_n - \mathcal{G} x_n| \rightarrow \infty \text{ as } n \rightarrow \infty, & \forall \mathcal{G} \in G \setminus G', \end{cases}$$

and there exists $x_\infty \in \mathbb{R}^d$ such that

$$x_n - \tilde{x}_n \rightarrow x_\infty \text{ as } n \rightarrow \infty.$$

Since $\|\varphi_n\|_{H^1}$ is bounded, there exists $\psi \in H^1(\mathbb{R}^d)$ such that, after taking a subsequence, $e^{-it_n \Delta} \tau_{-x_n} \varphi_n \rightharpoonup \psi / (\#G / \#G')$ in $H^1(\mathbb{R}^d)$ as $n \rightarrow \infty$. Here, we note that ψ is G' -invariant since φ_n is G -invariant and $x_n = \mathcal{G}' x_n$ for all $\mathcal{G}' \in G'$. We prove (5). Now, we have $w_n := e^{-i\tilde{t}_n \Delta} \tau_{-\tilde{x}_n} \varphi_n \rightharpoonup \tau_{x_\infty} \psi / (\#G / \#G')$ in $H^1(\mathbb{R}^d)$ as $n \rightarrow \infty$. Since $e^{it \Delta}$ commutes with the convolution with χ_ρ , we find that $e^{-i\tilde{t}_n \Delta} (\chi_\rho * \varphi_n)(\tilde{x}_n) = \chi_\rho * w_n(0)$. By (4.8) and (4.12), we have

$$(4a)^{-\frac{d-2\Lambda}{2\Lambda}} \left(\frac{A}{4} \right)^{\frac{d}{2\Lambda}} \leq \left| \frac{\chi_\rho * \psi(-x_\infty)}{\#G / \#G'} \right| \leq \kappa \rho^{\frac{d-2\Lambda}{2}} \frac{\|\psi\|_{\dot{H}^\Lambda}}{\#G / \#G'} \leq \kappa \rho^{\frac{d-2\Lambda}{2}} \frac{\|\psi\|_{H^1}}{\#G / \#G'}.$$

Since we take $\rho = (4Ca/A)^{\frac{1}{1-\Lambda}}$, we obtain the statement (5). We set $W_n := \varphi_n - e^{it_n \Delta} \tau_{x_n} \psi$. Since φ_n is G -invariant, we see that

$$\varphi_n = \sum_{\mathcal{G} \in G} \frac{\mathcal{G} \varphi_n}{\#G} = \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(e^{it_n \Delta} \tau_{x_n} \psi + W_n)}{\#G} = \sum_{\mathcal{G} \in G} \frac{e^{it_n \Delta} \mathcal{G}(\tau_{x_n} \psi)}{\#G} + \sum_{\mathcal{G} \in G} \frac{\mathcal{G} W_n}{\#G}.$$

This is the statement (4.6). Moreover, W_n is G' -invariant since φ_n and $\tau_{x_n} \psi$ are G' -invariant. We check the statement (1). The first statement $e^{-it_n \Delta} \tau_{-\mathcal{G} x_n} \varphi_n \rightharpoonup \mathcal{G} \psi / (\#G / \#G')$ in $H^1(\mathbb{R}^d)$ follows from the definition of ψ and the G -invariance of φ_n . We prove the second statement $e^{-it_n \Delta} \tau_{-\mathcal{G} x_n} \tilde{W}_n \rightharpoonup 0$ in $H^1(\mathbb{R}^d)$ for all $\mathcal{G} \in G$, where we recall that $\tilde{W}_n = \sum_{\mathcal{G} \in G} \mathcal{G} W_n / \#G$. Let $\{\mathcal{G}_k\}_{k=1}^{\#G / \#G'}$ be the set of left coset representatives, that is, we have

$$G = \sum_{k=1}^{\#G / \#G'} \mathcal{G}_k G'.$$

Since W_n is G' -invariant, we find that

$$\tilde{W}_n = \sum_{\mathcal{G} \in G} \frac{\mathcal{G}W_n}{\#G} = \sum_{k=1}^{\#G/\#G'} \frac{\mathcal{G}_k W_n}{\#G/\#G'}.$$

Let $\mathcal{G} = \mathcal{G}_l \mathcal{G}'$ for some $l \in \{1, 2, \dots, \#G/\#G'\}$ and $\mathcal{G}' \in G'$. Then, by the definition of W_n and the first statement in (1), we obtain

$$\begin{aligned} e^{-it_n \Delta} \tau_{-\mathcal{G}_n} \tilde{W}_n &= e^{-it_n \Delta} \tau_{-\mathcal{G}_l x_n} \varphi_n - \sum_{k=1}^{\#G/\#G'} \frac{\tau_{-\mathcal{G}_l x_n + \mathcal{G}_k x_n} \mathcal{G}_k \psi}{\#G/\#G'} \\ &\rightarrow \frac{\mathcal{G}_l \psi}{\#G/\#G'} - \frac{\mathcal{G}_l \psi}{\#G/\#G'} = 0, \end{aligned}$$

where we note that $|\mathcal{G}_l x_n + \mathcal{G}_k x_n| = |x_n + \mathcal{G}_l^{-1} \mathcal{G}_k x_n| \rightarrow \infty$ since $\mathcal{G}_l^{-1} \mathcal{G}_k \notin G'$ if $k \neq l$. Thus, we get the second statement in (1). Next, we prove (4). We set $\tilde{\psi}_n := \sum_{\mathcal{G} \in G} e^{it_n \Delta} \mathcal{G}(\tau_{x_n} \psi) / \#G$. We have

$$\begin{aligned} \|\varphi_n\|_{\dot{H}^\lambda}^2 &= \|\tilde{\psi}_n + \tilde{W}_n\|_{\dot{H}^\lambda}^2 = \|\tilde{\psi}_n\|_{\dot{H}^\lambda}^2 + \|\tilde{W}_n\|_{\dot{H}^\lambda}^2 + 2(\tilde{\psi}_n, \tilde{W}_n)_{\dot{H}^\lambda} \\ &= \|\psi\|_{\dot{H}^\lambda}^2 + \|\tilde{W}_n\|_{\dot{H}^\lambda}^2 + 2(\tilde{\psi}_n, \varphi_n - \tilde{\psi}_n)_{\dot{H}^\lambda}, \end{aligned}$$

where $(\cdot, \cdot)_{\dot{H}^\lambda}$ denotes the inner product in $\dot{H}^\lambda$. We calculate $(\tilde{\psi}_n, \varphi_n - \tilde{\psi}_n)_{\dot{H}^\lambda}$. Since $\tau_{x_n} \psi$ is G' -invariant, we observe that

$$\tilde{\psi}_n = \sum_{\mathcal{G} \in G} \frac{e^{it_n \Delta} \mathcal{G}(\tau_{x_n} \psi)}{\#G} = \sum_{k=1}^{\#G/\#G'} \frac{e^{it_n \Delta} \mathcal{G}_k(\tau_{x_n} \psi)}{(\#G/\#G')}.$$

By this observation, we have

$$\begin{aligned} &(\tilde{\psi}_n, \varphi_n - \tilde{\psi}_n)_{\dot{H}^\lambda} \\ &= \left(\sum_{k=1}^{\#G/\#G'} \frac{e^{it_n \Delta} \mathcal{G}_k(\tau_{x_n} \psi)}{\#G/\#G'}, \varphi_n - \sum_{l=1}^{\#G/\#G'} \frac{e^{it_n \Delta} \mathcal{G}_l(\tau_{x_n} \psi)}{\#G/\#G'} \right)_{\dot{H}^\lambda} \\ &= \frac{1}{(\#G/\#G')^2} \sum_{k=1}^{\#G/\#G'} \sum_{l=1}^{\#G/\#G'} (e^{it_n \Delta} \mathcal{G}_k(\tau_{x_n} \psi), \varphi_n - e^{it_n \Delta} \mathcal{G}_l(\tau_{x_n} \psi))_{\dot{H}^\lambda} \\ &= \frac{1}{(\#G/\#G')^2} \sum_{k,l=1}^{\#G/\#G'} \{ (e^{it_n \Delta} \mathcal{G}_k(\tau_{x_n} \psi), \varphi_n)_{\dot{H}^\lambda} - (e^{it_n \Delta} \mathcal{G}_k(\tau_{x_n} \psi), e^{it_n \Delta} \mathcal{G}_l(\tau_{x_n} \psi))_{\dot{H}^\lambda} \} \end{aligned}$$

For the first term, we find that, for all $k \in \{1, 2, \dots, \#G/\#G'\}$,

$$(4.13) \quad (e^{it_n \Delta} \mathcal{G}_k(\tau_{x_n} \psi), \varphi_n)_{\dot{H}^\lambda} = (\psi, e^{-it_n \Delta} \tau_{-x_n} \mathcal{G}_k^{-1} \varphi_n)_{\dot{H}^\lambda} \rightarrow \frac{\|\psi\|_{\dot{H}^\lambda}^2}{(\#G/\#G')}$$

since φ_n is G -invariant and $e^{-it_n\Delta}\tau_{-x_n}\varphi_n$ weakly converges to $\psi/(\#G/\#G')$ as $n \rightarrow \infty$ in $H^1(\mathbb{R}^d)$. For the second term, we obtain

$$\begin{aligned}
 (4.14) \quad (e^{it_n\Delta}\mathcal{G}_k(\tau_{x_n}\psi), e^{it_n\Delta}\mathcal{G}_l(\tau_{x_n}\psi))_{\dot{H}^\lambda} &= (\psi, \tau_{-x_n}\mathcal{G}_k^{-1}\mathcal{G}_l(\tau_{x_n}\psi))_{\dot{H}^\lambda} \\
 &= (\psi, \tau_{-x_n+\mathcal{G}_k^{-1}\mathcal{G}_lx_n}\mathcal{G}_k^{-1}\mathcal{G}_l\psi)_{\dot{H}^\lambda} \\
 &\rightarrow \begin{cases} \|\psi\|_{\dot{H}^\lambda}^2, & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}
 \end{aligned}$$

Combining (4.13) with (4.14), we get

$$\begin{aligned}
 &\sum_{k,l=1}^{\#G/\#G'} \{ (e^{it_n\Delta}\mathcal{G}_k(\tau_{x_n}\psi), \varphi_n)_{\dot{H}^\lambda} - (e^{it_n\Delta}\mathcal{G}_k(\tau_{x_n}\psi), e^{it_n\Delta}\mathcal{G}_l(\tau_{x_n}\psi))_{\dot{H}^\lambda} \} \\
 &\rightarrow \sum_{k,l=1}^{\#G/\#G'} \frac{\|\psi\|_{\dot{H}^\lambda}^2}{(\#G/\#G')} - \sum_{k=1}^{\#G/\#G'} \|\psi\|_{\dot{H}^\lambda}^2 = 0.
 \end{aligned}$$

This implies the first statement of (4). We set

$$f_n := \left| \|\varphi_n\|_{L^{p+1}}^{p+1} - \|\tilde{\psi}_n\|_{L^{p+1}}^{p+1} - \|\tilde{W}_n\|_{L^{p+1}}^{p+1} \right|.$$

We recall that, for every $P > 1$ and $l \geq 2$, there exists a constant $C = C_{P,l}$ such that

$$\left| \left| \sum_{j=1}^l z_j \right|^P - \sum_{j=1}^l |z_j|^P \right| \leq C \sum_{j \neq k} |z_j| |z_k|^{P-1},$$

for all $z_j \in \mathbb{C}$ for $1 \leq j \leq l$. This implies that

$$||z_1 + z_2|^{p+1} - |z_1|^{p+1} - |z_2|^{p+1}| \leq C|z_1||z_2|(|z_1|^{p-1} + |z_2|^{p-1}).$$

