

Cover times and extrema of local times for random walks on graphs

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Chapter 1

Introduction

1.1 General introduction

Simple random walks (SRWs) on graphs are discrete models of a particle moving in a medium as follows: at each position, we have an exponential clock of unit rate and at each step, the particle stays at the current position until the clock rings and moves to one of the neighbors uniformly at random.

This thesis deals with local times of SRWs on graphs, which measure how long SRWs spend at each vertex up to a given time. The main focus is on the study of extrema of local times for SRWs on the b -ary tree and on the two-dimensional discrete torus, and the *cover time*, that is, the first time at which the minimum of local times becomes positive, for the SRW on the supercritical percolation cluster on the lattice.

Percolation on the lattice describes how porous materials become wet upon immersion in water. The model is defined as follows: Fix $p \in [0, 1]$ and $d \geq 2$. Each edge of $\mathbb{Z}^d$ is retained with probability p and removed with probability $1 - p$ independently of all other edges. The resulting random graph consists of some connected components (clusters) and has a phase transition as p varies: there exists $p_c(d) \in (0, 1)$ such that if $p > p_c(d)$, there exists a unique infinite connected component almost surely, and if $p < p_c(d)$, all the connected components are finite almost surely. The regime $p > p_c(d)$ is called *supercritical*. It is known that in the supercritical regime, the SRW on the cluster has a lot of properties similar to those of the SRW on the original lattice (see [18, 65]).

Let $G = (V, E)$ be a finite connected graph with the vertex set V and the

set of unoriented edges E and $X = (X_t, t \geq 0, P_x, x \in V)$ be the SRW on G . The local time at $v \in V$ and time t is defined by

$$L_t^G(v) := \frac{1}{\deg(v)} \int_0^t 1_{\{X_s=v\}} ds,$$

where $\deg(v)$ is the degree of v . Fix a vertex $v_0 \in V$ and set

$$\tau_{v_0}(t) := \inf\{s \geq 0 : L_s^G(v_0) > t\}.$$

One of the remarkable properties of the local time process is that it has a close relationship with a Gaussian process called the *discrete Gaussian free field* (DGFF) on G : this is a $\mathbb{R}^V$ -valued Gaussian random variable $(h_v^G)_{v \in V}$ with $\mathbb{E}(h_v^G) = 0$ and $\mathbb{E}[h_u^G h_v^G] = g(u, v)$ for all $u, v \in V$, where $g(\cdot, \cdot)$ is the Green function given by $g(u, v) = E_u[L_{T_{v_0}}^G(v)]$ (T_v is the hitting time of v by X). Eisenbaum, Kaspi, Marcus, Rosen, and Shi [47] proved that for all $t > 0$,

$$\text{the law of } \left\{ L_{\tau_{v_0}(t)}^G(v) + \frac{1}{2}(h_v^G)^2 : v \in V \right\} \text{ under } P_{v_0} \otimes \mathbb{P}$$

is the same as

$$\text{that of } \left\{ \frac{1}{2} \left(h_v^G + \sqrt{2t} \right)^2 : v \in V \right\} \text{ under } \mathbb{P}. \quad (1.1)$$

In particular, we have as $t \rightarrow \infty$

$$\left(\frac{L_{\tau_{v_0}(t)}^G(v) - t}{\sqrt{2t}} \right)_{v \in V} \rightarrow (h_v^G)_{v \in V} \text{ in law.} \quad (1.2)$$

The relation (1.1) is called the *generalized second Ray-Knight theorem* or the *Dynkin isomorphism* due to Dynkin [46] (originated by Symanzik [92]). The expected cover time of X is defined by

$$t_{\text{cov}}^G := \max_{v \in V} E_v [\tau_{\text{cov}}^G],$$

where $\tau_{\text{cov}}^G := \inf \{t > 0 : \min_{v \in V} L_t^G(v) > 0\}$. The maximal hitting time is given by $t_{\text{hit}}^G := \max_{u, v \in V} E_u[T_v]$. (Recall that T_v is the hitting time of v by X .) For each set S , we will write $|S|$ to denote the cardinality of S . It is

known that the expected cover time satisfies the following general bounds (see, for example, [70, Chapter 11]):

$$t_{\text{hit}}^G \leq t_{\text{cov}}^G \leq 2t_{\text{hit}}^G \log |V|. \quad (1.3)$$

In most cases, the estimate of the expected cover time is reduced to that of the *effective resistance*: for each $f \in \mathbb{R}^V$, we define the Dirichlet energy by

$$\mathcal{E}(f) := \frac{1}{2} \sum_{\substack{u,v \in V \\ \{u,v\} \in E}} (f(u) - f(v))^2,$$

(recall that edges are unoriented) and the effective resistance between u and v by

$$R_{\text{eff}}^G(u, v)^{-1} := \inf\{\mathcal{E}(f) : f \in \mathbb{R}^V, f(u) = 1, f(v) = 0\}.$$

It is known that t_{hit}^G can be estimated via the volume $|E|$ and the effective resistance as follows (see, for example, [70, Proposition 10.6]):

$$|E| \max_{u,v \in V} R_{\text{eff}}^G(u, v) \leq t_{\text{hit}}^G \leq 2|E| \max_{u,v \in V} R_{\text{eff}}^G(u, v). \quad (1.4)$$

By (1.3) and (1.4), we have general bounds of t_{cov}^G by means of the effective resistance. Since the cover time is defined in terms of the minimum of the local times and we have the relation (1.1), it is plausible to expect that the cover time has a connection with the DGFF. This was first confirmed in a general setting by Ding, Lee, and Peres [40], and Zhai [95] strengthened the work as follows: there exist universal constants $c_1, c_2 \in (0, \infty)$ such that

$$1 - c_1 \sqrt{\frac{t_{\text{hit}}^G}{t_{\text{cov}}^G}} \leq \frac{t_{\text{cov}}^G}{|E| \cdot (\mathbb{E} \max_{v \in V} h_v^G)^2} \leq 1 + c_2 \sqrt{\frac{t_{\text{hit}}^G}{t_{\text{cov}}^G}}. \quad (1.5)$$

DGFFs are also related to Poisson point processes of Markovian loops [59, 68] and random interacements [94, 88].

The two-dimensional DGFF receives special attention because its continuous version plays important role in physics, for example, in the theory of the two-dimensional Liouville quantum gravity [43, 52] which has a connection with studies of the Gaussian multiplicative chaos [60, 44, 87]. See also [20].

DGFFs on trees are known as *branching random walks* (BRWs) which describes how particles moving on the line evolves. The continuous counterpart, *branching Brownian motion* (BBM), has been studied extensively since it has a close relationship with the partial differential equation, called the Kolmogorov-Petrovsky-Piscounov or Fisher equation [81, 25, 23].

1.2 Main results

This thesis consists of three papers:

- Chapter 2: Y. Abe. Effective resistances for supercritical percolation clusters in boxes. *Ann. Inst. H. Poincaré Probab. Statist.* **51** (2015), 935-946. DOI: 10.1214/14-AIHP604
- Chapter 3: Y. Abe. Maximum and minimum of local times for two-dimensional random walk. *Electron. Commun. Probab.* **20** (2015), no.22. DOI: 10.1214/ECP.v20-3877
- Chapter 4: Y. Abe. Extremes of local times for simple random walks on symmetric trees. Preprint.

We now summarize the main results of this thesis:

1.2.1 Results in Chapter 2

In Chapter 2, we study the expected cover time for the SRW on the supercritical percolation cluster on the lattice. Consider percolation on $\mathbb{Z}^d$, $d \geq 2$ in the supercritical regime $p > p_c(d)$. Let $\mathcal{C}^n$ be the largest connected component in $B(n) := [-n, n]^d \cap \mathbb{Z}^d$. Then, we show the following:

Theorem 1.2.1 (*Theorem 2.1.5 in Chapter 2*) *For all $d \geq 2$ and $p \in (p_c(d), 1)$, there exist $c_1, c_2 \in (0, \infty)$ such that the following holds $\mathbb{P}_p$ -almost surely, for large $n \in \mathbb{N}$ ($\mathbb{P}_p$ is the probability measure which governs the randomness of percolation):*

$$c_1 n^d (\log n)^2 \leq t_{\text{cov}}^{\mathcal{C}^n} \leq c_2 n^d (\log n)^2. \quad (1.6)$$

It is known that the expected cover time $t_{\text{cov}}^{B(n)}$ is comparable to $n^2 (\log n)^2$ in the case $d = 2$ and to $n^d \log n$ in the case $d \geq 3$, so (1.6) clarifies the quantitative difference between the SRWs on the supercritical percolation cluster and on the original lattice in three or higher dimensions. The proof of (1.6) is based on a sharp estimate of the maximal effective resistance on $\mathcal{C}^n$:

Theorem 1.2.2 (*Theorem 2.1.1 in Chapter 2*) *For all $d \geq 2$ and $p \in (p_c(d), 1)$, there exist $c_3, c_4 \in (0, \infty)$ such that the following holds $\mathbb{P}_p$ -almost surely, for large $n \in \mathbb{N}$:*

$$c_3 \log n \leq \max_{u, v \in \mathcal{C}^n} R_{\text{eff}}^{\mathcal{C}^n}(u, v) \leq c_4 \log n. \quad (1.7)$$

The following follows from (1.5) and (1.6):

Corollary 1.2.3 *(Corollary 2.1.7 in Chapter 2) For all $d \geq 2$ and $p \in (p_c(d), 1)$, there exist $c_5, c_6 \in (0, \infty)$ such that the following holds $\mathbb{P}_p$ -almost surely, for large $n \in \mathbb{N}$:*

$$c_5 \log n \leq \mathbb{E} \left[\max_{v \in \mathcal{C}^n} h_v^{\mathcal{C}^n} \right] \leq c_6 \log n.$$

The upper bound of (1.6) follows from (1.3), (1.4), and (1.7). In the proof of the lower bounds of (1.6) and (1.7), the key is to prove that $\mathbb{P}_p$ -almost surely, for large $n \in \mathbb{N}$, $\mathcal{C}^n$ has at least n^{c_7} dangling ends (one-dimensional objects) of length of order $\log n$ for some $c_7 > 0$. In the proof of the upper bound of (1.7), we construct the so-called “Kesten grid” on each two-dimensional slice of $\mathcal{C}^n$, which looks like the usual square lattice. It is known that in the case $d = 2$, the maximal effective resistance of $B(n)$ is comparable to $\log n$. Roughly speaking, the upper bound of (1.7) follows from this and the existence of the Kesten grid together with the triangle inequality of the effective resistance.

We describe related open problems. The following question has been asked by Marek Biskup:

Open Problem 1.2.4 *For $d \geq 2$ and $p \in (p_c(d), 1)$, prove or disprove that*

$$t_{cov}^{\mathcal{C}^n} \text{ is concentrated around } \mathbb{E}_p [t_{cov}^{\mathcal{C}^n}].$$

The following has been posed by Ryoki Fukushima:

Open Problem 1.2.5 *For $d \geq 2$ and $p \in (p_c(d), 1)$, do vertices covered last by the SRW on $\mathcal{C}^n$ lie on dangling ends?*

To resolve these open problems, we need deeper understanding of the geometric structure of $\mathcal{C}^n$.

1.2.2 Results in Chapter 3

In Chapter 3, we obtain leading orders of the maximum and minimum of local times for the SRW on the two-dimensional discrete torus $\mathbb{Z}_N^2 := (\mathbb{Z}/N\mathbb{Z})^2$.