Letting $g_n = |\tilde{\psi}_n|^{p-1} + |\tilde{W}_n|^{p-1}$, we get

$$\begin{aligned}
 f_n &\leq C \int_{\mathbb{R}^d} |\tilde{\psi}_n(x)| |\tilde{W}_n(x)| g_n(x) dx \\
 &\leq C \int_{\mathbb{R}^d} \left| \sum_{k=1}^{\#G/\#G'} \frac{e^{it_n\Delta}\mathcal{G}_k(\tau_{x_n}\psi)(x)}{\#G/\#G'} \right| |\tilde{W}_n(x)| g_n(x) dx \\
 &\leq C \sum_{k=1}^{\#G/\#G'} \int_{\mathbb{R}^d} |e^{it_n\Delta}\mathcal{G}_k(\tau_{x_n}\psi)(x)| |\tilde{W}_n(x)| g_n(x) dx \\
 &\leq C \sum_{k=1}^{\#G/\#G'} \int_{\mathbb{R}^d} |e^{it_n\Delta}\psi(x)| |\tau_{-x_n}\mathcal{G}_k^{-1}\tilde{W}_n(x)| |\tau_{-x_n}\mathcal{G}_k^{-1}g_n(x)| dx.
 \end{aligned}$$

Note that, by the triangle inequality and the Sobolev embedding,

$$\begin{aligned}
\|\tau_{-x_n} \mathcal{G}_k^{-1} g_n\|_{L^{\frac{p+1}{p-1}}} &= \|g_n\|_{L^{\frac{p+1}{p-1}}} \leq \left\| |\tilde{\psi}_n|^{p-1} \right\|_{L^{\frac{p+1}{p-1}}} + \left\| |\tilde{W}_n|^{p-1} \right\|_{L^{\frac{p+1}{p-1}}} \\
&= \left\| \tilde{\psi}_n \right\|_{L^{p+1}}^{p-1} + \left\| \tilde{W}_n \right\|_{L^{p+1}}^{p-1} \lesssim \left\| \tilde{\psi}_n \right\|_{H^1}^{p-1} + \left\| \tilde{W}_n \right\|_{H^1}^{p-1} \\
&\lesssim \|\psi\|_{H^1}^{p-1} + \|W_n\|_{H^1}^{p-1} < C
\end{aligned}$$

where we use $\{W_n\}$ is bounded in H^1 since $\{\varphi_n\}$ is bounded. And $\left\| \tau_{-x_n} \mathcal{G}_k^{-1} \tilde{W}_n \right\|_{L^{p+1}} = \left\| \tilde{W}_n \right\|_{L^{p+1}} < C$. Now, $\|e^{it_n \Delta} \psi\|_{L^{p+1}} \rightarrow 0$ as $n \rightarrow \infty$ since $t_n \rightarrow \pm\infty$ (see [7, Corollary 2.3.7]). Therefore, by the Hölder inequality, we get

$$f_n \lesssim \|e^{it_n \Delta} \psi\|_{L^{p+1}} \left\| \tau_{-x_n} \mathcal{G}_k^{-1} \tilde{W}_n \right\|_{L^{p+1}} \left\| \tau_{-x_n} \mathcal{G}_k^{-1} g_n \right\|_{L^{\frac{p+1}{p-1}}} \lesssim \|e^{it_n \Delta} \psi\|_{L^{p+1}} \rightarrow 0.$$

This means the second statement of (4).

If $A = 0$, then $\|e^{-it_n \Delta} \tau_{-x_n} \varphi_n\|_{L^{p+1}} = \|e^{-it_n \Delta} \varphi_n\|_{L^{p+1}} \leq \|e^{-it \Delta} \varphi_n\|_{L^\infty(\mathbb{R}; L^{p+1})} \rightarrow 0$. On the other hand, if $e^{-it_n \Delta} \tau_{-x_n} \varphi_n \rightarrow \psi / (\#G / \#G')$ as $n \rightarrow \infty$ in $H^1(\mathbb{R}^d)$, then $e^{-it_n \Delta} \tau_{-x_n} \varphi_n \rightarrow \psi / (\#G / \#G')$ as $n \rightarrow \infty$ in $L^{p+1}(B_R)$ for any $R > 0$ by a compactness argument. Combining them, we get $\psi = 0$.

Case2: Since $\{\tilde{t}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is bounded, we may assume $\tilde{t}_n \rightarrow \bar{t} \in \mathbb{R}$ as $n \rightarrow \infty$ taking a subsequence. Let $t_n := 0$ for all n . Minor modifications imply the statements, (1)–(3) and the first statement of (4). See the argument below (5.22) in [18] for the second statement of (4). $\square$

Proof of Proposition 4.1. We use an induction argument. By the boundedness of $\{\varphi_n\}$ in $H^1(\mathbb{R}^d)$, let $a := \limsup_{n \rightarrow \infty} \|\varphi_n\|_{H^1} < \infty$. Moreover, we define $A_1 := \limsup_{n \rightarrow \infty} \|e^{it \Delta} \varphi_n\|_{L^\infty(\mathbb{R}; L^{p+1})}$. Note that the Sobolev embedding and the boundedness of $\{\varphi_n\}$ in $H^1(\mathbb{R}^d)$ give $A_1 < \infty$. Taking a subsequence, we may assume that $A_1 = \lim_{n \rightarrow \infty} \|e^{it \Delta} \varphi_n\|_{L^\infty(\mathbb{R}; L^{p+1})}$. By Lemma 4.2, we obtain a subsequence, which is still denoted by $\{\varphi_n\}_{n \in \mathbb{N}}$, a subgroup G^1 of G , $\psi^1 \in H_{G^1}^1$, sequences $\{t_n^1\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{x_n^1\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$, and $\{W_n^1\}_{n \in \mathbb{N}} \subset H_{G^1}^1$ such that

$$\varphi_n = \tilde{\psi}_n^1 + \tilde{W}_n^1,$$

where $\tilde{\psi}_n^1 := e^{it_n^1 \Delta} \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n^1} \psi^1) / \#G$ and $\tilde{W}_n^1 := \sum_{\mathcal{G} \in G} \mathcal{G}W_n^1 / \#G$ and the following statements hold.

$$e^{-it_n^1 \Delta} \tau_{-g x_n^1} \varphi_n \rightharpoonup \frac{\mathcal{G}\psi^1}{\#G/\#G^1}, \text{ and } e^{-it_n^1 \Delta} \tau_{-g x_n^1} \tilde{W}_n^1 \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^d) \text{ for all } \mathcal{G} \in G.$$

$$t_n^1 = 0 \text{ or } t_n^1 \rightarrow \pm\infty \text{ as } n \rightarrow \infty.$$

$$\mathcal{G}^1 x_n^1 = x_n^1 \text{ for all } \mathcal{G}^1 \in G^1 \text{ and } |x_n^1 - \mathcal{G}x_n^1| \rightarrow \infty \text{ for all } \mathcal{G} \in G \setminus G^1.$$

$$\|\varphi_n\|_{\dot{H}^\lambda}^2 - \|\tilde{\psi}_n^1\|_{\dot{H}^\lambda}^2 - \|\tilde{W}_n^1\|_{\dot{H}^\lambda}^2 \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for all } 0 \leq \lambda \leq 1.$$

$$\|\varphi_n\|_{L^{p+1}}^{p+1} - \|\tilde{\psi}_n^1\|_{L^{p+1}}^{p+1} - \|\tilde{W}_n^1\|_{L^{p+1}}^{p+1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\|\psi^1\|_{H^1} \geq \nu A_1^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} \left(\limsup_{n \rightarrow \infty} \|\varphi_n\|_{H^1} \right)^{-\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}},$$

where $\nu > 0$ is the constant appearing in Lemma 4.2. We call these properties 1st properties. We note that $\tilde{W}_n^1$ is G -invariant and $A_2 := \limsup_{n \rightarrow \infty} \|e^{it\Delta} \tilde{W}_n^1\|_{L^\infty(\mathbb{R}; L^{p+1})} < \infty$. Taking a subsequence, we may assume that $A_2 = \lim_{n \rightarrow \infty} \|e^{it\Delta} \tilde{W}_n^1\|_{L^\infty(\mathbb{R}; L^{p+1})}$.

Applying Lemma 4.2 as $\varphi_n = \tilde{W}_n^1$, we obtain a subsequence, which is still denoted by $\{\tilde{W}_n^1\}$, a subgroup G^2 of G , $\psi^2 \in H_{G^2}^1$, sequences $\{t_n^2\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{x_n^2\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$, and $\{W_n^2\}_{n \in \mathbb{N}} \subset H_{G^2}^1$ such that

$$\tilde{W}_n^1 = \tilde{\psi}_n^2 + \tilde{W}_n^2,$$

where $\tilde{\psi}_n^2 := e^{it_n^2 \Delta} \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n^2} \psi^2) / \#G$ and $\tilde{W}_n^2 := \sum_{\mathcal{G} \in G} \mathcal{G}W_n^2 / \#G$ and the following statements hold.

$$e^{-it_n^2 \Delta} \tau_{-g x_n^2} \tilde{W}_n^1 \rightharpoonup \frac{\mathcal{G}\psi^2}{\#G/\#G^2}, \text{ and } e^{-it_n^2 \Delta} \tau_{-g x_n^2} \tilde{W}_n^2 \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^d) \text{ for all } \mathcal{G} \in G.$$

$$t_n^2 = 0 \text{ or } t_n^2 \rightarrow \pm\infty \text{ as } n \rightarrow \infty.$$

$$\mathcal{G}^2 x_n^2 = x_n^2 \text{ for all } \mathcal{G}^2 \in G^2 \text{ and } |x_n^2 - \mathcal{G}x_n^2| \rightarrow \infty \text{ for all } \mathcal{G} \in G \setminus G^2.$$

$$\|\tilde{W}_n^1\|_{\dot{H}^\lambda}^2 - \|\tilde{\psi}_n^2\|_{\dot{H}^\lambda}^2 - \|\tilde{W}_n^2\|_{\dot{H}^\lambda}^2 \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for all } 0 \leq \lambda \leq 1.$$

$$\|\tilde{W}_n^1\|_{L^{p+1}}^{p+1} - \|\tilde{\psi}_n^2\|_{L^{p+1}}^{p+1} - \|\tilde{W}_n^2\|_{L^{p+1}}^{p+1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\|\psi^2\|_{H^1} \geq \nu A_2^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} \left(\limsup_{n \rightarrow \infty} \|\tilde{W}_n^1\|_{H^1} \right)^{-\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}},$$

where $\nu > 0$ is the constant appearing in Lemma 4.2. Notice that the 1st properties hold for the subsequence. Repeating this procedure, we obtain a subsequence, which is still denoted by $\{\tilde{W}_n^{j-1}\}$, a subgroup G^j of G , $\psi^j \in H_{G^j}^1$, sequences $\{t_n^j\}_{n \in \mathbb{N}} \subset \mathbb{R}$, $\{x_n^j\}_{n \in \mathbb{N}} \subset \mathbb{R}^d$, and $\{W_n^j\}_{n \in \mathbb{N}} \subset H_{G^j}^1$ such that

$$\tilde{W}_n^{j-1} = \tilde{\psi}_n^j + \tilde{W}_n^j,$$

where $\tilde{\psi}_n^j := e^{it_n^j \Delta} \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n^j} \psi^j) / \#G$ and $\tilde{W}_n^j := \sum_{\mathcal{G} \in G} \mathcal{G} W_n^j / \#G$ and the following statements hold.