We consider the local time process $\{L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x) : x \in \mathbb{Z}_N^2\}$ for the SRW on $\mathbb{Z}_N^2$, where o is the origin of $\mathbb{Z}_N^2$ and

$$t_\theta := \theta \cdot \frac{1}{\pi} (\log N)^2, \quad \theta > 0.$$

It is known that under P_o

$$\lim_{N \rightarrow \infty} \tau_o(t_\theta) / t_{\text{cov}}^{\mathbb{Z}_N^2} = \theta \quad \text{in probability.} \quad (1.8)$$

Since local times are sums of Poissonian (with mean of order t_θ) number of independent exponential random variables and $E_o[L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x)] = t_\theta$ for all $x \in \mathbb{Z}_N^2$, in view of the central limit theorem, it is reasonable to normalize local times as follows (see also (1.2)):

$$M_N := \frac{\max_{x \in \mathbb{Z}_N^2} L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x) - t_\theta}{\sqrt{2t_\theta}}, \quad m_N := \frac{\min_{x \in \mathbb{Z}_N^2} L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x) - t_\theta}{\sqrt{2t_\theta}}.$$

Then, we show the following:

Theorem 1.2.6 (*Corollary 3.1.3 in Chapter 3*) (i) For all $\theta > 0$ and $\varepsilon > 0$, the following holds with P_o -probability tending to 1 as $N \rightarrow \infty$:

$$\left(1 + \frac{1}{2\sqrt{\theta}} - \varepsilon\right) \sqrt{\frac{2}{\pi}} \log N \leq M_N \leq \left(1 + \frac{1}{2\sqrt{\theta}} + \varepsilon\right) \sqrt{\frac{2}{\pi}} \log N. \quad (1.9)$$

(ii) For all $\theta > 1$ and $\varepsilon > 0$, the following holds with P_o -probability tending to 1 as $N \rightarrow \infty$:

$$-\left(1 - \frac{1}{2\sqrt{\theta}} + \varepsilon\right) \sqrt{\frac{2}{\pi}} \log N \leq m_N \leq -\left(1 - \frac{1}{2\sqrt{\theta}} - \varepsilon\right) \sqrt{\frac{2}{\pi}} \log N.$$

Note that (1.8) implies $\min_{x \in \mathbb{Z}_N^2} L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x) = 0$ with P_o -probability tending to 1 as $N \rightarrow \infty$ if $\theta \in (0, 1)$. The estimate (1.9) is closely related to the work [22] in the context of the two-dimensional DGFF. Let $V_N := [1, N]^2 \cap \mathbb{Z}^2$ and $(h_x^{V_N})_{x \in V_N}$ be the DGFF on V_N with zero boundary conditions. Bolthausen, Deuschel, and Giacomin [22] proved that for all $\varepsilon > 0$, the following holds with probability tending to 1 as $N \rightarrow \infty$:

$$(1 - \varepsilon) \sqrt{\frac{2}{\pi}} \log N \leq \max_{x \in V_N} h_x^{V_N} \leq (1 + \varepsilon) \sqrt{\frac{2}{\pi}} \log N. \quad (1.10)$$

The estimates (1.9) and (1.10) imply that there is a slight difference between the two by the factor $(2\sqrt{\theta})^{-1}$.

We also estimate the number of vertices with local times proportional to the maximum (or the minimum) by a factor between 0 and 1. For $\eta > 0$ and $\theta > 0$, set

$$\mathcal{L}_N^+(\eta, \theta) := \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x) - t_\theta}{\sqrt{2t_\theta}} \geq \eta \cdot \sqrt{\frac{2}{\pi}} \log N \right\},$$

$$\mathcal{L}_N^-(\eta, \theta) := \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x) - t_\theta}{\sqrt{2t_\theta}} \leq -\eta \cdot \sqrt{\frac{2}{\pi}} \log N \right\}.$$

Then, we prove the following:

Theorem 1.2.7 (*Theorem 3.1.2 in Chapter 3*) (i) For all $\theta > 0$, $\varepsilon > 0$, and $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$, the following holds with P_o -probability tending to 1 as $N \rightarrow \infty$:

$$N^{2-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2-\varepsilon} \leq |\mathcal{L}_N^+(\eta, \theta)| \leq N^{2-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2+\varepsilon}. \quad (1.11)$$

(ii) For all $\theta > 1$, $\varepsilon > 0$, and $\eta \in (0, 1 - \frac{1}{2\sqrt{\theta}})$, the following holds with P_o -probability tending to 1 as $N \rightarrow \infty$:

$$N^{2-2(\sqrt{\theta}-\sqrt{\theta-2\eta\sqrt{\theta}})^2-\varepsilon} \leq |\mathcal{L}_N^-(\eta, \theta)| \leq N^{2-2(\sqrt{\theta}-\sqrt{\theta-2\eta\sqrt{\theta}})^2+\varepsilon}.$$

The estimate (1.11) corresponds to the work by Daviaud [35] who showed that for all $\varepsilon > 0$ and $\eta \in (0, 1)$, with probability tending to 1 as $N \rightarrow \infty$,

$$N^{2(1-\eta^2)-\varepsilon} \leq \left| \left\{ x \in V_N : h_x^{V_N} \geq \eta \cdot \sqrt{\frac{2}{\pi}} \log N \right\} \right| \leq N^{2(1-\eta^2)+\varepsilon}. \quad (1.12)$$

Comparing (1.11) and (1.12), we have slightly different exponents in these estimates.

Roughly speaking, the upper bound of (1.9) follows from (1.1) and (1.10). In the proof of the lower bound of (1.9), we use a technique developed by Dembo, Peres, Rosen and Zeitouni [38]: their method reduces the analysis of the local time at each point x to that of the number of crossings of annuli

around x by the SRW on $\mathbb{Z}_N^2$. Since the SRW on $\mathbb{Z}_N^2$ behaves like an one-dimensional SRW in the radial direction, the study of the number of crossings is reduced to that of edge local times of the one-dimensional SRW. We give a notion that a point is “successful” in terms of the number of crossings around the vertex and show that local times at successful points are close to the maximum. Using the above argument, we show that we have at least one successful point, which implies the lower bound of (1.9).

Belius and Kistler [13] recently obtained the subleading order of the cover time for Brownian motion on $(\mathbb{R}/\mathbb{Z})^2$. We believe that applying their method, one could prove the following:

Open Problem 1.2.8 *For each $\theta > 0$, prove that*

$$\max_{x \in \mathbb{Z}_N^2} \sqrt{L_{\tau_o(t_\theta)}^{\mathbb{Z}_N^2}(x)} - (\sqrt{\theta} + 1) \frac{1}{\sqrt{\pi}} \log N + \frac{3}{8\sqrt{\pi}} \log(\log N) \text{ is tight.}$$

Similar result for the two-dimensional DGFF are known by Bramson and Zeitouni [28]. The keys in [28] are the Gibbs-Markov property and the hierarchical structure of the two-dimensional DGFF. The local time process on $\mathbb{Z}_N^2$ has similar properties, but they are too coarse for us to resolve Open problem 1.2.8. As the first step, we need to polish these.

1.2.3 Results in Chapter 4

In Chapter 4, we study maxima of local times for the SRW on b -ary tree. Fix an integer $b \geq 2$ and consider the b -ary tree T with the root ρ : this is a tree rooted at ρ whose vertices have exactly b children. Let T_i be the i -th generation of T and set $T_{\leq n} := \cup_{i=0}^n T_i$. We show that the properly centered maximum of local times over leaves weakly converges to a randomly shifted Gumbel distribution:

Theorem 1.2.9 *(Corollary 4.1.3 in Chapter 4) There exist $c_1 > 0$ and a finite positive random variable D_∞ such that for all $\lambda \in \mathbb{R}$ and any sequence $(t_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = \theta \in [0, \infty]$ and $t_n \geq c_1 n \log n$ for each $n \in \mathbb{N}$,*

$$\lim_{n \rightarrow \infty} P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau_\rho(t_n)}^{T_{\leq n}}}(v) \leq \sqrt{t_n} + a_n(t_n) + \lambda \right) = \mathbb{E} \left[e^{-\alpha_*(\theta) D_\infty e^{-2\sqrt{\log b} \lambda}} \right], \quad (1.13)$$

where $\alpha_*(\theta)$ is a positive constant depending on θ , and

$$a_n(t_n) := \sqrt{\log b} \, n - \frac{3}{4\sqrt{\log b}} \log n - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t_n} + n}{\sqrt{t_n}} \right). \quad (1.14)$$

One can check that the DGFF $(h_v^{T_{\leq n}})_{v \in T_{\leq n}}$ pinned to zero at ρ is a BRW on the b -ary tree with standard Gaussian increments. The random variable D_∞ in (1.13) is given by the limit of the so-called *derivative martingale* of the BRW: set

$$D_n := \sum_{v \in T_n} \left(\sqrt{\log b} \, n - \frac{h_v}{\sqrt{2}} \right) e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, n - \frac{h_v}{\sqrt{2}} \right)}.$$

It is known that $\lim_{n \rightarrow \infty} D_n$ exists almost surely and D_∞ is this limit. The constant $\alpha_*(\theta)$ in (1.13) is given by

$$\alpha_*(\theta) := \frac{4}{\sqrt{\pi}} \sqrt{\frac{\theta+1}{\theta+\sqrt{\log b}}} \cdot \lim_{\ell \rightarrow \infty} \int_{\ell^{\frac{2}{3}}}^{\ell} z e^{2\sqrt{\log b} z} \mathbb{P} \left(\max_{v \in T_\ell} \frac{h_v}{\sqrt{2}} > \sqrt{\log b} \, \ell + z \right) dz.$$

(We regard $\sqrt{\frac{\theta+1}{\theta+\sqrt{\log b}}} = 1$ in the case $\theta = \infty$.) Results similar to (1.13) are known for the BBM [66], for the BRW [9, 2], and for the two-dimensional DGFF [26, 19].

Now that we have exact information about the maximum of local times over leaves, the next subjects are statistics of leaves whose local times attain local maxima. To study this, we embed leaves in the interval $[0, 1]$ as follows: to each vertex $v \in T_k, k \geq 1$, we assign the label $(\bar{v}_1, \dots, \bar{v}_k)$ with $\bar{v}_i \in \{0, \dots, b-1\}$ for each $1 \leq i \leq k$ so that the vertices with the labels $(\bar{v}_1, \dots, \bar{v}_k, \ell), \ell \in \{0, \dots, b-1\}$ are children of v . For each $v \in T_n$, we define the location by

$$\sigma(v) := \sum_{i=1}^n \frac{\bar{v}_i}{b^i}.$$

We will write $\delta_{(x,h)}$ to denote the Dirac measure at $(x, h) \in [0, 1] \times \mathbb{R}$. Inspired by [24], we define the point process on $[0, 1] \times \mathbb{R}$ by

$$\Xi_{n,t_n}^{(m)} := \sum_{v \in T_m} \delta \left(\sigma(u_v^{(m)}), \max_{u \in T_{n-m}^v} \sqrt{L_{\tau_\rho(t_n)}^{T_{\leq n}}(u) - \sqrt{t_n} - a_n(t_n)} \right), \quad m < n, \quad (1.15)$$

where T_{n-m}^v is the $(n-m)$ -th generation of the subtree rooted at v and $u_v^{(m)}$ is the vertex on T_{n-m}^v with the maximal local time over T_{n-m}^v . We show the following:

Theorem 1.2.10 (*Theorem 4.1.1 in Chapter 4*) *There exist $c_2 > 0$ and a random Borel measure Z_∞ on $[0, 1]$ whose total mass has the same law as D_∞ such that the following holds for all $(t_n)_{n \geq 1}$ with $t_n \geq c_2 n \log n$ for each $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sqrt{t_n}/n =: \theta \in [0, \infty]$, and for all $(r_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} r_n = \infty$ and $\lim_{n \rightarrow \infty} r_n/n \in [0, 1)$: as $n \rightarrow \infty$,*

the point process $\Xi_{n,t_n}^{(n-r_n)}$ weakly converges to

a Cox process with intensity measure $\alpha_(\theta)Z_\infty(dx) \otimes 2\sqrt{\log b} e^{-2\sqrt{\log b}h} dh$,*

where D_∞ and $\alpha_(\theta)$ are the ones in (1.13).*

Similar results are already known for various models such as the BBM [7] and the two-dimensional DGFF [19], but there was no related work on local times. The novelty of our result is that we reveal the dependence of the centering term $a_n(t_n)$ in (1.14) on the time parameter t_n which we need to obtain the non-trivial limit of the point process.

In the proof of the Theorem 1.2.10, the key is to obtain a sharp right tail probability estimate of the maximum of local times over the leaves: this enables us to compute the Laplace functional of the point process. In the proof of the tail estimate, following a technique developed by [26], we use an estimate on the probability of the event that the local time process along the path from the root to a leaf does not go over a curve.

What we have proved so far is the convergence of the extremal process encoding only *local* maxima of local times. The convergence of the *full* extremal process has been established for the BBM [3, 8] and the BRW [75]. We believe that a similar result holds for the full extremal process of the local times on the b -ary tree. It is interesting to study similar tasks on local times in $\mathbb{Z}_N^2$, but as we saw in Open Problem 1.2.8, this is very challenging.

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Chapter 2

Effective resistances for supercritical percolation clusters in boxes

Let $\mathcal{C}^n$ be the largest open cluster for supercritical Bernoulli bond percolation in $[-n, n]^d \cap \mathbb{Z}^d$ with $d \geq 2$. We obtain a sharp estimate for the effective resistance on $\mathcal{C}^n$. As an application we show that the cover time for the simple random walk on $\mathcal{C}^n$ is comparable to $n^d(\log n)^2$. Noting that the cover time for the simple random walk on $[-n, n]^d \cap \mathbb{Z}^d$ is of order $n^d \log n$ for $d \geq 3$ (and of order $n^2(\log n)^2$ for $d = 2$), this gives a quantitative difference between the two random walks for $d \geq 3$.

2.1 Introduction

The effective resistance is a fundamental measurement of the conductivity for the electrical network. It has close connections with many subjects of reversible Markov chains such as transience/recurrence, heat kernels, mixing times and cover times. We refer to [45], [65], [70], [74] for the introduction of the theory of reversible Markov chains.