$$e^{-it_n^j \Delta} \tau_{-\mathcal{G} x_n^j} \tilde{W}_n^{j-1} \rightharpoonup \frac{\mathcal{G} \psi^j}{\#G / \#G^j}, \text{ and } e^{-it_n^j \Delta} \tau_{-\mathcal{G} x_n^j} \tilde{W}_n^j \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^d) \text{ for all } \mathcal{G} \in G.$$

$$t_n^j = 0 \text{ or } t_n^j \rightarrow \pm\infty \text{ as } n \rightarrow \infty.$$

$$\mathcal{G}^j x_n^j = x_n^j \text{ for all } \mathcal{G}^j \in G^j \text{ and } |x_n^j - \mathcal{G} x_n^j| \rightarrow \infty \text{ for all } \mathcal{G} \in G \setminus G^j.$$

$$\left\| \tilde{W}_n^{j-1} \right\|_{\dot{H}^\lambda}^2 - \left\| \tilde{\psi}_n^j \right\|_{\dot{H}^\lambda}^2 - \left\| \tilde{W}_n^j \right\|_{\dot{H}^\lambda}^2 \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for all } 0 \leq \lambda \leq 1.$$

$$\left\| \tilde{W}_n^{j-1} \right\|_{L^{p+1}}^{p+1} - \left\| \tilde{\psi}_n^j \right\|_{L^{p+1}}^{p+1} - \left\| \tilde{W}_n^j \right\|_{L^{p+1}}^{p+1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\left\| \psi^j \right\|_{H^1} \geq \nu A_j^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} \left(\limsup_{n \rightarrow \infty} \left\| \tilde{W}_n^{j-1} \right\|_{H^1} \right)^{-\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}},$$

where $\nu > 0$ is the constant appearing in Lemma 4.2. We call these properties j th properties. Here, we regard $\tilde{W}_n^0$ as φ_n . Combining 1st, 2nd, $\dots$, and J th properties, we obtain the statements (1), (2), and (5). Note that the orthogonalities of the functionals S_ω and K follows from the orthogonalities in norms. We prove (4). By the orthogonality of H^1 -norm, we have $\limsup_{n \rightarrow \infty} \left\| \tilde{W}_n^j \right\|_{H^1} \leq \limsup_{n \rightarrow \infty} \left\| \varphi_n \right\|_{H^1} = a$ for all j . Thus we see that

$$(4.15) \quad \left\| \psi^j \right\|_{H^1} \geq \nu A_j^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} a^{-\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}}.$$

By the orthogonality of H^1 -norm and Lemma A.3, we also have $\sum_{j=1}^J \left\| \psi^j \right\|_{H^1} \leq a$. Taking the limit $J \rightarrow \infty$, we obtain $\sum_{j=1}^\infty \left\| \psi^j \right\|_{H^1} \leq a$. Combining this with (4.15), we obtain

$$\nu a^{-\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}} \sum_{j=1}^\infty A_j^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} \leq \sum_{j=1}^\infty \left\| \psi^j \right\|_{H^1} \leq a.$$

This gives us $\sum_{j=1}^\infty A_j^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} \lesssim a^{1+\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}} < \infty$ so that $A_J \rightarrow 0$ as $J \rightarrow \infty$. By the Hölder inequality and the Strichartz estimate (3.2), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\| e^{it\Delta} \tilde{W}_n^J \right\|_{L^\alpha(\mathbb{R}; L^r)} &\leq \limsup_{n \rightarrow \infty} \left(\left\| e^{it\Delta} \tilde{W}_n^J \right\|_{L^q(\mathbb{R}; L^r)}^\theta \left\| e^{it\Delta} \tilde{W}_n^J \right\|_{L^\infty(\mathbb{R}; L^r)}^{1-\theta} \right) \\ &\lesssim \limsup_{n \rightarrow \infty} \left\| \tilde{W}_n^J \right\|_{H^1}^\theta \limsup_{n \rightarrow \infty} \left\| e^{it\Delta} \tilde{W}_n^J \right\|_{L^\infty(\mathbb{R}; L^r)}^{1-\theta} \\ &\lesssim a^\theta \limsup_{n \rightarrow \infty} \left\| e^{it\Delta} \tilde{W}_n^J \right\|_{L^\infty(\mathbb{R}; L^r)}^{1-\theta} \rightarrow 0 \end{aligned}$$

as $J \rightarrow \infty$ where $\theta = 2\{4 - (d-2)(p-1)\} / \{d(p-1)^2\} \in (0, 1)$. Thus, we obtain (4). At last, we prove (3). By the statement (6) in Lemma 4.2, $\psi^J = 0$ for some J implies that $\psi^j = 0$ for $j \geq J$. Therefore, there exists $J \in \mathbb{N} \cup \{\infty\}$ such that $\psi^j \neq 0$ for $j \leq J$ and $\psi^j = 0$ for $j > J$. In the case of $J = 1$, there is nothing to prove. We suppose $J \geq 2$. By 1st properties, we have

$$e^{-it_n^1 \Delta} \tau_{-\mathcal{G} x_n^1} \tilde{W}_n^1 \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^d) \text{ for all } \mathcal{G} \in G$$

and

$$e^{-it_n^2 \Delta} \tau_{-\mathcal{G}'x_n^2} \tilde{W}_n^1 \rightharpoonup \frac{\mathcal{G}'\psi^2}{\#G/\#G^2} \neq 0 \text{ in } H^1(\mathbb{R}^d) \text{ for all } \mathcal{G}' \in G.$$

By Lemma A.5 as $f_n = e^{-it_n^1 \Delta} \tau_{-\mathcal{G}x_n^1} \tilde{W}_n^1$, we get $|t_n^1 - t_n^2| + |\mathcal{G}x_n^1 - \mathcal{G}'x_n^2| \rightarrow \infty$ for all $\mathcal{G}, \mathcal{G}' \in G$. We suppose $J \geq 3$ and that the statement (3) is true for $1 \leq j \neq h \leq \mu-1$ for some $\mu < J$. For $j \in \{1, 2, \dots, \mu-1\}$, we have

$$\begin{aligned} \tilde{W}_n^{\mu-1} - \tilde{W}_n^{j-1} &= \tilde{W}_n^{\mu-2} - \tilde{\psi}_n^{\mu-1} - \tilde{W}_n^{j-1} = \tilde{W}_n^{\mu-3} - \tilde{\psi}_n^{\mu-2} - \tilde{\psi}_n^{\mu-1} - \tilde{W}_n^{j-1} \\ &= \dots = - \sum_{\nu=j}^{\mu-1} \tilde{\psi}_n^\nu. \end{aligned}$$

Therefore, we see that

$$(4.16) \quad e^{-it_n^j \Delta} \tau_{-\mathcal{G}x_n^j} \left(\tilde{W}_n^{\mu-1} - \tilde{W}_n^{j-1} \right) = -e^{-it_n^j \Delta} \tau_{-\mathcal{G}x_n^j} \tilde{\psi}_n^j - \sum_{\nu=j+1}^{\mu-1} e^{-it_n^j \Delta} \tau_{-\mathcal{G}x_n^j} \tilde{\psi}_n^\nu.$$

Let $\{\mathcal{G}_k^{(j)}\}_{k=1}^{\#G/\#G^j}$ be the set of left coset representatives and $\mathcal{G} = \mathcal{G}_l^{(j)} \mathcal{G}^j$ for some $l \in \{1, 2, \dots, \#G/\#G^j\}$ and $\mathcal{G}^j \in G^j$. For the first term of the right hand side in (4.16), we have

$$\begin{aligned} e^{-it_n^j \Delta} \tau_{-\mathcal{G}x_n^j} \tilde{\psi}_n^j &= \tau_{-\mathcal{G}x_n^j} \sum_{k=1}^{\#G/\#G^j} \frac{\mathcal{G}_k^{(j)}(\tau_{x_n^j} \psi^j)}{\#G/\#G^j} \\ &= \frac{\mathcal{G}\psi^j}{\#G/\#G^j} + \sum_{\substack{k=1 \\ k \neq l}}^{\#G/\#G^j} \frac{\tau_{-\mathcal{G}_l^{(j)}x_n^j + \mathcal{G}_k^{(j)}x_n^j} \mathcal{G}_k^{(j)}\psi^j}{\#G/\#G^j} \rightharpoonup \frac{\mathcal{G}\psi^j}{\#G/\#G^j}. \end{aligned}$$

The second term of the right hand side in (4.16) weakly converges to 0 in $H^1(\mathbb{R}^d)$ since we suppose that the statement (3) holds for $1 \leq j \neq h \leq \mu-1$. Since $e^{-it_n^j \Delta} \tau_{-\mathcal{G}x_n^j} \tilde{W}_n^{j-1} \rightharpoonup \mathcal{G}\psi^j/(\#G/\#G^j)$, we have

$$e^{-it_n^j \Delta} \tau_{-\mathcal{G}x_n^j} \tilde{W}_n^{\mu-1} \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^d) \text{ for all } \mathcal{G} \in G,$$

and

$$e^{-it_n^\mu \Delta} \tau_{-\mathcal{G}'x_n^\mu} \tilde{W}_n^{\mu-1} \rightharpoonup \frac{\mathcal{G}'\psi^2}{\#G/\#G^2} \neq 0 \text{ in } H^1(\mathbb{R}^d) \text{ for all } \mathcal{G}' \in G.$$

By Lemma A.5 as $f_n = e^{-it_n^j \Delta} \tau_{-\mathcal{G}x_n^j} \tilde{W}_n^{\mu-1}$, we get $|t_n^j - t_n^\mu| + |\mathcal{G}x_n^j - \mathcal{G}'x_n^\mu| \rightarrow \infty$ for all $\mathcal{G}, \mathcal{G}' \in G$. Therefore, we obtain the statement (3). This completes the proof. $\square$

Lemma 4.3. *Let k be a nonnegative integer and $\varphi_j \in H_G^1$ for $j \in \{1, 2, \dots, k\}$ satisfy*

$$\begin{aligned} S_\omega(\sum_{j=1}^k \varphi_j) &\leq l_\omega^G - \delta, & S_\omega(\sum_{j=1}^k \varphi_j) &\geq \sum_{j=1}^k S_\omega(\varphi_j) - \varepsilon, \\ K(\sum_{j=1}^k \varphi_j) &\geq -\varepsilon, & K(\sum_{j=1}^k \varphi_j) &\leq \sum_{j=1}^k K(\varphi_j) + \varepsilon, \end{aligned}$$

for δ, ε satisfying $(1 + d/2)\varepsilon < \delta$. Then we have $0 \leq S_\omega(\varphi_j) < l_\omega^G$ and $K(\varphi_j) \geq 0$ for all $j \in \{1, 2, \dots, k\}$. Namely, we see that $\varphi_j \in \mathcal{K}_{G,\omega}^+$ for all $j \in \{1, 2, \dots, k\}$.

Proof. Let $J_\omega := S_\omega - dK/4$. First, we prove the following equality.

$$l_\omega^G = \inf \{ J_\omega(\varphi) : \varphi \in H_G^1 \setminus \{0\}, K(\varphi) \leq 0 \}.$$

Let l' denote the right hand side. By the definition of l_ω^G and J_ω , we have

$$l_\omega^G = \inf \{ J_\omega(\varphi) : \varphi \in H_G^1 \setminus \{0\}, K(\varphi) = 0 \}.$$

Therefore, we have $l_\omega^G \geq l'$. We prove $l_\omega^G \leq l'$. Take $\varphi \in H_G^1$ such that $K(\varphi) \leq 0$. Then, there exists $\lambda_0 \leq 0$ such that $K(\varphi^{\lambda_0}) = 0$. Thus, we see that

$$l_\omega^G \leq S_\omega(\varphi^{\lambda_0}) = J_\omega(\varphi^{\lambda_0}) \leq J_\omega(\varphi).$$

Taking the infimum for $\varphi \in H_G^1$ such that $K(\varphi) \leq 0$, we obtain $l_\omega^G \leq l'$. Thus, we have $l_\omega^G = l'$. Next, we prove the statement of the present lemma. We assume that there exists an $j \in \{1, 2, \dots, k\}$ such that $K(\varphi_j) < 0$. By $l_\omega^G = l'$ and the positivity of J_ω , we obtain

$$\begin{aligned} l_\omega^G &\leq \sum_{j=1}^k J_\omega(\varphi_j) \\ &= \sum_{j=1}^k \left(S_\omega(\varphi_j) - \frac{d}{4} K(\varphi_j) \right) \\ &= \sum_{j=1}^k S_\omega(\varphi_j) - \sum_{j=1}^k \frac{d}{4} K(\varphi_j) \\ &\leq S_\omega \left(\sum_{j=1}^k \varphi_j \right) + \varepsilon - \frac{d}{4} \left(K \left(\sum_{j=1}^k \varphi_j \right) - \varepsilon \right) \\ &\leq l_\omega^G - \delta + \varepsilon + \frac{d}{2} \varepsilon \\ &< l_\omega^G. \end{aligned}$$

This is a contradiction. So, $K(\varphi_j) \geq 0$ for all $j \in \{1, 2, \dots, k\}$. Moreover, for any $j \in \{1, 2, \dots, k\}$, we have

$$S_\omega(\varphi_j) = J_\omega(\varphi_j) + \frac{d}{4} K(\varphi_j) \geq 0,$$

and

$$S_\omega(\varphi_j) \leq \sum_{j=1}^k S_\omega(\varphi_j) \leq S_\omega \left(\sum_{j=1}^k \varphi_j \right) + \varepsilon \leq l_\omega^G - \delta + \varepsilon < l_\omega^G.$$

This completes the proof. $\square$

We collect lemmas for nonlinear profiles.

Lemma 4.4. *Let $\{x_n\}$ be a sequence, $\psi \in H^1$, and U be a solution of (NLS) with the initial data ψ . Then, we have*

$$U_n(t) = e^{it\Delta} \tau_{x_n} \psi + i \int_0^t e^{i(t-s)\Delta} (|U_n(s)|^{p-1} U_n(s)) ds,$$

where $U_n(t, x) := U(t, x - x_n)$.

Lemma 4.4 follows from the space translation invariance of the equation (NLS).

Lemma 4.5. *Let $\{t_n\}$ satisfy $t_n \rightarrow \pm\infty$, $\{x_n\}$ be a sequence, $\psi \in H^1$, and U be a solution of (NLS) satisfying*

$$\|U_{\pm}(t) - e^{it\Delta}\psi\|_{H^1} \rightarrow 0 \text{ as } t \rightarrow \pm\infty$$

Then, we have

$$U_{\pm,n}(t) = e^{it\Delta}e^{it_n\Delta}\tau_{x_n}\psi + i \int_0^t e^{i(t-s)\Delta}(|U_{\pm,n}(s)|^{p-1}U_{\pm,n}(s))ds + e_{\pm,n}(t),$$

where $U_{\pm,n}(t, x) := U_{\pm}(t + t_n, x - x_n)$ and $\|e_{\pm,n}\|_{L^\alpha(\mathbb{R}; L^r)} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Since $U_{\pm,n}$ is a solution of (NLS) with the initial data $\tau_{x_n}U_{\pm}(t_n)$ by the time and space translation invariance, we have

$$\begin{aligned} e_{\pm,n}(t) &= U_{\pm,n}(t) - e^{it\Delta}e^{it_n\Delta}\tau_{x_n}\psi - i \int_0^t e^{i(t-s)\Delta}(|U_{\pm,n}(s)|^{p-1}U_{\pm,n}(s))ds \\ &= e^{it\Delta}\tau_{x_n}U_{\pm}(t_n) - e^{it\Delta}e^{it_n\Delta}\tau_{x_n}\psi. \end{aligned}$$

By the Strichartz estimate,

$$\|e_{\pm,n}\|_{L^\alpha(\mathbb{R}; L^r)} \lesssim \|U_{\pm}(t_n) - e^{it_n\Delta}\psi\|_{H^1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This completes the proof. $\square$

4.2. Construction of a critical element and Rigidity. By the definition of $\mathbf{S}_\omega^G$, we have $\mathbf{S}_\omega^G \leq l_\omega^G$. Lemma 2.1 and Proposition 3.3 give $\mathbf{S}_\omega^G > 0$. We prove $\mathbf{S}_\omega^G = \min\{m_\omega^G, l_\omega^G\}$ by contradiction argument so that we suppose $\mathbf{S}_\omega^G < \min\{m_\omega^G, l_\omega^G\}$.

Proposition 4.6. *Assume $\mathbf{S}_\omega^G < \min\{m_\omega^G, l_\omega^G\}$. Then, there exists a global solution u^c to (NLS) with G -invariance such that $S_\omega(u^c) = \mathbf{S}_\omega^G$ and $\|u^c\|_{L^\alpha(\mathbb{R}; L^r)} = \infty$.*

We call u^c a critical element.

Proof. By the definition of $\mathbf{S}_\omega^G$ and the assumption of $\mathbf{S}_\omega^G < \min\{m_\omega^G, l_\omega^G\}$, there exists a sequence $\{\varphi_n\} \in \mathcal{K}_{G,\omega}^+$ satisfying $\mathbf{S}_\omega^G < S_\omega(\varphi_n) < \min\{m_\omega^G, l_\omega^G\}$, $S_\omega(\varphi_n) \searrow \mathbf{S}_\omega^G$, and $u_n \notin L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$, where u_n is a global solution with the initial data φ_n . Since $\{\varphi_n\}$ is bounded in $H^1(\mathbb{R}^d)$, we apply the linear profile decomposition with G -invariance, Proposition 4.1, to the sequence $\{\varphi_n\}$ and then we obtain

$$\varphi_n = \sum_{j=1}^J \tilde{\psi}_n^j + \tilde{W}_n^J,$$

where we recall that $\tilde{\psi}_n^j = \sum_{g \in G} e^{it_n^j \Delta} \mathcal{G}(\tau_{x_n^j} \psi^j) / \#G$ and $\tilde{W}_n^J = \sum_{g \in G} \mathcal{G}W_n^J / \#G$. We also see that

$$\begin{aligned} S_\omega(\varphi_n) &= \sum_{j=1}^J S_\omega(\tilde{\psi}_n^j) + S_\omega(\tilde{W}_n^J) + o(1), \\ K(\varphi_n) &= \sum_{j=1}^J K(\tilde{\psi}_n^j) + K(\tilde{W}_n^J) + o(1), \end{aligned}$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. By these decompositions, we have

$$\begin{aligned} S_\omega(\varphi_n) &\leq l_\omega^G - \delta, \\ S_\omega(\varphi_n) &\geq \sum_{j=1}^J S_\omega(\tilde{\psi}_n^j) + S_\omega(\tilde{W}_n^J) - \varepsilon, \\ K(\varphi_n) &\geq 0 > -\varepsilon, \\ K(\varphi_n) &\leq \sum_{j=1}^J K(\tilde{\psi}_n^j) + K(\tilde{W}_n^J) + \varepsilon, \end{aligned}$$

for large n where $\delta = l_\omega^G - S_\omega(\varphi_1)$ and $\varepsilon > 0$ satisfies $(1 + d/2)\varepsilon < \delta$. Therefore, Lemma 4.3 gives us that $\tilde{\psi}_n^j \in \mathcal{H}_{G,\omega}^+$ for all $j \in \{1, 2, \dots, J\}$ and $\tilde{W}_n^J \in \mathcal{H}_{G,\omega}^+$. Thus, for any J , we obtain

$$\mathbf{S}_\omega^G = \lim_{n \rightarrow \infty} S_\omega(\varphi_n) \geq \sum_{j=1}^J \limsup_{n \rightarrow \infty} S_\omega(\tilde{\psi}_n^j).$$

We prove $\mathbf{S}_\omega^G = \limsup_{n \rightarrow \infty} S_\omega(\tilde{\psi}_n^j)$ for some j by a contradiction argument. We assume that $\mathbf{S}_\omega^G = \limsup_{n \rightarrow \infty} S_\omega(\tilde{\psi}_n^j)$ fails for all j . Namely, we assume that $\limsup_{n \rightarrow \infty} S_\omega(\tilde{\psi}_n^j) < \mathbf{S}_\omega^G$ for all j . By reordering, we can choose $0 \leq J_1 \leq J_2 \leq J$ such that

$$\begin{aligned} 1 \leq j \leq J_1 : & \quad t_n^j = 0, \quad \forall n \\ J_1 + 1 \leq j \leq J_2 : & \quad \lim_{n \rightarrow \infty} t_n^j = -\infty, \\ J_2 + 1 \leq j \leq J : & \quad \lim_{n \rightarrow \infty} t_n^j = +\infty. \end{aligned}$$

Above we are assuming that if $a > b$ then there is no j such that $a \leq j \leq b$. For $j \in [0, J_1]$, by the assumption of the contradiction argument and $t_n^j = 0$, we have $0 < \limsup_{n \rightarrow \infty} S_\omega(\sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n^j} \psi^j) / \#G) < \mathbf{S}_\omega^G$. By the choice of $\{x_n^j\}$ and Lemma A.3,

$$\limsup_{n \rightarrow \infty} S_\omega \left(\sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right) = \frac{\#G}{\#G^j} S_\omega \left(\frac{\psi^j}{\#G / \#G^j} \right) < \mathbf{S}_\omega^G < m_\omega^G.$$

Therefore, $S_\omega(\psi^j / (\#G / \#G^j)) < \mathbf{S}_\omega^{G^j}$. By the definition of $\mathbf{S}_\omega^{G^j}$, the solution U^j to (NLS) with the initial data $\psi^j / (\#G / \#G^j)$ belongs to $L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$.

For $j \in [J_1 + 1, J_2]$, we have

$$\begin{aligned}
m_\omega^G &> \mathbf{S}_\omega^G > \limsup_{n \rightarrow \infty} S_\omega \left(\tilde{\psi}_n^j \right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{1}{2} \left\| \sum_{g \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right\|_{\dot{H}^1}^2 + \frac{\omega}{2} \left\| \sum_{g \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right\|_{L^2}^2 \right) \\
&\quad - \frac{1}{p+1} \lim_{n \rightarrow \infty} \left\| e^{it_n^j \Delta} \sum_{g \in G} \frac{\mathcal{G}(\tau_{x_n^j} \psi^j)}{\#G} \right\|_{L^{p+1}}^{p+1} \\
&= \frac{\#G}{\#G^j} \left(\frac{1}{2} \left\| \frac{\psi^j}{\#G/\#G^j} \right\|_{\dot{H}^1}^2 + \frac{\omega}{2} \left\| \frac{\psi^j}{\#G/\#G^j} \right\|_{L^2}^2 \right),
\end{aligned}$$

where we use $\|e^{it_n \Delta} \phi\|_{L^{p+1}} \rightarrow 0$ as $n \rightarrow \infty$ (see [7, Corollary 2.3.7]) and Lemma A.3. This inequality implies that $\psi^j/(\#G/\#G^j)$ satisfies the assumption of Lemma 3.4 as $G = G^j$, where we note that $\mathbf{S}_\omega^{G^j} \leq l_\omega^{G^j}$. Thus, we obtain the global solution U_-^j to (NLS) such that $U_-^j(0) \in \mathcal{K}_{G^j, \omega}^+$ and

$$\left\| U_-^j(t) - e^{it\Delta} \frac{\psi^j}{\#G/\#G^j} \right\|_{H^1} \rightarrow 0 \text{ as } t \rightarrow -\infty.$$

Moreover, U_-^j belongs to $L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$ by the definition of $\mathbf{S}_\omega^{G^j}$ since we have

$$S_\omega(U_-^j) = \frac{1}{2} \left\| \frac{\psi^j}{\#G/\#G^j} \right\|_{\dot{H}^1}^2 + \frac{\omega}{2} \left\| \frac{\psi^j}{\#G/\#G^j} \right\|_{L^2}^2 < \mathbf{S}_\omega^{G^j}.$$