Effective resistances for percolation clusters in $\mathbb{Z}^d$ have been studied for a long time. Grimmett, Kesten and Zhang [54] showed almost-sure finiteness of the effective resistance from a fixed point to infinity on the infinite supercritical percolation cluster in $\mathbb{Z}^d$ for $d \geq 3$. (This is equivalent to almost-sure transience of the simple random walk on the cluster.) The result was ex-

tended to a series of works on more general energy of flows on percolation clusters [4, 16, 55, 57, 58, 69]. The study of effective resistances for boxes on $\mathbb{Z}^d$ under percolation goes back to 1980's [29, 53, 62]. These results showed that critical phenomena occur at the critical percolation probability for effective resistances between opposing faces of boxes (see Remark 2.1.3 below). In [21], Boivin and Rau estimated the effective resistances from a fixed point to boundaries of boxes in the infinite supercritical percolation cluster on $\mathbb{Z}^2$ (see Remark 2.1.4 below). In [14], general upper bounds for effective resistances on general graphs are given by using isoperimetric inequalities. The estimates are used to obtain upper bounds of effective point-to-point resistances on supercritical percolation clusters in boxes on $\mathbb{Z}^2$; however, there seems to be a gap in this part (see Remark 2.1.2 below).

Many properties of simple random walks on supercritical percolation clusters in $\mathbb{Z}^d$ have been investigated such as transience [4, 16, 54, 55], mixing times [15, 31], heat kernel decays [10, 77], invariance principles [12, 76, 90], collisions of two independent simple random walks [11, 30], and the existence of the harmonic measure [21]. These properties are very similar to those of simple random walks on $\mathbb{Z}^d$. A famous folk conjecture is that most important properties of simple random walks survive for simple random walks on supercritical percolation clusters (see, for example, [83]).

In this paper, we consider the largest supercritical percolation cluster in $[-n, n]^d \cap \mathbb{Z}^d$. We obtain the correct order of the maximum of the effective resistances between vertices in the giant cluster. Applying the result, we obtain a sharp estimate of the cover time for the simple random walk on the largest cluster; the result shows that it is much larger than the cover time for the simple random walk on $[-n, n]^d \cap \mathbb{Z}^d$ when $d \geq 3$. The author should mention that the referee pointed out that this result was a conjecture.

In order to describe our results more precisely, we begin with some notation. Let $|\cdot|_1$ be the ℓ^1 -distance on $\mathbb{Z}^d$. We define the set of edges between all nearest-neighbour pairs on $\mathbb{Z}^d$ by $E(\mathbb{Z}^d) := \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}$. For $p \in [0, 1]$, let $\mathbb{P}_p$ be the product Bernoulli measure on $\{0, 1\}^{E(\mathbb{Z}^d)}$ with $\mathbb{P}_p(\omega(e) = 1) = 1 - \mathbb{P}_p(\omega(e) = 0) = p$ for each $e \in E(\mathbb{Z}^d)$. We say that an edge e is open for $\omega \in \{0, 1\}^{E(\mathbb{Z}^d)}$ if $\omega(e) = 1$. For $A \subseteq \mathbb{Z}^d$, we define a random set of edges by

$$\mathcal{O}_A = \mathcal{O}_A(\omega) := \{\{x, y\} \in E(\mathbb{Z}^d) : x, y \in A, \omega(\{x, y\}) = 1\}. \quad (2.1)$$

An open cluster in A is a connected component of the graph $(A, \mathcal{O}_A)$. We

define the critical probability $p_c(\mathbb{Z}^d)$ by

$$\inf\{p \in [0, 1] : \mathbb{P}_p(\text{the cluster in } \mathbb{Z}^d \text{ containing the origin is infinite}) > 0\}. \quad (2.2)$$

We focus our attention to clusters in the box

$$B(n) := [-n, n]^d \cap \mathbb{Z}^d. \quad (2.3)$$

It is known that for $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, we have the unique largest open cluster in $B(n)$ whose size is proportional to n^d , $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$. For $d \geq 3$, see [84, Theorem 1.2]. The $d = 2$ case is well-known, and [32] will serve as a convenient reference. We write $\mathcal{C}^n$ to denote the largest open cluster in $B(n)$.

Let $G = (V(G), E(G))$ be a finite connected graph; the set $V(G)$ is the vertex set and $E(G)$ is the edge set. We define the Dirichlet energy by

$$\mathcal{E}(f) := \frac{1}{2} \sum_{\substack{u, v \in V(G) \\ \{u, v\} \in E(G)}} (f(u) - f(v))^2, f \in \mathbb{R}^{V(G)}.$$

For $A, B \subset V(G)$ with $A \cap B = \emptyset$, the effective resistance between A and B for G is defined by

$$R_{\text{eff}}^G(A, B)^{-1} := \inf\{\mathcal{E}(f) : f \in \mathbb{R}^{V(G)}, \mathcal{E}(f) < \infty, f|_A = 1, f|_B = 0\}.$$

We write $R_{\text{eff}}^G(x, y)$ to denote $R_{\text{eff}}^G(\{x\}, \{y\})$. We now state our result on the effective resistance for $\mathcal{C}^n$.

Theorem 2.1.1 *For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$,*

$$c_1 \cdot \log n \leq \max_{x, y \in \mathcal{C}^n} R_{\text{eff}}^{\mathcal{C}^n}(x, y) \leq c_2 \cdot \log n.$$

Remark 2.1.2 *Corollary 3.1 of [14] says that for $d = 2$ and $p > p_c(\mathbb{Z}^2)$, there exists $c_1 > 0$ such that $\lim_{n \rightarrow \infty} \mathbb{P}_p(\max_{x, y \in \mathcal{C}^n} R_{\text{eff}}^{\mathcal{C}^n}(x, y) \leq c_1 \cdot \log n) = 1$. The proof is based on general upper bounds for effective resistances given in Theorem 2.1 of [14] and an isoperimetric profile for $\mathcal{C}^n$ studied in [15]. However, according to the latest arXiv version of their paper [14] and [64], the proof only implies the following: there exists $c_1 > 0$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}_p \left(\max_{x, y \in \mathcal{C}^n} R_{\text{eff}}^{\mathcal{C}^n}(x, y) \leq c_1 \cdot (\log n)^2 \right) = 1.$$

This is due to bad isoperimetry of some small subsets of $\mathcal{C}^n$.

Remark 2.1.3 Fix $d \geq 2$. Let us consider a random graph G_n which has the vertex set $[0, n]^d \cap \mathbb{Z}^d$ and the edge set $\mathcal{O}_{[0, n]^d \cap \mathbb{Z}^d}$. We write R_n to denote the effective resistance

$$R_{\text{eff}}^{G_n}([0, n]^{d-1} \times \{0\}, [0, n]^{d-1} \times \{n\}).$$

In [29, 53], it was shown that the following holds: there exist $c_1, c_2 > 0$ such that if $p < p_c(\mathbb{Z}^d)$,

$$R_n = \infty \text{ for large } n \in \mathbb{N}, \quad \mathbb{P}_p - a.s.,$$

and if $p_c(\mathbb{Z}^d) < p \leq 1$,

$$c_1 \leq \liminf_{n \rightarrow \infty} n^{d-2} R_n \leq \limsup_{n \rightarrow \infty} n^{d-2} R_n \leq c_2, \quad \mathbb{P}_p - a.s.$$

Remark 2.1.4 Let $\mathcal{C}_\infty$ be the infinite Bernoulli bond percolation cluster in $\mathbb{Z}^2$. Define spheres in $\mathcal{C}_\infty$ by

$$\partial B_{\mathcal{C}_\infty}(x, r) = \{y \in \mathcal{C}_\infty : d_{\mathcal{C}_\infty}(x, y) = r\}, \quad x \in \mathcal{C}_\infty, r \in \mathbb{N},$$

where $d_{\mathcal{C}_\infty}$ is the graph distance of $\mathcal{C}_\infty$. In Proposition 4.3 of the published version of [21], Boivin and Rau proved the following: for $p \in (p_c(\mathbb{Z}^2), 1]$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$ -a.s., for all $x_0 \in \mathcal{C}_\infty$, for large $n \in \mathbb{N}$,

$$c_1 \log n \leq R_{\text{eff}}^{\mathcal{C}_\infty}(x_0, \partial B_{\mathcal{C}_\infty}(x_0, n)) \leq c_2 \log n.$$

For a finite connected graph $G = (V(G), E(G))$ and $x \in V(G)$, let $\tau_x(G)$ be the hitting time of x by the simple random walk on G . We define the cover time by

$$t_{\text{cov}}(G) := \max_{x \in V(G)} E_x \left(\max_{y \in V(G)} \tau_y(G) \right).$$

Applying Theorem 2.1.1, we can obtain an improvement of Proposition 3.3 of the arXiv version of [1].

Theorem 2.1.5 For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$,

$$c_1 \cdot n^d (\log n)^2 \leq t_{\text{cov}}(\mathcal{C}^n) \leq c_2 \cdot n^d (\log n)^2.$$

Remark 2.1.6 *It is known that $t_{\text{cov}}(B(n))$ is comparable to $n^2(\log n)^2$ when $d = 2$ and to $n^d \log n$ when $d \geq 3$ (see, for example, Section 11.3.2 of [70]). Therefore, Theorem 2.1.5 implies that the simple random walk on $\mathcal{C}^n$ exhibits anomalous behavior when $d \geq 3$. This is due to irregularity of $\mathcal{C}^n$; the cluster $\mathcal{C}^n$ contains a lot of one-dimensional objects (see Lemma 2.2.6 below).*

Let $G = (V(G), E(G))$ be any finite connected graph. Let us define the Gaussian free field on G . This is a centered Gaussian process $\{\eta_x^G\}_{x \in V(G)}$ with $\eta_{x_0}^G = 0$ for some fixed vertex $x_0 \in V(G)$ and the covariances are given by

$$\mathbf{E}(\eta_x^G \eta_y^G) = \frac{1}{2} \left(R_{\text{eff}}^G(x, x_0) + R_{\text{eff}}^G(y, x_0) - R_{\text{eff}}^G(x, y) \right), \quad x, y \in V(G).$$

We define the expected maximum of the Gaussian free field by

$$M_G := \mathbf{E} \left(\max_{x \in V(G)} \eta_x^G \right).$$

Note that M_G does not depend on the choice of x_0 . For a set S , we will write $|S|$ to denote the cardinality of S . In [40], Ding, Lee and Peres proved the following: there exist universal constants $c_1, c_2 > 0$ such that

$$c_1 \cdot |E(G)| \cdot (M_G)^2 \leq t_{\text{cov}}(G) \leq c_2 \cdot |E(G)| \cdot (M_G)^2. \quad (2.4)$$

By (2.4) and Theorem 2.1.5, we obtain the following estimate of $M_{\mathcal{C}^n}$ immediately.

Corollary 2.1.7 *For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$,*

$$c_1 \cdot \log n \leq M_{\mathcal{C}^n} \leq c_2 \cdot \log n.$$

Remark 2.1.8 *By Remark 2.1.6 and (2.4), $M_{B(n)}$ is comparable to $\log n$ when $d = 2$ and to $\sqrt{\log n}$ when $d \geq 3$. Thus there is a marked quantitative difference between $M_{\mathcal{C}^n}$ and $M_{B(n)}$ when $d \geq 3$.*

Let us describe the outline of the paper. Section 2.2 gives preliminary lemmas about percolation estimates. In Subsection 2.3.1, we give a proof of Theorem 2.1.5 via Theorem 2.1.1. The upper bound is due to Theorem 2.1.1 and the Matthews bound [78]. The lower bound is based on the general

bound on the cover time via the Gaussian free field given in [40]. The key of the proof of the lower bound is counting the number of one-way open paths of length of order $\log n$. In Subsection 2.3.2, we construct the so-called “Kesten grid” (the terminology comes from Mathieu and Remy [77]). This is an analogue of the square lattice and consists of “white” sites on a renormalized lattice; the concept of “white” sites is from the renormalization argument of Antal and Pisztora [5]. In Subsection 2.3.3, we prove Theorem 2.1.1. The lower bound is immediately followed by the fact that $\mathcal{C}^n$ has a one-way open path of length of order $\log n$. In the proof of the upper bound, we construct unit flows between vertices of $\mathcal{C}^n$ with energy of order $\log n$. Thanks to Kesten grids, we can let the flows run almost on some “sheets” and adapt an argument of estimating upper bounds of effective resistances for $\mathbb{Z}^2$. We use a result on the chemical distance for $\mathcal{C}^n$ based on [31]; this guarantees that every vertex of $\mathcal{C}^n$ can be connected with a Kesten grid decorated with small boxes by an open path in $\mathcal{C}^n$ of length of order $\log n$.