For $j \in [J_2 + 1, J]$, by the similar argument, we obtain a global solution U_+^j such that $U_+^j(0) \in \mathcal{K}_{G, \omega}^+$, $U_+^j \in L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$, and

$$\left\| U_+^j(t) - e^{it\Delta} \frac{\psi^j}{\#G/\#G^j} \right\|_{H^1} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

We define

$$U^j := \begin{cases} U^1, & \text{if } j = 1, \\ U_-^j, & \text{if } j \in [2, J_2], \\ U_+^j, & \text{if } j \in [J_2 + 1, J], \end{cases} \quad \text{and } U_n^j(t, x) := U^j(t + t_n^j, x - x_n^j).$$

Moreover, we define

$$u_n^J := \sum_{j=1}^J \sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)} U_n^j,$$

where $\{\mathcal{G}_k^{(j)}\}_{k=1}^{\#G/\#G^j}$ be the set of left coset representatives. Then u_n^J satisfies

$$\begin{aligned}
i\partial_t u_n^J + \Delta u_n^J + |u_n^J|^{p-1} u_n^J &= e_n^J, \\
e_n^J &= |u_n^J|^{p-1} u_n^J - \sum_{j=1}^J \sum_{k=1}^{\#G/\#G^j} \left| \mathcal{G}_k^{(j)} U_n^j \right|^{p-1} \mathcal{G}_k^{(j)} U_n^j.
\end{aligned}$$

Moreover, we have

$$u_n(0) - u_n^J(0) = \sum_{j=1}^J \sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)} \left(e^{it_n^j \Delta} \tau_{x_n^j} \frac{\psi^j}{\#G/\#G^j} - \tau_{x_n^j} U^j(t_n^j) \right) + \tilde{W}_n^J.$$

To apply the perturbation lemma, Lemma 3.5, we prove the following inequalities hold for large n .

$$(4.17) \quad \|u_n^J\|_{L^\alpha(\mathbb{R}; L^r)} \leq A,$$

$$(4.18) \quad \|e_n^J\|_{L^{\beta'}(R; L^{r'})} \leq \varepsilon(A),$$

$$(4.19) \quad \|e^{it\Delta}(u_n(0) - u_n^J(0))\|_{L^\alpha(\mathbb{R}; L^r)} \leq \varepsilon(A).$$

We prove (4.17). By the definition of U_n^j , we have

$$\begin{aligned} u_n^J(t) &= \sum_{j=1}^J \sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)} U_n^j(t) \\ &= \sum_{j=1}^{J_1} \sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)}(\tau_{x_n^j} U^j(t)) + \sum_{j=J_1+1}^{J_2} \sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)}(\tau_{x_n^j} U_-^j(t + t_n^j)) \\ &\quad + \sum_{j=J_2+1}^J \sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)}(\tau_{x_n^j} U_+^j(t + t_n^j)). \end{aligned}$$

Let v_n^j denote $\sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)}(\tau_{x_n^j} U^j(t))$ when $1 \leq j \leq J_1$, $\sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)}(\tau_{x_n^j} U_-^j(t + t_n^j))$ when $J_1 + 1 \leq j \leq J_2$, and $\sum_{k=1}^{\#G/\#G^j} \mathcal{G}_k^{(j)}(\tau_{x_n^j} U_+^j(t + t_n^j))$ when $J_2 + 1 \leq j \leq J$. Thus, we have

$$u_n^J = \sum_{j=1}^J v_n^j.$$

By (5) in Proposition 4.1 and Lemma A.3, we have

$$\|\varphi_n\|_{H^1}^2 = \sum_{j=1}^J \frac{\#G}{\#G^j} \left\| \frac{\psi^j}{\#G/\#G^j} \right\|_{H^1}^2 + \|\tilde{W}_n^J\|_{H^1}^2 + o_n(1).$$

Therefore, $\sup_{n \in \mathbb{N}} \|\varphi_n\|_{H^1}^2 < \infty$ implies that there exists a finite set $\mathcal{J}$ such that $\|\psi^j / (\#G / \#G^j)\|_{H^1} < \varepsilon_{sd}$ for $j \notin \mathcal{J}$, where ε_{sd} is a constant appearing in Proposition 3.3. Thus, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|u_n^J\|_{L^\alpha(\mathbb{R}; L^r)} &= \limsup_{n \rightarrow \infty} \left\| \sum_{j=1}^J v_n^j \right\|_{L^\alpha(\mathbb{R}; L^r)} \\ &\leq \limsup_{n \rightarrow \infty} \left\| \sum_{j \in \mathcal{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}; L^r)} + \limsup_{n \rightarrow \infty} \left\| \sum_{j \notin \mathcal{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}; L^r)} \\ &\leq \limsup_{n \rightarrow \infty} \sum_{j \in \mathcal{J}} \|v_n^j\|_{L^\alpha(\mathbb{R}; L^r)} + \limsup_{n \rightarrow \infty} \left\| \sum_{j \notin \mathcal{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}; L^r)} \end{aligned}$$

Using $|t_n^j - t_n^h| \rightarrow \infty$ or $|\mathcal{G}x_n^j - \mathcal{G}'x_n^h| \rightarrow \infty$ for all $\mathcal{G}, \mathcal{G}' \in G$ if $j \neq h$ and $|\mathcal{G}_k^{(j)}x_n^j - \mathcal{G}_l^{(j)}x_n^j| \rightarrow \infty$ if $k \neq l$, the first term is estimated as follows.

$$(4.20) \quad \limsup_{n \rightarrow \infty} \sum_{j \in \mathcal{J}} \|v_n^j\|_{L^\alpha(\mathbb{R}; L^r)} = \sum_{j \in \mathcal{J}} \|U^j\|_{L^\alpha(\mathbb{R}; L^r)} < A_1 < \infty.$$

Next, we estimate the second term. By the Gagliardo–Nirenberg inequality $\|f\|_{L^r} \leq C \|\nabla f\|_{L^2}^\eta \|f\|_{L^\gamma}^{1-\eta}$, where $\eta = n^2/4 - n(n+2)/\{2(p+1)\}$ and it satisfies $1 - \eta > \gamma/a$, and the Sobolev embedding $\|f\|_{L^\gamma} \leq C \|f\|_{H^1}$, we obtain

$$(4.21) \quad \|f\|_{L^a(\mathbb{R}; L^r)} \leq C \|f\|_{L^a(\mathbb{R}; H^1)}^{\frac{a-\gamma}{a}} \|f\|_{L^\gamma(\mathbb{R}; L^\gamma)}^{\frac{\gamma}{a}}.$$

We have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\| \sum_{j \notin \mathcal{J}} v_n^j \right\|_{L^\gamma(\mathbb{R}; L^\gamma)}^\gamma &\leq \limsup_{n \rightarrow \infty} \sum_{j \notin \mathcal{J}} \|v_n^j\|_{L^\gamma(\mathbb{R}; L^\gamma)}^\gamma \\ &\quad + C_J \limsup_{n \rightarrow \infty} \sum_{j \notin \mathcal{J}, j \neq k} \int_{\mathbb{R} \times \mathbb{R}^d} |v_n^j| |v_n^k|^{\gamma-1} dx dt \end{aligned}$$

By Proposition 3.3 and $\gamma > 2$, we get

$$\limsup_{n \rightarrow \infty} \sum_{j \notin \mathcal{J}} \|v_n^j\|_{L^\gamma(\mathbb{R}; L^\gamma)}^\gamma \leq C \limsup_{n \rightarrow \infty} \sum_{j \notin \mathcal{J}} \|U^j(t_n^j)\|_{H^1}^2 \leq C \varepsilon_{sd}.$$

And we have

$$\int_{\mathbb{R} \times \mathbb{R}^d} |v_n^j| |v_n^k|^{\gamma-1} dx dt \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where we use $|t_n^j - t_n^h| \rightarrow \infty$ or $|\mathcal{G}x_n^j - \mathcal{G}'x_n^h| \rightarrow \infty$ for all $\mathcal{G}, \mathcal{G}' \in G$ if $j \neq h$ and $|\mathcal{G}_k^{(j)}x_n^j - \mathcal{G}_l^{(j)}x_n^j| \rightarrow \infty$ if $k \neq l$. See Lemma 4.5 and (6.38) in [18]. Thus, we obtain

$$\limsup_{n \rightarrow \infty} \left\| \sum_{j \notin \mathcal{J}} v_n^j \right\|_{L^\gamma(\mathbb{R}; L^\gamma)} < C < \infty.$$

Moreover, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\| \sum_{j \notin \mathcal{J}} v_n^j \right\|_{L^\infty(\mathbb{R}; H^1)}^2 &\leq \limsup_{n \rightarrow \infty} \sum_{j \notin \mathcal{J}} \|v_n^j\|_{H^1}^2 \\ &\quad + 2 \limsup_{n \rightarrow \infty} \sum_{j \notin \mathcal{J}, j \neq k} \langle v_n^j | v_n^k \rangle_{H^1} \end{aligned}$$

The second term tends to 0 as $n \rightarrow \infty$ and the first term is estimated as follows by Proposition 3.3.

$$\limsup_{n \rightarrow \infty} \sum_{j \notin \mathcal{J}} \|v_n^j\|_{H^1}^2 \leq C \varepsilon_{sd}.$$

And thus, we get

$$\limsup_{n \rightarrow \infty} \left\| \sum_{j \notin \mathcal{J}} v_n^j \right\|_{L^\infty(\mathbb{R}; H^1)} < C < \infty.$$

Therefore, by (4.21), we have

$$(4.22) \quad \limsup_{n \rightarrow \infty} \left\| \sum_{j \notin \mathcal{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}; L^r)} < A_2 < \infty.$$

Combining (4.20) and (4.22), we get

$$\limsup_{n \rightarrow \infty} \|u_n^J\|_{L^\alpha(\mathbb{R}; L^r)} < A_1 + A_2 =: A < \infty$$

We prove (4.19). By the triangle inequality, the Strichartz estimate, the definition of U^j , (4) in Proposition 4.1, and Lemmas 4.4 and 4.5, we have

$$\begin{aligned} &\|e^{it\Delta}(u_n(0) - u_n^J(0))\|_{L^\alpha(\mathbb{R}; L^r)} \\ &\leq \sum_{j=1}^J \sum_{k=1}^{\#G/\#G^j} \left\| e^{it\Delta} \left(\tau_{x_n^j} U^j(t_n^j) - e^{it_n^j \Delta} \tau_{x_n^j} \frac{\psi^j}{\#G/\#G^j} \right) \right\|_{L^\alpha(\mathbb{R}; L^r)} + \|e^{it\Delta} \tilde{W}_n^J\|_{L^\alpha(\mathbb{R}; L^r)} \\ &\leq \sum_{j=1}^J \sum_{k=1}^{\#G/\#G^j} \left\| \tau_{x_n^j} U^j(t_n^j) - e^{it_n^j \Delta} \tau_{x_n^j} \frac{\psi^j}{\#G/\#G^j} \right\|_{H^1} + \|e^{it\Delta} \tilde{W}_n^J\|_{L^\alpha(\mathbb{R}; L^r)} \\ &\leq \varepsilon \leq \varepsilon(A), \end{aligned}$$

for large n and J . We prove (4.18). In general, the following inequality holds.

$$\left| \left| \sum_{j=1}^J z^j \right|^{p-1} \sum_{j=1}^J z^j - \sum_{j=1}^J |z^j|^{p-1} z^j \right| \leq C_J \sum_{1 \leq j \neq h \leq J} |z^j|^{p-1} |z^h|.$$

This implies that

$$\|e_n^J\|_{L^{\beta'}(\mathbb{R}; L^{r'})} \leq C_J \sum_{1 \leq j \neq h \leq J} \| |U_n^j|^{p-1} |U_n^h| \|_{L^{\beta'}(\mathbb{R}; L^{r'})}.$$