Throughout the paper, we will write c, c', c'' to denote positive constants depending only on the dimension of the lattice and the percolation parameter. Values of c, c' and c'' will change from line to line. We use $c_1, c_2, \dots$ to denote constants whose values are fixed within each argument. If we cite elsewhere the constant c_1 in Lemma 2.2.4, we write it as $c_{2.2.4.1}$, for example.

2.2 Percolation estimates

In this Section, we collect some useful results on percolation estimates. We need the following facts to prove Theorem 2.1.1 and Theorem 2.1.5.

(1) Size and connectivity pattern of $\mathcal{C}^n$

A path is a sequence $(x_0, x_1, \dots, x_\ell)$ satisfying that $|x_{i-1} - x_i|_1 = 1$ for all $1 \leq i \leq \ell$. An open path is a path all of whose edges are open. We write $|\cdot|_\infty$ to denote the ℓ^∞ -distance on $\mathbb{Z}^d$. The diameter of an open path $(x_0, x_1, \dots, x_\ell)$ is defined by $\max_{0 \leq i, j \leq \ell} |x_i - x_j|_\infty$. For $\kappa > 0$, let A_κ^n be the event that every open path in $B(n)$ with diameter larger than $\kappa \log n$ is contained in $\mathcal{C}^n$. The following fact follows easily from [84, Theorem 1.2 and Theorem 3.1] for $d \geq 3$ and [32, Theorem 1 and Theorem 9] for $d = 2$.

Lemma 2.2.1 *Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. There exist $c_1, c_2 > 0$ and $\kappa_0, n_0 \in$*

$\mathbb{N}$ such that for all $\kappa \geq \kappa_0$ and $n \geq n_0$,

$$\mathbb{P}_p(A_\kappa^n \cap \{|\mathcal{C}^n| \geq c_1 n^d\}) \geq 1 - 2 \exp(-c_2 \kappa \log n).$$

Remark 2.2.2 By Lemma 2.2.1 and the Borel-Cantelli lemma, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$, the event $A_{c_1}^n \cap \{|\mathcal{C}^n| \geq c_2 n^d\}$ holds.

(2) Chemical distance of $\mathcal{C}^n$

Let $d_{\mathcal{C}^n}$ be the graph distance of the graph $(\mathcal{C}^n, \mathcal{O}_{\mathcal{C}^n})$. The following result is immediately followed by Lemma 3.2 and (4.1) in [31].

Lemma 2.2.3 Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. There exist $c_1 > 0$ and $\kappa_0 \in \mathbb{N}$ such that the following holds for all $\kappa \geq \kappa_0$, $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$: for all $x, y \in \mathcal{C}^n$ with $|x - y|_1 \leq \kappa \log n$,

$$d_{\mathcal{C}^n}(x, y) \leq c_1 \kappa \log n.$$

(3) Crossing probabilities

In (3), we restrict our attention to Bernoulli site percolation on the square lattice. Let $\mathbb{Q}_q$ be the product Bernoulli measure on $\{0, 1\}^{\mathbb{Z}^2}$ with $\mathbb{Q}_q(\omega(x) = 1) = 1 - \mathbb{Q}_q(\omega(x) = 0) = q$ for each $x \in \mathbb{Z}^2$. For $\omega \in \{0, 1\}^{\mathbb{Z}^2}$ and $x \in \mathbb{Z}^2$, we will say that x is occupied if $\omega(x) = 1$. The critical probability $q_c(\mathbb{Z}^2)$ is defined similarly to (2.2). We write $x \cdot y$ to denote the inner product of x and y . A self-avoiding path is a path all of whose vertices are distinct. Let e_1, e_2 be the standard basis for $\mathbb{Z}^2$. Fix $m, n \in \mathbb{N}$. A horizontal crossing in $([0, m] \times [0, n]) \cap \mathbb{Z}^2$ is a self-avoiding path $(x_0, x_1, \dots, x_\ell)$ with $x_0 \cdot e_1 = 0$ and $x_\ell \cdot e_1 = m$.

Lemma 2.2.4 [62, Theorem 11.1] For $d = 2$ and $q > q_c(\mathbb{Z}^2)$, there exist $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned} & \mathbb{Q}_q \left(\text{there exist at least } c_1 n \text{ disjoint horizontal crossings} \right. \\ & \quad \left. \text{consisting of occupied sites in } ([0, m] \times [0, n]) \cap \mathbb{Z}^2 \right) \\ & \geq 1 - c_2 \cdot m \exp(-c_3 n). \end{aligned}$$

(4) Renormalization argument

We recall the renormalization argument of [5]. We will write $a, b, \dots$ rather

than $x, y, \dots$ to denote vertices of the renormalized lattice. Fix a positive integer K . To $a \in \mathbb{Z}^d$, we associate boxes

$$B_a(K) := (2K + 1)a + [-K, K]^d \cap \mathbb{Z}^d, \quad (2.5)$$

$$B'_a(K) := (2K + 1)a + \left[-\frac{5}{4}K, \frac{5}{4}K\right]^d \cap \mathbb{Z}^d. \quad (2.6)$$

We will say that an open cluster crosses a box if the cluster intersects all the faces of the box. For $a \in \mathbb{Z}^d$, we define an event R_a^K satisfying the following:

- There exists a unique open cluster $\mathcal{C}$ in $B'_a(K)$ crossing $B'_a(K)$.
- The open cluster $\mathcal{C}$ crosses all the subboxes of $B'_a(K)$ of side length larger than $\frac{K}{10}$.
- Any open paths in $B'_a(K)$ of diameter larger than $\frac{K}{10}$ are contained in $\mathcal{C}$.

We say $a \in \mathbb{Z}^d$ is white for $\omega \in \{0, 1\}^{E(\mathbb{Z}^d)}$ if $\mathbf{1}_{R_a^K}(\omega) = 1$. By [84, Theorem 3.1] for $d \geq 3$ and [32, Theorem 9] for $d = 2$, if $p > p_c(\mathbb{Z}^d)$, we have for all $a \in \mathbb{Z}^d$

$$\lim_{K \rightarrow \infty} \mathbb{P}_p(R_a^K) = 1.$$

By [71, Theorem 0.0 (ii)], the following holds:

Lemma 2.2.5 *For $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, there exists a function $q : \mathbb{N} \rightarrow [0, 1]$ with $\lim_{K \rightarrow \infty} q(K) = 1$ such that*

*the process $(\mathbf{1}_{R_a^K})_{a \in \mathbb{Z}^d}$ stochastically dominates
a Bernoulli site percolation process in $\mathbb{Z}^d$ with parameter $q(K)$.*

(5) Special open paths

Let $e_1, \dots, e_d$ be the standard basis for $\mathbb{Z}^d$. We define a one-sided boundary of $B(n)$ by

$$\partial_1 B(n) := \{x \in B(n) : x \cdot e_1 = n\}.$$

We will say that $x \in \partial_1 B(n)$ is m -special if it is a base point of a special open path; that is, the edge $\{x + ie_1, x + (i + 1)e_1\}$ is open for each $0 \leq i \leq m - 1$ and the vertex $x + ie_1$ does not have any other open edges for each $1 \leq i \leq m$. See Figure 2.1. The following is a key to prove the lower bounds of Theorem 2.1.1 and Theorem 2.1.5.

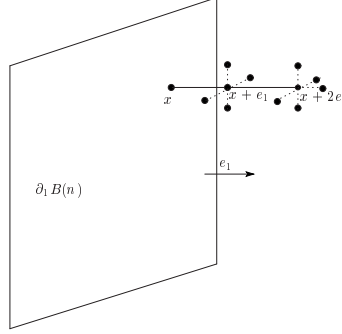


Figure 2.1: An illustration of a 2-special vertex x and the corresponding special open path. The dotted line segments at $x + ie_1 (i = 1, 2)$ are closed edges. The solid line segment between x and $x + 2e_1$ are composed of open edges.

Lemma 2.2.6 *For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$,*

$$|\{x \in \partial_1 B(n) \cap \mathcal{C}^n : x \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\}| \geq n^{c_2}.$$

Proof. Fix a large constant $K > 0$. We define a subset of $\partial_1 B(n)$ by

$$\partial_{1,2}^K B(n) := \left\{ x \in \partial_1 B(n) : |x \cdot e_i| \leq \frac{5}{4}K \text{ for all } 3 \leq i \leq d \right\}.$$

Let S_n^K be the “square” defined by

$$\left\{ a \in \mathbb{Z}^d : |a \cdot e_1| \leq \left\lceil \frac{n}{2K+1} \right\rceil, |a \cdot e_2| \leq \left\lfloor \frac{n-5K/4}{2K+1} \right\rfloor, a \cdot e_i = 0 \text{ for } 3 \leq i \leq d \right\}.$$

Note that S_n^K is isomorphic to $[0, 2\lceil \frac{n}{2K+1} \rceil] \times [0, 2\lfloor \frac{n-5K/4}{2K+1} \rfloor] \cap \mathbb{Z}^2$. Let $\varepsilon > 0$ be a small constant. By Lemma 2.2.4 and Lemma 2.2.5, the following holds: there exist $c, c' > 0$ such that with $\mathbb{P}_p$ -probability at least $1 - c \cdot n \exp(-c'n)$, we have more than εn disjoint crossings of white sites in S_n^K . By this fact and the definition of white sites together with Lemma 2.2.1, the event $G_n^K := \{|\partial_{1,2}^K B(n) \cap \mathcal{C}^n| < \varepsilon n\}$ satisfies the following: there exist $c, c' > 0$ and $\kappa_0 \in \mathbb{N}$ such that for all $\kappa \geq \kappa_0$ and sufficiently large $n \in \mathbb{N}$,

$$\mathbb{P}_p(G_n^K) \leq c \exp(-c' \kappa \log n). \quad (2.7)$$

Take $c_1 > 0$ small enough to make $r := 1 - c_1(\log \frac{1}{p} + 2(d-1)\log \frac{1}{1-p})$ positive. Fix $0 < c_2 < r$. We write $\text{Bin}(m, q)$ to denote a binomial random variable with parameter m and q . Let q_n be the probability of the event that a fixed vertex in $\partial_1 B(n)$ is $\lfloor c_1 \log n \rfloor$ -special. Consider the σ -field generated by finite-dimensional cylinders associated with configurations restricted to $B(n)$. Note that conditioned on the σ -field, events $\{x \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\}$ and $\{y \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\}$ are independent for $x, y \in \partial_1 B(n)$ with $|x - y|_\infty \geq 2$. By conditioning on the σ -field, we have for some $c, c' > 0$ and some $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned} & \mathbb{P}_p \left(\left\{ |\{x \in \partial_1 B(n) \cap \mathcal{C}^n : x \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\}| \leq n^{c_2} \right\} \cap (G_n^K)^c \right) \\ & \leq \mathbb{P}_p \left(\text{Bin}(\lfloor \varepsilon' n \rfloor, q_n) \leq n^{c_2} \right) \\ & \leq c \exp(-c' n^r). \end{aligned} \tag{2.8}$$

In the last inequality, we have used the Chebyshev inequality and the fact that $nq_n \geq cn^r$ for some $c > 0$. By (2.7) and (2.8) together with the Borel-Cantelli lemma, we obtain the conclusion. $\square$

2.3 Proof of Theorem 2.1.1 and Theorem 2.1.5

2.3.1 Cover time estimate

In this Subsection we prove Theorem 2.1.5 via Theorem 2.1.1. We begin with general bounds on cover times based on [40, 78]. The following is immediately followed by [70, Proposition 10.6 and Theorem 11.2] (see also [33, 78]) and [40, Theorem 1.1 and Lemma 1.11].

Lemma 2.3.1 *Let $G = (V(G), E(G))$ be any finite connected graph.*

(1) *There exists a universal constant $c_1 > 0$ such that*

$$t_{\text{cov}}(G) \leq c_1 \cdot |E(G)| \cdot \left(\max_{x, y \in V(G)} R_{\text{eff}}^G(x, y) \right) \cdot \log |V(G)|.$$

(2) *There exists a universal constant $c_2 > 0$ such that for all subset $\tilde{V} \subset$*

$V(G)$,

$$t_{cov}(G) \geq c_2 \cdot |E(G)| \cdot \left(\min_{\substack{x, y \in \tilde{V} \\ x \neq y}} R_{eff}^G(x, y) \right) \cdot \log |\tilde{V}|.$$

Proof of Theorem 2.1.5 via Theorem 2.1.1. The upper bound is immediately followed by Lemma 2.3.1 (1) and Theorem 2.1.1. From now, we prove the lower bound. By Remark 2.2.2, we have $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$,

$$\mathcal{C}^{n - \lceil c_{2.2.6.1} \log n \rceil} \subseteq \mathcal{C}^n. \quad (2.9)$$

Set $m_n := n - \lceil c_{2.2.6.1} \log n \rceil$. We define a set V_n of tips of special open paths by

$$\{x + \lfloor c_{2.2.6.1} \log m_n \rfloor e_1 : x \in \partial_1 B(m_n) \cap \mathcal{C}^{m_n} \text{ and } x \text{ is } \lfloor c_{2.2.6.1} \log m_n \rfloor\text{-special}\}.$$

By (2.9), Lemma 2.2.6 and the Nash-Williams inequality (see, for example, [70, Proposition 9.15]), we have the following $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$: for all $x, y \in V_n$ with $x \neq y$,

$$V_n \subseteq \mathcal{C}^n, |V_n| \geq (m_n)^{c_{2.2.6.2}} \text{ and } R_{eff}^{\mathcal{C}^n}(x, y) \geq \lfloor c_{2.2.6.1} \log m_n \rfloor.$$

Therefore, by Remark 2.2.2 and Lemma 2.3.1 (2) with $\tilde{V} = V_n$, we obtain the lower bound. $\square$

2.3.2 Kesten grid

In this Subsection, we construct the so-called “Kesten grids” on the renormalized lattice (recall Section 2.2 (4)) in order to obtain the upper bound of Theorem 2.1.1. The terminology is due to Mathieu and Remy [77]. We note that the construction of the Kesten grid is based on Theorem 11.1 of [62] (see Lemma 2.2.4).

Let us define some notations. Recall that $e_1, \dots, e_d$ is the standard basis for $\mathbb{Z}^d$. Fix a sufficiently large positive integer K . Recall the notation (2.6). For a subset S of the renormalized lattice, we define a fattened version of S by

$$W(S) := \bigcup_{a \in S} B'_a(K). \quad (2.10)$$

Let α be a positive constant. We will choose α sufficiently large later. Let ℓ_n be the largest integer ℓ satisfying that

$$(2K + 1)((2\lceil \alpha \log n \rceil + 1)\ell + \lceil \alpha \log n \rceil) + \frac{5}{4}K \leq n,$$

and set

$$\tilde{\ell}_n := (2\lceil \alpha \log n \rceil + 1)\ell_n + \lceil \alpha \log n \rceil. \quad (2.11)$$

Note that we have $W(B(\tilde{\ell}_n)) \subset B(n)$ (recall the notation (2.3)). Thus we can regard $B(\tilde{\ell}_n)$ as a box in the renormalized lattice corresponding to the original box $B(n)$. The number of two-dimensional sections in $B(\tilde{\ell}_n)$ is $\frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2}$; each of them is isomorphic to the square $[-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2$. In Subsection 2.3.3 below, we will focus our attention on one of them, namely

$$F_1 := \{k_1 e_1 + k_2 e_2 : -\tilde{\ell}_n \leq k_1, k_2 \leq \tilde{\ell}_n\}. \quad (2.12)$$

We write

$$F_i, \quad 2 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2} \quad (2.13)$$

to denote the other two-dimensional sections of $B(\tilde{\ell}_n)$. For $-\ell_n \leq m \leq \ell_n$, we define the m -th horizontal strip of F_1

$$R_1^n(m) := \{k_1 e_1 + k_2 e_2 : -\tilde{\ell}_n \leq k_1 \leq \tilde{\ell}_n, |k_2 - (2\lceil \alpha \log n \rceil + 1)m| \leq \lceil \alpha \log n \rceil\} \quad (2.14)$$

and the m -th vertical strip of F_1

$$R_2^n(m) := \{k_1 e_1 + k_2 e_2 : -\tilde{\ell}_n \leq k_2 \leq \tilde{\ell}_n, |k_1 - (2\lceil \alpha \log n \rceil + 1)m| \leq \lceil \alpha \log n \rceil\}. \quad (2.15)$$

We define a horizontal (respectively, vertical) crossing of F_1 as a self-avoiding path with endvertices a, b satisfying $a \cdot e_1 = -\tilde{\ell}_n$ and $b \cdot e_1 = \tilde{\ell}_n$ (respectively, $a \cdot e_2 = -\tilde{\ell}_n$ and $b \cdot e_2 = \tilde{\ell}_n$). We define strips and crossings for the other two-dimensional sections of $B(\tilde{\ell}_n)$ in a similar fashion. We note that each strip is isomorphic to $([0, 2\tilde{\ell}_n] \times [0, 2\lceil \alpha \log n \rceil]) \cap \mathbb{Z}^2$. Taking α large enough, the following holds immediately by Lemma 2.2.4 and Lemma 2.2.5.

Corollary 2.3.2 *Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. The following holds $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$: for all $1 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2}$ and $-\ell_n \leq m \leq \ell_n$, there exist at least $2c_{2.2.4.1}\lceil \alpha \log n \rceil$ disjoint horizontal (respectively, vertical) crossings consisting of white sites in the m -th horizontal (respectively, vertical) strip of F_i .*

Set $L_n := \lceil 2c_{2.2.4.1}\lceil \alpha \log n \rceil \rceil$. Fix $1 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2}$ and a configuration satisfying the event of Corollary 2.3.2. We can choose L_n disjoint crossings of white sites in each strip of F_i . Since F_i is isomorphic to $[-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2$, the horizontal and the vertical crossings intersect (see Figure 2.2) and form a grid on F_i . We will call it a Kesten grid. Note that Kesten grids on F_i and F_j do not intersect if $i \neq j$ in general.

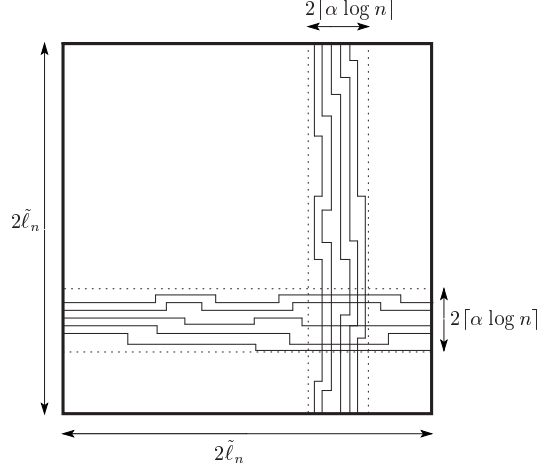


Figure 2.2: An illustration of disjoint crossings of white sites (thin solid lines) in strips (rectangles with dotted boundaries) of F_1 (square with thick solid boundary). These horizontal and vertical crossings intersect since F_1 is isomorphic to $[-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2$.

2.3.3 Effective resistance estimate

In this Subsection we prove Theorem 2.1.1. To prove it, we only need to estimate the upper bound of effective resistances for pairs of vertices on $W(F_i) \cap \mathcal{C}^n$, $1 \leq i \leq \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2}$ (recall the notations (2.10)-(2.13)). In the following proposition, we will focus our attention to $W(F_1) \cap \mathcal{C}^n$. Recall the constant α in Subsection 2.3.2. To $k_1, k_2 \in \mathbb{Z}$, we associate the “square” defined by

$$S_{k_1, k_2}^n := \{s_1 e_1 + s_2 e_2 : |s_i - (2\lceil \alpha \log n \rceil + 1)k_i| \leq \lceil \alpha \log n \rceil, i = 1, 2\}.$$

Proposition 2.3.3 *Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. Let $m_1, m_2, \tilde{m}_2$ be integers satisfying that $m_2 < \tilde{m}_2$, $\tilde{m}_2 - m_2$ is even, and $m_1 + \frac{\tilde{m}_2 - m_2}{2} \leq \ell_n$. Then there exists $c_1 > 0$ (not depending on choices of $m_1, m_2, \tilde{m}_2$) such that $\mathbb{P}_p$ -a.s., for large $n \in \mathbb{N}$, the following holds: for all $x \in W(S_{m_1, m_2}^n) \cap \mathcal{C}^n$ and $y \in W(S_{m_1, \tilde{m}_2}^n) \cap \mathcal{C}^n$,*

$$R_{\text{eff}}^n(x, y) \leq c_1 \cdot \log n.$$

The upper bound of Theorem 2.1.1 follows from a similar argument of the proof of Proposition 2.3.3 below together with Lemma 2.2.3 and the

triangle inequality for the effective resistance (see, for instance, [70, Corollary 10.8]). We omit the details. The lower bound of Theorem 2.1.1 holds by Lemma 2.2.6, (2.9) and the Nash-Williams inequality. In the rest of the article, we focus on proving Proposition 2.3.3. Fix some integers r, r' with $r < r'$. A left-to-right (respectively, right-to-left) path in $[r, r'] \times \mathbb{Z}^{d-1}$ is a path in $[r, r'] \times \mathbb{Z}^{d-1}$ whose initial and final vertices a, b satisfy $a \cdot e_1 = r$ and $b \cdot e_1 = r'$ (respectively, $a \cdot e_1 = r'$ and $b \cdot e_1 = r$). A bottom-to-top path in $\mathbb{Z} \times [r, r'] \times \mathbb{Z}^{d-2}$ is a path in $\mathbb{Z} \times [r, r'] \times \mathbb{Z}^{d-2}$ whose initial and final vertices a, b satisfy $a \cdot e_2 = r$ and $b \cdot e_2 = r'$. For a path Γ and $a, a' \in \Gamma$ (we assume that a appears before a' in Γ), we write $\Gamma[a, a']$ to denote a part of Γ from a to a' . In particular, when a' is the final vertex of Γ , we write $\Gamma[a]$ in the place of $\Gamma[a, a']$.

Proof of Proposition 2.3.3. We fix a configuration satisfying the events of Remark 2.2.2, Corollary 2.3.2, and Lemma 2.2.3 with κ large enough. We choose vertices x and y as in Proposition 2.3.3. We will construct a unit flow between x and y . Our argument is based on [74, Proposition 2.15] and Section 4.1 of the arXiv version of [21].

We first construct a random self-avoiding open path from x to y , most of whose parts lie on $W(F_1) \cap \mathcal{C}^n$. Recall the notations (2.14) and (2.15). A Kesten grid guarantees the existence of the following self-avoiding paths:

- disjoint left-to-right (respectively, right-to-left) self-avoiding paths of white sites $H_1^m, \dots, H_{L_n}^m$ contained in $R_1^n(m_2 + m)$ for $1 \leq m \leq \frac{\tilde{m}_2 - m_2}{2}$ (respectively, $\frac{\tilde{m}_2 - m_2}{2} < m < \tilde{m}_2 - m_2$),
- disjoint bottom-to-top self-avoiding paths of white sites $V_1^m, \dots, V_{L_n}^m$ contained in $R_2^n(m_1 + m)$ for $0 \leq m \leq \frac{\tilde{m}_2 - m_2}{2}$,
- left-to-right self-avoiding paths of white sites H_x and H_y of length at most $c \log n$ for some $c > 0$, contained in S_{m_1, m_2}^n and in $S_{m_1, \tilde{m}_2}^n$ respectively.

From now, we construct an open path from x to y corresponding to a fixed label (i, j) with $i \in \{0, \dots, \frac{\tilde{m}_2 - m_2}{2}\}$ and $j \in \{1, \dots, L_n\}$ by the following three steps.

Step 1: For convenience, we will associate a vertex $(k_1, k_2) \in \mathbb{Z}^2$ to S_{k_1, k_2}^n . Let $f : \{0, \dots, 2i\} \rightarrow \{0, \dots, i\}$ be a function defined by $f(r) = r$ for $0 \leq r \leq i$ and $f(r) = 2i - r$ for $i < r \leq 2i$. Take a self-avoiding path Γ_i as follows:

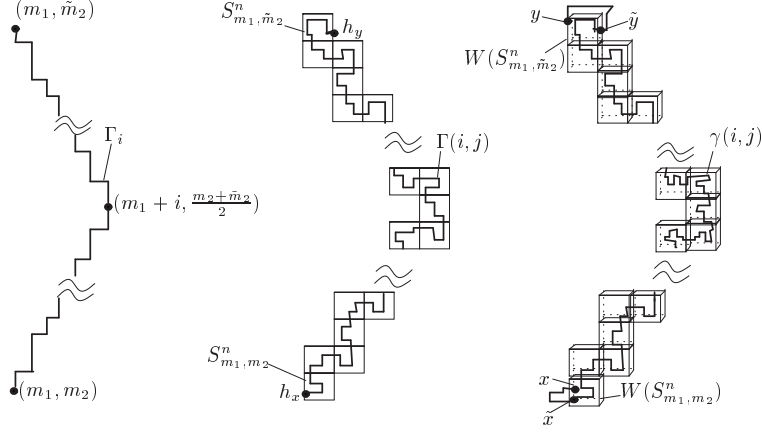


Figure 2.3: Illustrations of paths Γ_i , $\Gamma(i, j)$ and $\gamma(i, j)$.