An approximation argument and $|t_n^j - t_n^h| \rightarrow \infty$ or $|\mathcal{G}x_n^j - \mathcal{G}'x_n^h| \rightarrow \infty$ for all $\mathcal{G}, \mathcal{G}' \in G$ if $j \neq h$ and also use $|\mathcal{G}_k^{(j)}x_n^j - \mathcal{G}_l^{(j)}x_n^j| \rightarrow \infty$ if $k \neq l$ give us $\| |U_n^j|^{p-1} |U_n^h| \|_{L^{\beta'}(\mathbb{R}; L^{r'})} \rightarrow 0$ as $n \rightarrow \infty$. Thus, we obtain (4.18). Applying Lemma 3.5, we conclude that u_n scatters. However, this contradicts the definition of $\{\varphi_n\}$. Therefore, there exists j such that $\mathbf{S}_\omega^G = \limsup_{n \rightarrow \infty} S_\omega(\tilde{\psi}_n^j)$. We may assume $j = 1$. The linear profile decomposition as $J = 1$ and $\tilde{W}_n^1 \in \mathcal{K}_{G,\omega}^+$ imply $\|\tilde{W}_n^1\|_{L^\infty(\mathbb{R}; H^1)} \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 2.1. Therefore, we see that

$$\begin{aligned} \varphi_n &= \tilde{\psi}_n^1 + \tilde{W}_n^1, \\ \|\tilde{W}_n^1\|_{L^\infty(\mathbb{R}; H^1)} &\rightarrow 0, \\ \mathbf{S}_\omega^G &= \lim_{n \rightarrow \infty} S_\omega(\tilde{\psi}_n^1). \end{aligned}$$

We assume that there exists $G^1 \subsetneq G$ such that $x_n^1 = \mathcal{G}^1 x_n^1$ for all $\mathcal{G}^1 \in G^1$ and $|x_n^1 - \mathcal{G}x_n^1| \rightarrow \infty$ for all $\mathcal{G} \in G \setminus G^1$. Let U be a global solution of (NLS) with the initial data $\psi^1/(\#G/\#G^1)$ if $t_n^1 = 0$ or the final data $\psi^1/(\#G/\#G^1)$ if $|t_n^1| \rightarrow \infty$. Then, by the definition of $\mathbf{S}_\omega^{G^1}$, U belongs to $L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$ since we have, by Lemma A.3,

$$\lim_{n \rightarrow \infty} S_\omega(\tilde{\psi}_n^1) = \lim_{n \rightarrow \infty} \frac{\#G}{\#G^1} S_\omega \left(e^{it_n^1 \Delta} \frac{\psi^1}{\#G/\#G^1} \right) = \mathbf{S}_\omega^G < m_\omega^G \leq \frac{\#G}{\#G^1} \mathbf{S}_\omega^{G^1}.$$

By Lemma 3.5 again, this contradicts that u_n does not belong to $L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$. Thus, $G^1 = G$. This means that ψ^1 and W_n^1 are G -invariant, $x_n^1 = \mathcal{G}x_n^1$ for all $\mathcal{G} \in G$, and we see that

$$\varphi_n = e^{it_n^1 \Delta} \tau_{x_n^1} \psi^1 + W_n^1.$$

Let u^c be a global solution of (NLS) with the initial data ψ^1 if $t_n^1 = 0$ or the final data ψ^1 if $|t_n^1| \rightarrow \infty$. Then, u^c is G -invariant. We prove $\|u^c\|_{L^\alpha(\mathbb{R}; L^r)} = \infty$. Suppose that $\|u^c\|_{L^\alpha(\mathbb{R}; L^r)} < \infty$. We observe that $\varphi_n - \tau_{x_n^1} u^c(t_n^1) = e^{it_n^1 \Delta} \tau_{x_n^1} \psi^1 - \tau_{x_n^1} u^c(t_n^1) + W_n^1$, so that we have

$$\|e^{it_n^1 \Delta} (\varphi_n - \tau_{x_n^1} u^c(t_n^1))\|_{L^\alpha(\mathbb{R}; L^r)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By Lemma 3.5, we see that $u_n \in L^\alpha(\mathbb{R} : L^r(\mathbb{R}))$ for large n , which is absurd. Thus, we get $\|u^c\|_{L^\alpha(\mathbb{R}; L^r)} = \infty$. Moreover, we have $S_\omega(u^c) = \lim_{n \rightarrow \infty} S_\omega(e^{it_n^1 \Delta} \psi^1) = \mathbf{S}_\omega^G$. Thus, we get a critical element u^c . $\square$

We say that the solution u is a forward critical element if u is a critical element and satisfies $\|u\|_{L^\alpha([0, \infty); L^r)} = \infty$. In the same manner, we define a backward critical element. We only prove extinction of the forward critical element since that of the backward critical element can be obtained by the similar argument based on time reversibility. The extinction contradicts Proposition 4.6.

Lemma 4.7. *Let u be a forward critical element. There exists a continuous function $x : [0, \infty) \rightarrow \mathbb{R}^d$ such that $\mathcal{G}x(t) = x(t)$ for all $\mathcal{G} \in G$ and $\{u(t, \cdot - x(t)) : t \in [0, \infty)\}$ is precompact in $H^1(\mathbb{R}^d)$.*

The above lemma can be obtained by the same argument as in [12, Proposition 3.2] noting u is G -invariant and $\{x_n^1\}$, which appears in the profile decomposition, satisfies $\mathcal{G}x_n^1 = x_n^1$ for all $\mathcal{G} \in G$.

Lemma 4.8. *Let u be a solution to (NLS) satisfying that there exists a continuous function $x : [0, \infty) \rightarrow \mathbb{R}^d$ such that $\{u(t, \cdot - x(t)) : t \in [0, \infty)\}$ is precompact in $H^1(\mathbb{R}^d)$. Then, for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that*

$$(4.23) \quad \int_{|x+x(t)| > R} |\nabla u(t, x)|^2 + |u(t, x)|^2 + |u(t, x)|^{p+1} dx < \varepsilon \text{ for any } t \in [0, \infty).$$

It can be obtained by using directly the argument of [12, Corollary 3.3].

Lemma 4.9. *Let u be a forward critical element. Then, the momentum must be 0, i.e. $P(u) = 0$.*

Proof. First, we prove $\mathcal{G}P(u) = P(u)$ for all $\mathcal{G} \in G$. By the G -invariance of u , we see that

$$\begin{aligned} P(u) &= P(\mathcal{G}^{-1}u) = \operatorname{Im} \int_{\mathbb{R}^d} \overline{e^{i\theta}u(\mathcal{G}x)} \nabla \{e^{i\theta}u(\mathcal{G}x)\} dx \\ &= \mathcal{G} \operatorname{Im} \int_{\mathbb{R}^d} \overline{u(\mathcal{G}x)} \nabla u(\mathcal{G}x) dx = \mathcal{G} \operatorname{Im} \int_{\mathbb{R}^d} \overline{u(x)} \nabla u(x) dx = \mathcal{G}P(u). \end{aligned}$$

Therefore, the Galilean transformation

$$u_{\xi_0}(t, x) := e^{i(x \cdot \xi_0 - |\xi_0|^2 t)} u(t, x - 2t\xi_0),$$

where $\xi_0 = -P(u)/M(u)$, conserves the G -invariance of the solution. The rest of the proof is same as in [12, Proposition 4.1] and [1, Proposition 4.1 (iii)]. $\square$

We use the following lemma to prove the rigidity lemma, Lemma 4.11.

Lemma 4.10. *Let u be a solution to (NLS) on $[0, \infty)$ such that $P(u) = 0$ and there exists a continuous $x : [0, \infty) \rightarrow \mathbb{R}^d$ such that, for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that*

$$\int_{|x+x(t)| > R} |\nabla u(t, x)|^2 + |u(t, x)|^2 + |u(t, x)|^{p+1} dx < \varepsilon \text{ for any } t \in [0, \infty).$$

Then, we have

$$\frac{x(t)}{t} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

This follows from [12, Lemma 5.1], [18, Proof of Theorem 7.1, Step1].

Lemma 4.11 (Rigidity). *If the solution u with G -invariance satisfies the following properties, then $u = 0$.*

- (1) $u_0 \in \mathcal{K}_{G, \omega}^+$.
- (2) $P(u) = 0$.
- (3) *There exists a continuous $x : [0, \infty) \rightarrow \mathbb{R}^d$ such that $\mathcal{G}x(t) = x(t)$ for all $t \in [0, \infty)$ and $\mathcal{G} \in G$ and, for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that*

$$\int_{|x+x(t)| > R} |\nabla u(t, x)|^2 + |u(t, x)|^2 + |u(t, x)|^{p+1} dx < \varepsilon \text{ for any } t \in [0, \infty).$$

For the proof of Lemma 4.11, see [12, Theorem 6.1] and [18, Theorem 7.1].

Combining Lemmas 4.7, 4.8, and 4.9, the forward critical element satisfies the assumption (1)–(3) in Lemma 4.11. The result by Lemma 4.11 contradicts $S_\omega(u) = \mathbf{S}_\omega^G > 0$. This contradiction comes from the assumption $\mathbf{S}_\omega^G < \min\{m_\omega^G, l_\omega^G\}$ in Proposition 4.6. This means $\mathbf{S}_\omega^G = \min\{m_\omega^G, l_\omega^G\}$, which completes the proof of Proposition 1.3 (i).

4.3. Proof of Theorem 1.2. First, we prove the following lemma.

Lemma 4.12. *Let $G \subset \mathbb{R}/2\pi\mathbb{Z} \times O(d)$ be a finite group and G' be a subgroup of G satisfying (*). Then, we have*

$$l_\omega^G \leq \frac{\#G}{\#G'} l_\omega^{G'}.$$

Proof. By the definition of $l_\omega^{G'}$, for large $N \in \mathbb{N}$, there exists $Q'_N \in H_{G'}^1$ such that

$$S_\omega(Q'_N) = l_\omega^{G'} + \frac{1}{N} \text{ and } K(Q'_N) = 0.$$

And there exists a sequence $\{x_n\} \subset \mathbb{R}^d$ such that

$$\begin{cases} \{x_n - \mathcal{G}'x_n\} \text{ is bounded for all } \mathcal{G}' \in G', \\ |x_n - \mathcal{G}x_n| \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for all } \mathcal{G} \in G \setminus G'. \end{cases}$$

We define

$$Q_n := \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n} Q'_N)}{\#G'} \text{ and } \lambda_n := \frac{\frac{4}{d} \|\nabla Q_n\|_{L^2}^2}{\frac{2(p-1)}{p+1} \|Q_n\|_{L^{p+1}}^{p+1}}.$$

Then, $K(\lambda_n Q_n) = 0$ and $\lambda_n Q_n$ is G -invariant. Moreover, Lemma A.3 implies that

$$\|Q_n\|_{H^s}^2 \rightarrow \frac{\#G}{\#G'} \|Q'_N\|_{H^s}^2 \text{ and } \|Q_n\|_{L^{p+1}}^{p+1} \rightarrow \frac{\#G}{\#G'} \|Q'_N\|_{L^{p+1}}^{p+1},$$

where $s = 0, 1$. Therefore, we obtain $\lambda_n \rightarrow 1$ as $n \rightarrow \infty$. This implies

$$S_\omega(\lambda_n Q_n) \rightarrow \frac{\#G}{\#G'} \left(l_\omega^{G'} + \frac{1}{N} \right).$$

This means that $l_\omega^G \leq \#G l_\omega^{G'} / \#G'$. □

Next, we prove $\mathbf{S}_\omega^G = l_\omega^G$ for any finite group G by an induction argument.