- Let $(s_r)_{0 \leq r \leq 2i+1}$ be a sequence with $0 = s_0 < s_1 < \dots < s_{2i+1} = \tilde{m}_2 - m_2$. For $0 \leq r \leq 2i$, set $v_{2r} = (m_1 + f(r), m_2 + s_r)$ and $v_{2r+1} = (m_1 + f(r), m_2 + s_{r+1})$. The self-avoiding path Γ_i is obtained from the sequence $(v_r)_{0 \leq r \leq 4i+1}$ by linear interpolation.
- The path Γ_i lies within 1 in ℓ^∞ -distance from the piecewise line segments connecting (m_1, m_2) , $(m_1 + i, \frac{m_2 + \tilde{m}_2}{2})$ and $(m_1, \tilde{m}_2)$.

See the left-hand side of Figure 2.3.

Step 2: Recall the notation before the proof of Proposition 2.3.3. Let h_x (respectively, h_y) be the initial (respectively, the final) vertex of H_x (respectively, H_y). We define vertices $a_1, \dots, a_{4i+2}$ as follows:

- For $0 \leq r \leq 2i$, a_{2r+1} is the last-visited vertex of $H_j^{s_r}[a_{2r}]$ by $V_j^{f(r)}$, where $a_0 := h_x$ and $H_j^{s_0} := H_x$.
- For $0 \leq r \leq 2i$, a_{2r+2} is the last-visited vertex of $V_j^{f(r)}[a_{2r+1}]$ by $H_j^{s_{r+1}}$, where $H_j^{s_{2i+1}} := H_y$.

See Figure 2.4. We can obtain a self-avoiding path $\Gamma(i, j)$ of white sites from h_x to h_y in $\cup_{(k_1, k_2) \in \Gamma_i} S_{k_1, k_2}^n$ by connecting segments

$$H_j^{s_r}[a_{2r}, a_{2r+1}], V_j^{f(r)}[a_{2r+1}, a_{2r+2}], \quad 0 \leq r \leq 2i, \quad \text{and} \quad H_y[a_{4i+2}, h_y].$$

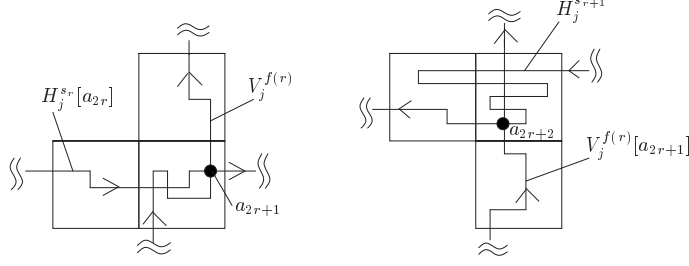


Figure 2.4: Illustrations of a_{2r+1} and a_{2r+2} for $0 \leq r \leq 2i$.

See the middle of Figure 2.3.

Step 3: Recall the notation (2.5). Fix vertices $\tilde{x} \in B_{h_x}(K) \cap \mathcal{C}^n$ and $\tilde{y} \in B_{h_y}(K) \cap \mathcal{C}^n$. Indeed, we can take such $\tilde{x}$ and $\tilde{y}$ by Remark 2.2.2. By the definition of white sites, we can take a self-avoiding open path $\tilde{\gamma}(i, j)$ from $\tilde{x}$ to $\tilde{y}$ contained in $W(\Gamma(i, j))$. By Lemma 2.2.3, we have self-avoiding open paths γ_x and γ_y in $\mathcal{C}^n$ from x to $\tilde{x}$ and from $\tilde{y}$ to y respectively of length at most $c \log n$ for some $c > 0$. We can get a self-avoiding open path $\gamma(i, j)$ from x to y in $\mathcal{C}^n$ via $\gamma_x, \tilde{\gamma}(i, j)$ and γ_y . See the right-hand side of Figure 2.3.

Let X and Y be random variables distributed uniformly on $\{0, \dots, \frac{\tilde{m}_2 - m_2}{2}\}$ and on $\{1, \dots, L_n\}$ respectively; they are defined on a probability space with probability measure $\mathbf{P}$. From now, we construct a unit flow from x to y via the random open path $\gamma(X, Y)$. Recall the notation (2.1). We define a random function ψ on $\mathcal{C}^n \times \mathcal{C}^n$ by

$$\psi(u, v) := \begin{cases} 1 & \text{if } \{u, v\} \in \mathcal{O}_{\mathcal{C}^n} \text{ and } \gamma(X, Y) \text{ passes } u, v \text{ in this order,} \\ -1 & \text{if } \{u, v\} \in \mathcal{O}_{\mathcal{C}^n} \text{ and } \gamma(X, Y) \text{ passes } v, u \text{ in this order,} \\ 0 & \text{otherwise.} \end{cases}$$

We define a function θ on $\mathcal{C}^n \times \mathcal{C}^n$ by

$$\theta(u, v) := \mathbf{E}(\psi(u, v)) \text{ for } u, v \in \mathcal{C}^n.$$

Since ψ is a unit flow from x to y , θ is a unit flow from x to y . In order to bound θ , let us define the following function on $\mathcal{C}^n \times \mathcal{C}^n$:

$$p(u, v) := \begin{cases} \mathbf{P}(\gamma(X, Y) \text{ passes the edge } \{u, v\}) & \text{if } \{u, v\} \in \mathcal{O}_{\mathcal{C}^n}, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 \leq r \leq \tilde{m}_2 - m_2$, we define a set of labels of the r -th level by

$$D_r := \begin{cases} \{(m_1 + s, m_2 + r) \in \mathbb{Z}^2 : 0 \leq s \leq r + 1\} & \text{if } r \leq \frac{\tilde{m}_2 - m_2}{2}, \\ \{(m_1 + s, m_2 + r) \in \mathbb{Z}^2 : 0 \leq s \leq \tilde{m}_2 - m_2 - r + 1\} & \text{otherwise.} \end{cases}$$

Let U be the union of γ_x , γ_y , $W(H_x)$ and $W(H_y)$. By Thomson's principle (see, for example, [70, Theorem 9.10]) and the construction of $\gamma(X, Y)$, we have

$$\begin{aligned} R_{\text{eff}}^{\mathcal{C}^n}(x, y) &\leq \frac{1}{2} \sum_{u, v \in \mathcal{C}^n} \theta(u, v)^2 \\ &\leq \sum_{u, v \in U} p(u, v)^2 + \sum_{r=0}^{(\tilde{m}_2 - m_2)/2} \sum_{(k_1, k_2) \in D_r} \sum_{\substack{u, v \in W(S_{k_1, k_2}^n) \\ v \notin U}} p(u, v)^2 \\ &\quad + \sum_{r=(\tilde{m}_2 - m_2)/2 + 1}^{\tilde{m}_2 - m_2} \sum_{(k_1, k_2) \in D_r} \sum_{\substack{u, v \in W(S_{k_1, k_2}^n) \\ v \notin U}} p(u, v)^2. \end{aligned} \quad (2.16)$$

Since γ_x, γ_y, H_x and H_y are self-avoiding paths of length of order $\log n$, the first term on the right-hand side of (2.16) is bounded by $c \log n$ for some $c > 0$. Fix $1 \leq r \leq \frac{\tilde{m}_2 - m_2}{2}$, $(k_1, k_2) \in D_r$ and $u, v \in W(S_{k_1, k_2}^n)$ with $v \notin U$ and $\{u, v\} \in \mathcal{O}_{\mathcal{C}^n}$. By the construction of the random open path $\gamma(X, Y)$, we have for some $c > 0$,

$$p(u, v) \leq \frac{c}{r \cdot L_n}.$$

So we have for some $c, c' > 0$,

$$\sum_{r=0}^{(\tilde{m}_2 - m_2)/2} \sum_{(k_1, k_2) \in D_r} \sum_{\substack{u, v \in W(S_{k_1, k_2}^n) \\ v \notin U}} p(u, v)^2 \leq c \sum_{r=1}^{(\tilde{m}_2 - m_2)/2} \frac{1}{r} \leq c' \log n.$$

By a similar argument, the last term on the right-hand side of (2.16) is bounded by $c \log n$ for some $c > 0$. Therefore by (2.16), we obtain the conclusion. $\square$

Chapter 3

Maximum and minimum of local times for two-dimensional random walk

We obtain the leading orders of the maximum and the minimum of local times for the simple random walk on the two-dimensional torus at time proportional to the cover time. We also estimate the number of points with large (or small) values of the local times. These are analogues of estimates on the two-dimensional Gaussian free fields by Bolthausen, Deuschel, and Giacomin [Ann. Probab., **29** (2001)] and Daviaud [Ann. Probab., **34** (2006)], but we have different exponents from the case of the Gaussian free field.

3.1 Introduction

The theory of local times of random walks is very profound. It is well-known that local times of random walks have close relationships with the Gaussian free field (GFF). The connection goes back to [46]. Eisenbaum, Kaspi, Marcus, Rosen, and Shi [47] gave a powerful equivalence in law called “generalized second Ray-Knight theorem” (see Remark 3.1.1). Using the theorem, Ding, Lee, and Peres [40] established a useful connection between the expected maximum of the GFF and the cover time, and quite recently Zhai [95] strengthened the result by constructing a coupling of the occupation time field and the GFF (see Theorem 3.2.6).

Much efforts have been made to study local times of the simple random

walk on $\mathbb{Z}^2$. Erdős and Taylor [48] obtained an estimate on the maximum of local times of the simple random walk on $\mathbb{Z}^2$ by time n . Dembo, Peres, Rosen and Zeitouni [36] improved the result; they gave the leading order of the maximum and estimated the number of “favorite points” (see also [89]). Okada [82] obtained a corresponding estimate on frequently visited sites in the inner boundary of the random walk range. Sznitman [93] studied convergence of occupation time fields and related the fields to the GFF.

As mentioned above, works [48, 36, 89] are closely linked to the study of extremes of the two-dimensional GFF. Bolthausen, Deuschel, and Giacomin [22] obtained the leading order of its maximum (see Remark 3.1.4). Daviaud [35] estimated the number of points with large values of the GFF (see Remark 3.1.4).

In this paper, we study the maximum and the minimum of local times of the simple random walk on the two-dimensional torus at time proportional to the cover time. While similar work has been done in [22, 35] for the GFF, one cannot apply their results to deduce corresponding local time estimates, and indeed considerable amounts of efforts are needed to obtain such estimates. We also note that the exponents for the local times are different from those of the GFF (see Theorem 3.1.2 and Remark 3.1.4).

To state our results, we begin with some notation. We will write $\mathbb{Z}_N^2$ to denote the two-dimensional discrete torus with N^2 vertices. Let $X = (X_t)_{t \geq 0}$ be the continuous-time simple random walk on $\mathbb{Z}_N^2$ with exponential holding times of parameter 1. Let P_x be the law of X starting from $x \in \mathbb{Z}_N^2$. We define the local time of X by

$$L_t^N(x) := \int_0^t 1_{\{X_s=x\}} ds, \quad x \in \mathbb{Z}_N^2, \quad t \geq 0,$$

and the inverse local time by

$$\tau_t := \inf\{s \geq 0 : L_s^N(0) > t\}, \quad t \geq 0.$$

We will take the following time parameter

$$t_\theta = t_\theta(N) := \frac{4}{\pi} \theta (\log N)^2, \quad \theta > 0.$$

Note that τ_{t_θ} is approximated by $\theta \cdot \frac{4}{\pi} N^2 (\log N)^2$ and that $\frac{4}{\pi} N^2 (\log N)^2$ is close to the cover time of $\mathbb{Z}_N^2$ (see Lemma 3.2.4 and Theorem 1.1 in [37]).

We define sets of “thick points” and “thin points” by

$$\mathcal{L}_N^+(\eta, \theta) := \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \geq \eta \cdot 2\sqrt{2/\pi} \log N \right\}, \quad \eta, \theta > 0, \quad (3.1)$$

$$\mathcal{L}_N^-(\eta, \theta) := \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \leq -\eta \cdot 2\sqrt{2/\pi} \log N \right\}, \quad \eta, \theta > 0. \quad (3.2)$$

We will say that $(h_x^N)_{x \in \mathbb{Z}_N^2}$ is the GFF on $\mathbb{Z}_N^2$ if $(h_x^N)_{x \in \mathbb{Z}_N^2}$ is a centered Gaussian field with $h_0^N = 0$ and $\mathbb{E}[h_x^N h_y^N] = E_x[L_{T_0}(y)]$ for all $x, y \in \mathbb{Z}_N^2$, where $T_0 := \inf\{t \geq 0 : X_t = 0\}$.

Remark 3.1.1 *Let (h_x^N) be the GFF on $\mathbb{Z}_N^2$ with a measure $\mathbb{P}$. The generalized second Ray-Knight theorem [47] says that for all $t \geq 0$,*

$$\begin{aligned} & \text{the law of } \left\{ L_{\tau_t}^N(x) + \frac{1}{2}(h_x^N)^2 : x \in \mathbb{Z}_N^2 \right\} \text{ under } P_0 \times \mathbb{P} \\ & \text{is the same as that of } \left\{ \frac{1}{2}(h_x^N + \sqrt{2t})^2 : x \in \mathbb{Z}_N^2 \right\} \text{ under } \mathbb{P}. \end{aligned} \quad (3.3)$$

In particular, fixing N , we have

$$\left(\frac{L_{\tau_t}^N(x) - t}{\sqrt{2t}} \right)_{x \in \mathbb{Z}_N^2} \rightarrow (h_x^N)_{x \in \mathbb{Z}_N^2} \text{ in law as } t \rightarrow \infty. \quad (3.4)$$

By (3.4), one can expect that (3.1) and (3.2) will be close in law to corresponding level sets of the GFF (but not exactly, as we see in Theorem 3.1.2 and Remark 3.1.4). We note that one cannot deduce local time estimates corresponding to [22, 35] from (3.3) or (3.4).

We say that a sequence of events A_N holds with high probability if $\lim_{N \rightarrow \infty} P(A_N) = 1$. We write $|B|$ to denote the cardinality of $B \subset \mathbb{Z}_N^2$. We now state our results.

Theorem 3.1.2 *(i) For all $\theta > 0$, $\varepsilon > 0$, and $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$, the following holds with high probability (under P_0):*

$$N^{2-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2-\varepsilon} \leq |\mathcal{L}_N^+(\eta, \theta)| \leq N^{2-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2+\varepsilon}.$$

Furthermore, for all $\theta > 0$ and $\eta > 1 + \frac{1}{2\sqrt{\theta}}$,

$$|\mathcal{L}_N^+(\eta, \theta)| = 0, \quad \text{with high probability (under } P_0\text{)}.$$

(ii) For all $\theta > 1$, $\varepsilon > 0$, and $\eta \in (0, 1 - \frac{1}{2\sqrt{\theta}})$, the following holds with high probability (under P_0):

$$N^{2-2(\sqrt{\theta}-\sqrt{\theta-2\eta\sqrt{\theta}})^2-\varepsilon} \leq |\mathcal{L}_N^-(\eta, \theta)| \leq N^{2-2(\sqrt{\theta}-\sqrt{\theta-2\eta\sqrt{\theta}})^2+\varepsilon}.$$

Furthermore, for all $\theta > 1$ and $\eta > 1 - \frac{1}{2\sqrt{\theta}}$,

$$|\mathcal{L}_N^-(\eta, \theta)| = 0, \quad \text{with high probability (under } P_0\text{)}.$$

The next corollary follows immediately from Theorem 3.1.2.

Corollary 3.1.3 (i) For all $\theta > 0$ and $\varepsilon > 0$, the following holds with high probability (under P_0):

$$(1 + \frac{1}{2\sqrt{\theta}} - \varepsilon)2\sqrt{\frac{2}{\pi}} \log N \leq \frac{\max_{x \in \mathbb{Z}_N^2} L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \leq (1 + \frac{1}{2\sqrt{\theta}} + \varepsilon)2\sqrt{\frac{2}{\pi}} \log N.$$

(ii) For all $\theta > 1$ and $\varepsilon > 0$, the following holds with high probability (under P_0):

$$-(1 - \frac{1}{2\sqrt{\theta}} + \varepsilon)2\sqrt{\frac{2}{\pi}} \log N \leq \frac{\min_{x \in \mathbb{Z}_N^2} L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \leq -(1 - \frac{1}{2\sqrt{\theta}} - \varepsilon)2\sqrt{\frac{2}{\pi}} \log N.$$

Remark 3.1.4 Set $V_N := [1, N]^2 \cap \mathbb{Z}^2$. Let $(\tilde{h}_x^N)_{x \in V_N}$ be the GFF on V_N with zero boundary conditions. Bolthausen, Deuschel, and Giacomin [22] obtained the leading order of $\max_{x \in V_N} \tilde{h}_x^N$: for all $\varepsilon > 0$,

$$(1 - \varepsilon)2\sqrt{2/\pi} \log N \leq \max_{x \in V_N} \tilde{h}_x^N \leq (1 + \varepsilon)2\sqrt{2/\pi} \log N \quad \text{with high probability.}$$

Daviaud [35] showed that the following holds with high probability for all $\varepsilon > 0$ and $\eta \in (0, 1)$:

$$N^{2(1-\eta^2)-\varepsilon} \leq |\{x \in V_N : \tilde{h}_x^N \geq \eta \cdot 2\sqrt{2/\pi} \log N\}| \leq N^{2(1-\eta^2)+\varepsilon}. \quad (3.5)$$

Remark 3.1.5 By Lemma 3.2.4 below and Theorem 1.1 in [37], it is clear that for $\theta \in (0, 1)$, we have $\min_{x \in \mathbb{Z}_N^2} L_{\tau_{t_\theta}}^N(x) = 0$.

We can obtain the following estimate similar to (3.5) for the GFF on $\mathbb{Z}_N^2$ by using Theorem 3.1.2 and Theorem 3.2.5, 3.2.6 below. The proof is given in Section 3.2.

Corollary 3.1.6 *Let $(h_x^N)_{x \in \mathbb{Z}_N^2}$ be the GFF on $\mathbb{Z}_N^2$. For all $\varepsilon > 0$ and $\eta \in (0, 1)$, the following holds with high probability:*

$$N^{2(1-\eta^2)-\varepsilon} \leq |\{x \in \mathbb{Z}_N^2 : h_x^N \geq \eta \cdot 2\sqrt{2/\pi} \log N\}| \leq N^{2(1-\eta^2)+\varepsilon}.$$

In order to give an intuitive explanation of the exponent in Theorem 3.1.2(i), let us give additional notation. Let $d(\cdot, \cdot)$ be the ℓ^2 -distance in $\mathbb{Z}_N^2$. Set $D(x, r) := \{y \in \mathbb{Z}_N^2 : d(x, y) < r\}$. Fix a subset $A \subset \mathbb{Z}_N^2$. We define its boundary by $\partial A := \{y \in \mathbb{Z}_N^2 : y \in \mathbb{Z}_N^2 \setminus A, d(x, y) = 1 \text{ for some } x \in A\}$, and the hitting time of A by $T_A := \inf\{t \geq 0 : X_t \in A\}$. We will write T_x to denote $T_{\{x\}}$ for $x \in \mathbb{Z}_N^2$. Set $G_A(x, y) := E_x[L_{T_{\partial A}}^N(y)]$, $x, y \in \mathbb{Z}_N^2$. Fix $x \in \mathbb{Z}_N^2$ and $0 < r < R < \frac{N}{2}$. We define a sequence of stopping times as follows:

$$\tau_x^{(0)}[r, R] := \inf\{t \geq 0 : X_t \in \partial D(x, r)\},$$

$$\sigma_x^{(j)}[r, R] := \inf\{t \geq 0 : X_t \circ \theta_{\sum_{i=0}^{j-1} \tau_x^{(i)}[r, R]} \in \partial D(x, R)\}, \quad j \geq 1,$$

$$\tau_x^{(j)}[r, R] := \inf\{t > \sigma_x^{(j)}[r, R] : X_t \circ \theta_{\sum_{i=0}^{j-1} \tau_x^{(i)}[r, R]} \in \partial D(x, r)\}, \quad j \geq 1,$$

where $\theta_t, t \geq 0$ is the shift operator. Although the definition of $\tau_x^{(0)}[r, R]$ does not depend on R , we write R just for notational convenience when given the annulus $D(x, R) \setminus D(x, r)$. We define local times of excursions as follows:

$$L_x^{(j)}[r, R] := L_{T_{\partial D(x, R)}}^N(x) \circ \theta_{\sum_{i=0}^{j-1} \tau_x^{(i)}[r, R]}, \quad j \geq 1.$$

We now give heuristics about the exponent in Theorem 3.1.2(i). Let $K_n := n^b e^n n^{3n}$, where b is a positive constant. We will consider the simple random walk on $\mathbb{Z}_{K_n}^2$. Set $r_{n,k} := e^n n^{3(n-k)}$, $k = 0, \dots, n$. For $x \in \mathbb{Z}_{K_n}^2$ and $0 \leq \ell \leq n-1$, we write N_ℓ^x to denote the number of excursions from $\partial D(x, r_{n,\ell+1})$ to $\partial D(x, r_{n,\ell})$ up to time τ_{t_θ} . By concentration estimates (see Lemma 3.2.2 and 3.2.4),

$$(K_n)^2 t_\theta \approx \tau_{t_\theta} \approx \sum_{j=0}^{N_0^x} \tau_x^{(j)}[r_{n,1}, r_{n,0}] \approx \frac{2}{\pi} (K_n)^2 \log \left(\frac{r_{n,0}}{r_{n,1}} \right) N_0^x.$$

Thus, we have

$$N_0^x \approx 6\theta n^2 \log n. \quad (3.6)$$

By the law of large numbers, if $\frac{L_{\tau_t^\theta}^{K_n}(x) - t_\theta}{\sqrt{2t_\theta}} \approx \eta \cdot 2\sqrt{\frac{2}{\pi}} \log K_n$, then we have

$$\begin{aligned} (\theta + 2\eta\sqrt{\theta}) \frac{4}{\pi} (\log K_n)^2 &\approx \sum_{j=1}^{N_{n-1}^x} L_x^{(j)}[r_{n,n}, r_{n,n-1}] \\ &\approx N_{n-1}^x \cdot G_{D(x, r_{n,n-1})}(y, x) \approx N_{n-1}^x \cdot \frac{2}{\pi} \log \left(\frac{r_{n,n-1}}{r_{n,n}} \right), \end{aligned}$$

where y is a fixed point in $\partial D(x, r_{n,n})$, and we have used an estimate on Green's functions (see Lemma 3.2.1). Hence, we have

$$N_{n-1}^x \approx 6(\theta + 2\eta\sqrt{\theta})n^2 \log n. \quad (3.7)$$

To obtain the order of $|\mathcal{L}_{K_n}^+(\eta, \theta)|$, we should estimate the probability $P_0(N_{n-1}^x \approx 6(\theta + 2\eta\sqrt{\theta})n^2 \log n)$. Since for $1 \leq \ell \leq n-1$ and $y \in \partial D(x, r_{n,\ell})$, we have $P_y(T_{\partial D(x, r_{n,\ell+1})} < T_{\partial D(x, r_{n,\ell-1})}) \approx \frac{1}{2}$ (see Lemma 3.2.1), we can reduce the problem to the case of the simple random walk on $\{0, \dots, n\}$; we need to know the probability of the event that the walk traverses $6(\theta + 2\eta\sqrt{\theta})n^2 \log n$ times from n to $n-1$ until it crosses N_0^x times from 1 to 0. By this observation, (3.6) and a large deviation estimate, we have

$$P_0(N_{n-1}^x \approx 6(\theta + 2\eta\sqrt{\theta})n^2 \log n) \approx (K_n)^{-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2}.$$

Therefore, if $|\mathcal{L}_{K_n}^+(\eta, \theta)|$ is concentrated around its expectation, we have

$$|\mathcal{L}_{K_n}^+(\eta, \theta)| \approx (K_n)^{2-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2}.$$

The organization of the paper is the following. Section 3.2 gives preliminary lemmas and the proof of Corollary 3.1.6. In Section 3.3, we prove Theorem 3.1.2(i). The proof is based on the “refined second moment method” in [38, 89]. In Section 3.4, we prove Theorem 3.1.2(ii).

We will write $c_1, c_2, \dots$ to denote positive universal constants whose values are fixed within each argument. We use $c_1(\theta), c_2(\theta), \dots$ for positive constants which depend only on θ . Given a sequence $(\varepsilon_N)_{N \geq 0}$, we will write $\varepsilon_N = o(1_N)$ if $\lim_{N \rightarrow \infty} \varepsilon_N = 0$.

3.2 Preliminary lemmas and proof of Corollary 3.1.6

In this section, we collect useful lemmas and give the proof of Corollary 3.1.6. We will use the following basic estimates on the two-dimensional random walk. See, for example, Theorem 1.6.6, Proposition 1.6.7, and Exercise 1.6.8 in [67].

Lemma 3.2.1 (i) *There exist $c_1, c_2 > 0$ such that the following hold for all $0 < R < \frac{N}{2}$, $x \in \mathbb{Z}_N^2$, and $x_0 \in D(x, R)$:*

$$|G_{D(x,R)}(x, x) - \frac{2}{\pi} \log R| \leq c_1,$$

$$\left| G_{D(x,R)}(x_0, x) - \frac{2}{\pi} \log \left(\frac{R}{d(x_0, x)} \right) \right| \leq c_2 \left(\frac{1}{d(x_0, x)} + \frac{1}{R} \right).$$

(ii) *There exist $c_1, c_2 > 0$ such that for all $0 < r < R < \frac{N}{2}$, $x, x_0 \in \mathbb{Z}_N^2$ with $r < d(x_0, x) < R$,*

$$\frac{\log(R/d(x_0, x)) - c_1/r}{\log(R/r)} \leq P_{x_0}(T_{\partial D(x,r)} < T_{\partial D(x,R)}) \leq \frac{\log(R/d(x_0, x)) + c_2/r}{\log(R/r)}.$$

The following lemma relates time to the number of excursions.