Proof of Theorem 1.2 (1) (i). We use Noetherian induction. The set of finite groups in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ is well-founded by the binary relation $\subset$ and its minimal element is the trivial group.

Step1. By [18] and [1], we have $\mathbf{S}_\omega^G = l_\omega^G = l_\omega$ if $G = \{(0, \mathcal{I}_d)\}$.

Step2. Let G be a finite subgroup in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$. We assume that $\mathbf{S}_\omega^{G'} = l_\omega^{G'}$ for any subgroup G' of G . Then, by Lemma 4.12, we get

$$\begin{aligned} m_\omega^G &= \min_{G' \subsetneq G \text{ satisfying } (*)} \frac{\#G}{\#G'} \mathbf{S}_\omega^{G'} \\ &= \min_{G' \subsetneq G \text{ satisfying } (*)} \frac{\#G}{\#G'} l_\omega^{G'} \\ &\geq l_\omega^G. \end{aligned}$$

Therefore, by Proposition 1.3 (i), we obtain $\mathbf{S}_\omega^G = \min\{m_\omega^G, l_\omega^G\} = l_\omega^G$. Thus, Noetherian induction implies that $\mathbf{S}_\omega^G = l_\omega^G$ for any finite group G . This means that Theorem 1.2 (1) (i) holds. $\square$

5. PROOF OF THE SCATTERING RESULT FOR THE INFINITE GROUP INVARIANT SOLUTIONS

In this section, we prove Proposition 1.3 (1) (ii). Let a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ be infinite and satisfy that the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact throughout this section.

First, we prove that the sequence $\{\tilde{x}_n\}$ in (4.12) is bounded by a contradiction argument. We suppose that $\{\tilde{x}_n\}$ is unbounded. We may assume that $|\tilde{x}_n| \rightarrow \infty$ as $n \rightarrow \infty$. Since $\{\varphi_n\}$ is bounded in H^1 , we can find $\tilde{\psi} \in H_G^1$ and $\psi \in H^1(\mathbb{R}^d)$ such that

$$\begin{aligned} e^{-i\tilde{t}_n\Delta}\varphi_n &\rightharpoonup \tilde{\psi} \text{ in } H^1(\mathbb{R}^d), \\ e^{-i\tilde{t}_n\Delta}\tau_{-\tilde{x}_n}\varphi_n &\rightharpoonup \psi \text{ in } H^1(\mathbb{R}^d). \end{aligned}$$

Since the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact, we see that $e^{-i\tilde{t}_n\Delta}\varphi_n \rightarrow \tilde{\psi}$ in $L^{p+1}(\mathbb{R}^d)$. On the other hand, for any $R > 0$, we have $e^{-i\tilde{t}_n\Delta}\tau_{-\tilde{x}_n}\varphi_n \rightarrow \psi$ in $L^{p+1}(B_R)$, where B_R is a ball of radius R centered at the origin. These limits imply that

$$\begin{aligned} \left\| \tau_{-\tilde{x}_n}\tilde{\psi} - \psi \right\|_{L^{p+1}(B_R)} &\leq \left\| \tau_{-\tilde{x}_n}\tilde{\psi} - e^{-i\tilde{t}_n\Delta}\tau_{-\tilde{x}_n}\varphi_n \right\|_{L^{p+1}(B_R)} + \left\| e^{-i\tilde{t}_n\Delta}\tau_{-\tilde{x}_n}\varphi_n - \psi \right\|_{L^{p+1}(B_R)} \\ &\rightarrow 0. \end{aligned}$$

Since $|\tilde{x}_n| \rightarrow \infty$ as $n \rightarrow \infty$, we have $\|\tau_{-\tilde{x}_n}\psi\|_{L^{p+1}(B_R)} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we see that, for any $R > 0$,

$$\|\psi\|_{L^{p+1}(B_R)} \leq \left\| \tau_{-\tilde{x}_n}\tilde{\psi} - \psi \right\|_{L^{p+1}(B_R)} + \|\tau_{-\tilde{x}_n}\psi\|_{L^{p+1}(B_R)} \rightarrow 0.$$

This means that $\psi = 0$. However, we have $\psi \neq 0$ by (4.12). This is a contradiction.

Thus, we can take $x_n := 0$ for all $n \in \mathbb{N}$ in the linear profile decomposition lemma. The rest of the proof is same as in the radial case. See [21] for details.

APPENDIX A. LEMMAS

Lemma A.1. *Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and $\{\tilde{x}_n\}$ be a sequence. Then, there exists a subgroup G' of G such that the sequence $\{\tilde{x}_n - \mathcal{G}'\tilde{x}_n\}$ is bounded for all $\mathcal{G}' \in G'$ and $|\tilde{x}_n - \mathcal{G}\tilde{x}_n| \rightarrow \infty$ as $n \rightarrow \infty$ for all $\mathcal{G} \in G \setminus G'$.*

Proof. It is easy to check that there exists a subset G' of G such that the sequence $\{\tilde{x}_n - \mathcal{G}'\tilde{x}_n\}$ is bounded for $\mathcal{G}' \in G'$ and the sequence $\{\tilde{x}_n - \mathcal{G}\tilde{x}_n\}$ is unbounded for $\mathcal{G} \in G \setminus G'$. We prove that G' is a group. Let $\langle G' \rangle$ denote the group generated by G' . For any $\mathcal{G} \in \langle G' \rangle$, there exist $k \in \mathbb{N}$ and $\{\mathcal{G}'_1, \dots, \mathcal{G}'_k\} \subset G'$ such that $\mathcal{G} = (\mathcal{G}'_1)^{s_1}(\mathcal{G}'_2)^{s_2} \dots (\mathcal{G}'_k)^{s_k}$, where s_j denotes either 1 or -1 for $j \in \{1, 2, \dots, k\}$. By

the triangle inequality, we have

$$\begin{aligned}
|\tilde{x}_n - \mathcal{G}\tilde{x}_n| &= |\tilde{x}_n - (\mathcal{G}'_1)^{s_1}(\mathcal{G}'_2)^{s_2} \cdots (\mathcal{G}'_k)^{s_k} \tilde{x}_n| \\
&\leq |\tilde{x}_n - (\mathcal{G}'_1)^{s_1} \tilde{x}_n| + |(\mathcal{G}'_1)^{s_1} \tilde{x}_n - (\mathcal{G}'_1)^{s_1}(\mathcal{G}'_2)^{s_2} \tilde{x}_n| \\
&\quad + |(\mathcal{G}'_1)^{s_1}(\mathcal{G}'_2)^{s_2} \tilde{x}_n - (\mathcal{G}'_1)^{s_1}(\mathcal{G}'_2)^{s_2}(\mathcal{G}'_3)^{s_3} \tilde{x}_n| \\
&\quad + \cdots + |(\mathcal{G}'_1)^{s_1}(\mathcal{G}'_2)^{s_2} \cdots (\mathcal{G}'_{k-1})^{s_{k-1}} \tilde{x}_n - (\mathcal{G}'_1)^{s_1}(\mathcal{G}'_2)^{s_2} \cdots (\mathcal{G}'_k)^{s_k} \tilde{x}_n| \\
&\leq |\tilde{x}_n - (\mathcal{G}'_1)^{s_1} \tilde{x}_n| + |\tilde{x}_n - (\mathcal{G}'_2)^{s_2} \tilde{x}_n| + |\tilde{x}_n - (\mathcal{G}'_3)^{s_3} \tilde{x}_n| + \cdots + |\tilde{x}_n - (\mathcal{G}'_k)^{s_k} \tilde{x}_n|.
\end{aligned}$$

Therefore, $\{\tilde{x}_n - \mathcal{G}\tilde{x}_n\}$ is bounded. This means that $\langle G' \rangle \subset G'$ and thus G' is a group. $\square$

Lemma A.2. *Let $k \in \mathbb{N}$ and $\mathcal{A}$ be a $kd \times d$ -matrix. We assume that a sequence $\{\tilde{x}_n\} \subset \mathbb{R}^d$ satisfies that there exists $\bar{x} \in \mathbb{R}^{kd}$ such that $\mathcal{I}\tilde{x}_n - \mathcal{A}\tilde{x}_n \rightarrow \bar{x}$ where $\mathcal{I}$ is a $kd \times d$ -matrix such that*

$$\mathcal{I} = \left\{ \begin{pmatrix} \mathcal{I}_d \\ \mathcal{I}_d \\ \vdots \\ \mathcal{I}_d \end{pmatrix} \right\}_k.$$

Then, there exist $\{x_n\} \subset \mathbb{R}^d$ and $x_\infty \in \mathbb{R}^d$ such that

$$\begin{cases} \mathcal{A}x_n = \mathcal{I}x_n, \\ x_n - \tilde{x}_n \rightarrow x_\infty. \end{cases}$$

Proof. It is well known that there exist $kd \times kd$ -matrix $\mathcal{P}$ and $d \times d$ -matrix $\mathcal{Q}$ such that

$$(A.1) \quad \mathcal{P}(\mathcal{I} - \mathcal{A})\mathcal{Q} = \underbrace{\begin{pmatrix} 1 & & a_{11} & \cdots & a_{1d-r} \\ & \ddots & \vdots & & \vdots \\ & & 1 & a_{r1} & \cdots & a_{rd-r} \\ 0 & & & & 0 \end{pmatrix}}_r \underbrace{\quad}_{d-r} \Bigg\}_{kd-r}^r,$$

where $r = \text{rank}(\mathcal{I} - \mathcal{A})$. We set $\tilde{y}_n := \mathcal{Q}^{-1}\tilde{x}_n$, $\mathcal{B} := \mathcal{P}(\mathcal{I} - \mathcal{A})\mathcal{Q}$, and $\bar{y} := \mathcal{P}\bar{x}$. Then, we have $\mathcal{B}\tilde{y}_n \rightarrow \bar{y}$ since

$$\begin{aligned}
|\mathcal{B}\tilde{y}_n - \bar{y}| &= |\mathcal{P}(\mathcal{I} - \mathcal{A})\mathcal{Q}\tilde{y}_n - \mathcal{P}\bar{x}| = |\mathcal{P}(\mathcal{I} - \mathcal{A})\mathcal{Q}\mathcal{Q}^{-1}\tilde{x}_n - \mathcal{P}\bar{x}| \\
&\leq |\mathcal{P}||(\mathcal{I} - \mathcal{A})\tilde{x}_n - \bar{x}| \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

In particular, for $i \in \{1, 2, \dots, r\}$,

$$(A.2) \quad |\tilde{y}_n^i + (a_{i1}\tilde{y}_n^{r+1} + \cdots + a_{id-r}\tilde{y}_n^d) - \bar{y}^i| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where z^j denotes the j -th component of $z \in \mathbb{R}^d$. We take $\{y_n\} \subset \mathbb{R}^d$ satisfying the following properties.

$$\begin{cases} y_n^i := -(a_{i1}\tilde{y}_n^{r+1} + \cdots + a_{id-r}\tilde{y}_n^d), \\ y_n^j := \tilde{y}_n^j, \end{cases}$$

for $i \in \{1, 2, \dots, r\}$ and $j \in \{r+1, r+2, \dots, d\}$. Then, we have $\mathcal{B}y_n = 0$ for all $n \in \mathbb{N}$ by the definition of $\{y_n\}$. Moreover, we have for $i \in \{1, 2, \dots, r\}$ and

$$j \in \{r+1, r+2, \dots, d\},$$

$$\begin{aligned} \tilde{y}_n^i - y_n^i &= \tilde{y}_n^i + (a_{i1}\tilde{y}_n^{r+1} + \dots + a_{id-r}\tilde{y}_n^d) \underbrace{-(a_{i1}\tilde{y}_n^{r+1} + \dots + a_{id-r}\tilde{y}_n^d) - y_n^i}_{=0 \text{ by the definition of } y_n^i} \\ &\rightarrow \bar{y}^i \text{ as } n \rightarrow \infty. \end{aligned}$$

and $\tilde{y}_n^j - y_n^j = 0$. Therefore,

$$\frac{d\left\{ \begin{pmatrix} \tilde{y}_n \\ 0 \end{pmatrix} \right\}}{kd - d\left\{ \begin{pmatrix} y_n \\ 0 \end{pmatrix} \right\}} \rightarrow \bar{y} \text{ as } n \rightarrow \infty.$$

Note that $\bar{y}^{r+1} = \bar{y}^{r+2} = \dots = \bar{y}^{kd} = 0$. Define $x_n := \mathcal{Q}y_n$ by (A.1). Then, we have

$$0 = \mathcal{B}y_n = \mathcal{P}(\mathcal{I} - \mathcal{A})\mathcal{Q}y_n = \mathcal{P}(\mathcal{I} - \mathcal{A})x_n.$$

Multiplying $\mathcal{P}^{-1}$ from the left, we obtain $\mathcal{A}x_n = \mathcal{I}x_n$. Moreover,

$$\tilde{x}_n - x_n = \mathcal{Q}(\tilde{y}_n - y_n) \rightarrow \mathcal{Q} \begin{pmatrix} \bar{y}^1 \\ \vdots \\ \bar{y}^d \end{pmatrix} =: x_\infty.$$

This completes the proof. $\square$

Lemma A.3. *Let $f \in H_{G'}^1$ and $\{x_n\}$ satisfy $|x_n - \mathcal{G}x_n| \rightarrow \infty$ as $n \rightarrow \infty$ for $\mathcal{G} \in G \setminus G'$. We have the following identities.*

$$(A.3) \quad \left\| \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right\|_{\dot{H}^\lambda}^2 = \frac{\#G}{\#G'} \|\#G' f\|_{\dot{H}^\lambda}^2 + o(1),$$

$$(A.4) \quad \left\| \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right\|_{L^p}^p = \frac{\#G}{\#G'} \|\#G' f\|_{L^p}^p + o(1)$$

where $\lambda \in [0, 1]$, $p \geq 1$ and $o(1) \rightarrow 0$ as $n \rightarrow \infty$. In particular, the following identity holds for any $\omega > 0$.

$$S_\omega \left(\sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right) = \frac{\#G}{\#G'} S_\omega(\#G' f) + o(1).$$

To prove Lemma A.3, we need Refined Fatou's lemma. See [6] and [34, Theorem 1.9].

Lemma A.4 (Refined Fatou's lemma). *Let $\{f_n\} \subset L^p(\mathbb{R}^d)$ with $\limsup_{n \rightarrow \infty} \|f_n\|_{L^p} < \infty$. If $f_n \rightarrow f$ almost everywhere, then we have*

$$\int_{\mathbb{R}^d} ||f_n|^p - |f_n - f|^p - |f|^p| dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In particular, $\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p \rightarrow \|f\|_{L^p}^p$ as $n \rightarrow \infty$.

Proof of Lemma A.3. Let $\{\mathcal{G}_k\}_{k=1}^{\#G/\#G'}$ be the set of left coset representatives. First, we prove (A.3).

$$\begin{aligned}
\left\| \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right\|_{\dot{H}^\lambda}^2 &= (\#G')^2 \left\| \sum_{k=1}^{\#G/\#G'} \mathcal{G}_k(\tau_{x_n} f) \right\|_{\dot{H}^\lambda}^2 \\
&= (\#G')^2 \left\{ \sum_{k=1}^{\#G/\#G'} \|\mathcal{G}_k(\tau_{x_n} f)\|_{\dot{H}^\lambda}^2 + \sum_{\substack{k,l=1 \\ k \neq l}}^{\#G/\#G'} (\mathcal{G}_k(\tau_{x_n} f), \mathcal{G}_l(\tau_{x_n} f))_{\dot{H}^\lambda} \right\} \\
&= \frac{\#G}{\#G'} \|\#G' f\|_{\dot{H}^\lambda}^2 \\
&\quad + (\#G')^2 \sum_{\substack{k,l=1 \\ k \neq l}}^{\#G/\#G'} (\mathcal{G}_k(\tau_{x_n} f), \mathcal{G}_l(\tau_{x_n} f))_{\dot{H}^\lambda}
\end{aligned}$$

The second term tends to 0 as $n \rightarrow \infty$ since $|x_n - \mathcal{G}x_n| \rightarrow \infty$ as $n \rightarrow \infty$ for $\mathcal{G} \in G \setminus G'$ and $\mathcal{G}_k^{-1} \mathcal{G}_l \notin G'$ for $k \neq l$. This implies (A.3). Next, we prove (A.4). Without loss of generality, we may assume that $\mathcal{G}_1 = (0, \mathcal{I}_d)$.

$$\begin{aligned}
\left\| \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right\|_{L^p}^p &= (\#G')^p \left\| \sum_{k=1}^{\#G/\#G'} \mathcal{G}_k(\tau_{x_n} f) \right\|_{L^p}^p \\
&= (\#G')^p \left\| \tau_{x_n} f + \sum_{k=2}^{\#G/\#G'} \mathcal{G}_k(\tau_{x_n} f) \right\|_{L^p}^p \\
&= (\#G')^p \left\| f + \sum_{k=2}^{\#G/\#G'} \tau_{-x_n + \mathcal{G}_k x_n} \mathcal{G}_k f \right\|_{L^p}^p.
\end{aligned}$$

We set $f_n^1 := f + \sum_{k=2}^{\#G/\#G'} \tau_{-x_n + \mathcal{G}_k x_n} \mathcal{G}_k f$. Then, $\sup_{n \in \mathbb{N}} \|f_n^1\|_{L^p} < \infty$ and $f_n^1 \rightarrow f$ almost everywhere since $|x_n - \mathcal{G}x_n| \rightarrow \infty$ as $n \rightarrow \infty$ for $\mathcal{G} \in G \setminus G'$. Therefore, Refined Fatou's lemma gives us

$$\|f_n^1\|_{L^p}^p - \|f_n^1 - f\|_{L^p}^p - \|f\|_{L^p}^p \rightarrow 0.$$

Here, $f_n^1 - f = \sum_{k=2}^{\#G/\#G'} \tau_{-x_n+\mathcal{G}_k x_n} \mathcal{G}_k f$ and

$$\begin{aligned}
\left\| \sum_{k=2}^{\#G/\#G'} \tau_{-x_n+\mathcal{G}_k x_n} \mathcal{G}_k f \right\|_{L^p}^p &= \left\| \sum_{k=2}^{\#G/\#G'} \tau_{\mathcal{G}_k x_n} \mathcal{G}_k f \right\|_{L^p}^p \\
&= \left\| \tau_{\mathcal{G}_2 x_n} \mathcal{G}_2 f + \sum_{k=3}^{\#G/\#G'} \tau_{\mathcal{G}_k x_n} \mathcal{G}_k f \right\|_{L^p}^p \\
&= \left\| \mathcal{G}_2(\tau_{x_n} f) + \sum_{k=3}^{\#G/\#G'} \tau_{\mathcal{G}_k x_n} \mathcal{G}_k f \right\|_{L^p}^p \\
&= \left\| f + \sum_{k=3}^{\#G/\#G'} \tau_{-x_n+\mathcal{G}_2^{-1} \mathcal{G}_k x_n} \mathcal{G}_2^{-1} \mathcal{G}_k f \right\|_{L^p}^p
\end{aligned}$$

Let $f_n^2 := f + \sum_{k=3}^{\#G/\#G'} \tau_{-x_n+\mathcal{G}_2^{-1} \mathcal{G}_k x_n} \mathcal{G}_2^{-1} \mathcal{G}_k f$. Then, $\sup_{n \in \mathbb{N}} \|f_n^2\|_{L^p} < \infty$ and $f_n^2 \rightarrow f$ almost everywhere since $|x_n - \mathcal{G}_k x_n| \rightarrow \infty$ as $n \rightarrow \infty$ for $\mathcal{G} \in G \setminus G'$. Therefore, by Refined Fatou's lemma again, we get

$$\|f_n^2\|_{L^p}^p - \|f_n^2 - f\|_{L^p}^p - \|f\|_{L^p}^p \rightarrow 0.$$

By repeating this procedure, we get

$$\|f_n^1\|_{L^p}^p - \left\| f_n^{\#G/\#G'-1} - f \right\|_{L^p}^p - (\#G/\#G' - 1) \|f\|_{L^p}^p \rightarrow 0,$$

where $f_n^{\#G/\#G'-1} := f + \tau_{-x_n+\mathcal{G}_{\#G/\#G'-1}^{-1} \cdots \mathcal{G}_2^{-1} \mathcal{G}_{\#G/\#G' x_n}} \mathcal{G}_{\#G/\#G'-1}^{-1} \cdots \mathcal{G}_2^{-1} \mathcal{G}_k f$. Now, we have

$$\begin{aligned}
\left\| f_n^{\#G/\#G'-1} - f \right\|_{L^p}^p &= \left\| \tau_{-x_n+\mathcal{G}_{\#G/\#G'-1}^{-1} \cdots \mathcal{G}_2^{-1} \mathcal{G}_{\#G/\#G' x_n}} \mathcal{G}_{\#G/\#G'-1}^{-1} \cdots \mathcal{G}_2^{-1} \mathcal{G}_k f \right\|_{L^p}^p \\
&= \|f\|_{L^p}^p.
\end{aligned}$$

Thus, we obtain

$$\left\| \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right\|_{L^p}^p = (\#G')^p \|f_n^1\|_{L^p}^p \rightarrow (\#G')^p \frac{\#G}{\#G'} \|f\|_{L^p}^p.$$

This completes the proof. $\square$

Lemma A.5. *Let $\{t_n\} \subset \mathbb{R}$, $\{x_n\} \subset \mathbb{R}^d$, $\{f_n\} \subset H^1(\mathbb{R}^d)$ and $\psi \in H^1(\mathbb{R}^d) \setminus \{0\}$ satisfy*

$$f_n \rightharpoonup 0 \text{ and } e^{it_n \Delta} f_n(\cdot + x_n) \rightharpoonup \psi \text{ as } n \rightarrow \infty \text{ in } H^1(\mathbb{R}^d).$$

Then, $|t_n| \rightarrow \infty$ or $|x_n| \rightarrow \infty$ as $n \rightarrow \infty$ taking a subsequence.

Proof. If $|t_n| + |x_n|$ is bounded, then $f_n(\cdot + x_n) \rightharpoonup 0$ since $f_n \rightharpoonup 0$. This contradicts $\psi \neq 0$. See also [18, Lemma 5.3]. $\square$

APPENDIX B. APPLICATIONS

We introduce some applications of Theorem 1.2 in this appendix. Here, we only treat examples in two dimensional case. We only introduce three applications.

(1). We consider

$$G = \left\{ \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(\pi, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \left(\pi, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right), \left(0, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right\}.$$

By Theorem 1.2, we can classify the solutions with G -invariance satisfying

$$S_\omega < 4l_\omega,$$

where we can show $l_\omega^G = 4l_\omega$ easily.

(2). We consider

$$G = \left\{ \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(\pi, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right\}.$$

By Theorem 1.2, we classify the solutions with G -invariance satisfying

$$S_\omega < 2l_\omega,$$

where we have used the fact that $l_\omega^G = 2l_\omega$.

(3). We consider

$$G = \left\{ \left(\theta, \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right) : \theta \in [0, 2\pi) \right\}.$$

Then, the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact since Gx has infinitely many elements for $x \neq 0$. By Theorem 1.2, we can classify the solutions such that $S_\omega < l_\omega^G$. It is worth noting that l_ω^G is strictly larger than l_ω .

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO, KYOTO, 606-8502, JAPAN

E-mail address: inui@math.kyoto-u.ac.jp