Lemma 3.2.2 *There exist c_1, c_2, c_3 such that the following holds for all r, R with $0 < 2r < R < \frac{N}{2}$, $c_1(\frac{1}{r} + \frac{r}{R}) \leq \delta \leq c_2$, $x \in \mathbb{Z}_N^2$, and $M \in \mathbb{N}$:*

$$P_0 \left(\sum_{j=0}^M \tau_x^{(j)}[r, R] \geq (1 + \delta) \frac{2}{\pi} N^2 \log(R/r) M \right) \leq \exp \left(-c_3 \delta^2 \frac{\log(R/r)}{\log(N/r)} M \right).$$

Proof. The proof is almost the same as that of Lemma 3.2 in [38] since Lemma 3.1 of [38] holds even for the continuous-time simple random walk.

□

We will use the following moment estimate on local times.

Lemma 3.2.3 *Fix $x \in \mathbb{Z}_N^2$, $0 < R < \frac{N}{2}$, and $x_0 \in D(x, R)$. For all $\beta > 0$,*

$$E_{x_0} \left[\exp \left\{ -\frac{\beta}{G_{D(x,R)}(x, x)} L_{T_{\partial D(x,R)}}(x) \right\} \right] = 1 - \frac{G_{D(x,R)}(x_0, x)}{G_{D(x,R)}(x, x)} \cdot \frac{\beta}{1 + \beta}. \quad (3.8)$$

Proof. By Kac's moment formula (see, for example, (4) in [49]), we have for all $k \in \mathbb{N}$,

$$E_{x_0}[(L_{T_{\partial D(x,R)}}(x))^k] = k! \cdot G_{D(x,R)}(x_0, x) \cdot (G_{D(x,R)}(x, x))^{k-1}. \quad (3.9)$$

The equation (3.8) follows immediately from (3.9) for all $0 < \beta < 1$. Regarding both sides of (3.8) as analytic functions of β , we can show (3.8) even for all $\beta \geq 1$ by the uniqueness theorem of analytic functions. $\square$

The following is a special version of Lemma 2.1 in [39].

Lemma 3.2.4 *There exists $c_1 > 0$ such that for all $t > 0$ and $\lambda \geq 1$,*

$$P_0[|\tau_t - tN^2| \geq c_1(\sqrt{\lambda t \log N} + \lambda \log N)N^2] \leq 6e^{-\frac{\lambda}{16}}.$$

Proof. Note that the definition of the inverse local time in [39] is slightly different from ours; it corresponds to τ_{4t} in our notation. Since the effective resistances between vertices in $\mathbb{Z}_N^2$ are of order $\log N$ (see, for example, (10.20) in [70]), the statement follows from Lemma 2.1 of [39]. $\square$

The following theorem is about the number of “late points” of X .

Theorem 3.2.5 *For all $\varepsilon > 0$ and $\eta \in (0, 1)$, the following holds with high probability (under P_0):*

$$N^{2-2\eta-\varepsilon} \leq |\{x \in \mathbb{Z}_N^2 : T_x \geq \eta \cdot \frac{4}{\pi} N^2 (\log N)^2\}| \leq N^{2-2\eta+\varepsilon}.$$

Furthermore, for all $\eta > 1$,

$$|\{x \in \mathbb{Z}_N^2 : T_x \geq \eta \cdot \frac{4}{\pi} N^2 (\log N)^2\}| = 0 \quad \text{with high probability (under } P_0\text{)}.$$

Proof. Recall that the holding times of X are independent exponential variables with mean 1. Thus, it is clear that Theorem 3.2.5 follows immediately from Proposition 1.1 in [38], Theorem 1.1 in [37], and the law of large numbers for the variables. $\square$

The following theorem connects “thick points”, “thin points” and the GFF.

Theorem 3.2.6 (Theorem 3.1, [95]) *Let $(h_x^N)_{x \in \mathbb{Z}_N^2}$ be the GFF on $\mathbb{Z}_N^2$. For all $t > 0$,*

$$\{\sqrt{L_{\tau_t}^N(x)} : x \in \mathbb{Z}_N^2\} \preceq \frac{1}{\sqrt{2}} \{\max(h_x^N + \sqrt{2t}, 0) : x \in \mathbb{Z}_N^2\},$$

where $\preceq$ denotes the stochastic domination.

Finally, we give the proof of Corollary 3.1.6 by lemmas above and Theorem 3.1.2(i).

Proof of Corollary 3.1.6. Fix $\varepsilon > 0$ and $\eta \in (0, 1)$. We have for all $\lambda > 0$,

$$\begin{aligned} & \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : h_x^N \geq \eta \cdot 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \\ &= \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : h_x^N \leq -\eta \cdot 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \end{aligned} \quad (3.10)$$

$$\leq P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : L_{\tau_{t_{\eta^2}}}^N(x) = 0 \right\} \right| \geq \lambda \right) \quad (3.11)$$

$$= P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : T_x > \tau_{t_{\eta^2}} \right\} \right| \geq \lambda \right), \quad (3.12)$$

where we have used the symmetry of the GFF in (3.10) and Theorem 3.2.6 in (3.11). Take $\lambda = N^{2-2\eta^2+\varepsilon}$. By Lemma 3.2.4 and Theorem 3.2.5, the probability in (3.12) goes to 0 as $N \rightarrow \infty$. We have for all $\theta > 0$ and $\lambda > 0$,

$$\begin{aligned} & \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : h_x^N \geq \eta \cdot 2\sqrt{2/\pi} \log N \right\} \right| \leq \lambda \right) \\ &= \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : \frac{1}{\sqrt{2}}(h_x^N + \sqrt{2t_\theta}) \geq (\sqrt{\theta} + \eta) \cdot \frac{2}{\sqrt{\pi}} \log N \right\} \right| \leq \lambda \right) \\ &\leq P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \geq \left(\eta + \frac{\eta^2}{2\sqrt{\theta}} \right) \cdot 2\sqrt{2/\pi} \log N \right\} \right| \leq \lambda \right), \end{aligned} \quad (3.13)$$

where we have used Theorem 3.2.6 in (3.13). Take $\lambda = N^{2-2\eta^2-\varepsilon}$. By Theorem 3.1.2(i), the probability in (3.13) goes to 0 as $N \rightarrow \infty$. $\square$

3.3 Proof of Theorem 3.1.2(i)

Given Theorem 3.2.6, the upper bound of Theorem 3.1.2(i) is easy.

Proof of the upper bound of Theorem 3.1.2(i). Fix $\theta > 0$, $\varepsilon > 0$, and $\eta > 0$.

Let $(h_x^N)_{x \in \mathbb{Z}_N^2}$ be the GFF on $\mathbb{Z}_N^2$. We have for all $\lambda > 0$,

$$\begin{aligned} & P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \geq \eta \cdot 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \\ &\leq \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : h_x^N \geq (\sqrt{\theta} + 2\eta\sqrt{\theta} - \sqrt{\theta}) \cdot 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \end{aligned} \quad (3.14)$$

$$= \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : h_x^N \leq -(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta}) \cdot 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \quad (3.15)$$

$$\leq P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : L_{\tau_t_{(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2}}^N(x) = 0 \right\} \right| \geq \lambda \right) \quad (3.16)$$

$$= P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : T_x > \tau_{t_{(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2}} \right\} \right| \geq \lambda \right), \quad (3.17)$$

where we have used the symmetry of the GFF in (3.15) and Theorem 3.2.6 in (3.14) and (3.16). Take $\lambda = N^{2-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2+\varepsilon}$, $\varepsilon > 0$. By Lemma 3.2.4 and Theorem 3.2.5, the probability in (3.17) goes to 0 as $N \rightarrow \infty$. $\square$

From now on, we prove the lower bound of Theorem 3.1.2(i) by applying the methods in [38, 89]. First, we define the notion that a point is “successful”. Set

$$r_{n,k} := e^n n^{3(n-k)}, \quad 0 \leq k \leq n, \quad K_n := n^{\bar{\gamma}} r_{n,0}, \quad (3.18)$$

where $\bar{\gamma} \in [b, b+4]$ and b is a sufficiently large positive constant. Since K_n 's take values over all sufficiently large positive integers, we may only consider the subsequence. From now, we will consider the simple random walk on $\mathbb{Z}_{K_n}^2$. Given $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$, we set

$$n_\ell := \left\lceil 6 \left(1 - \frac{1}{n^{1/4}} \right) \left\{ \sqrt{\theta} + \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right) \frac{\ell}{n} \right\}^2 n^2 \log n \right\rceil, \quad \ell \leq n-1.$$

For $x \in \mathbb{Z}_{K_n}^2$ and $1 \leq \ell \leq n-1$,

$$N_\ell^x \text{ is the number of excursions from } \partial D(x, r_{n,\ell+1}) \text{ to } \partial D(x, r_{n,\ell}) \\ \text{up to random time } \sum_{j=0}^{n_0} \tau_x^{(j)}[r_{n,1}, r_{n,0}].$$

Definition 3.3.1 Fix $x \in \mathbb{Z}_{K_n}^2$. We will say that x is successful if

$$|N_\ell^x - n_\ell| \leq n, \quad \text{for } 1 \leq \ell \leq n-1.$$

Remark 3.3.2 We give an intuition about Definition 3.3.1. Assume that $\frac{L_{\tau_{t_\theta}}^{K_n}(x) - t_\theta}{\sqrt{2t_\theta}} \approx \eta \cdot 2\sqrt{2/\pi} \log K_n$. We already know that under this assumption, $N_0^x \approx 6\theta n^2 \log n$ and $N_{n-1}^x \approx 6(\theta + 2\eta\sqrt{\theta})n^2 \log n$ (recall (3.6) and

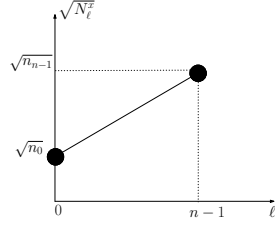


Figure 3.1: If x is successful, $(\sqrt{N_\ell^x})_{0 \leq \ell \leq n-1}$ behaves like a linear function.

(3.7)). Due to a recent work by Belius and Kistler [13], one expects that conditioned on $\sqrt{N_0^x} \approx \sqrt{n_0}$ and $\sqrt{N_{n-1}^x} \approx \sqrt{n_{n-1}}$, $(\sqrt{N_\ell^x})_{0 \leq \ell \leq n-1}$ behaves roughly like a Brownian bridge from $\sqrt{n_0}$ to $\sqrt{n_{n-1}}$. Therefore, we see that $(\sqrt{N_\ell^x})_{0 \leq \ell \leq n-1}$ would typically look like a linear function in ℓ with $\sqrt{N_0^x} \approx \sqrt{n_0}$ and $\sqrt{N_{n-1}^x} \approx \sqrt{n_{n-1}}$ (see Figure 3.1). We used this insight in Definition 3.3.1. Note that our framework is quite different from those in [38, 89] and so is the definition of “successful”.

The lower bound of Theorem 3.1.2(i) follows from the following three propositions.

Proposition 3.3.3 *For all $\theta > 0$ and $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$, the following holds with high probability (under P_0):*

$$\{x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0}) : x \text{ is successful}\} \subset \mathcal{L}_{K_n}^+((1 - 1/\log \log n)(1 - 1/n)^2 \eta, \theta).$$

Proposition 3.3.4 *For all $\theta > 0$, $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$, and $x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0})$,*

$$P_0(x \text{ is successful}) = (1 + o(1_n))q_n,$$

where q_n satisfies the following: there exists $c_1(\theta), c_2(\theta) > 0$ such that

$$e^{-c_1(\theta)n \log \log n} (K_n)^{-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2} \leq q_n \leq e^{-c_2(\theta)n \log \log n} (K_n)^{-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2}.$$

Proposition 3.3.5 *Let q_n be given in Proposition 3.3.4. Fix $\theta > 0$ and $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$. For $x, y \in \mathbb{Z}_{K_n}^2$, set*

$$\ell(x, y) := \min\{\ell : D(x, r_{n,\ell} + 1) \cap D(y, r_{n,\ell} + 1) = \emptyset\} \wedge n.$$

(i) *There exist $c_1(\theta), c_2(\theta) > 0$ such that for all $x, y \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0})$ with $1 \leq \ell(x, y) \leq n - 2$,*

$$\begin{aligned} & P_0(x \text{ and } y \text{ are successful}) \\ & \leq n^{c_1(\theta)} e^{c_2(\theta)\ell \log \log n} (q_n)^2 \cdot \exp \left\{ 6 \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right)^2 \ell \log n \right\}. \end{aligned}$$

(ii) *For all $x, y \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0})$ with $\ell(x, y) = 0$,*

$$P_0(x \text{ and } y \text{ are successful}) = (1 + o(1_n))(q_n)^2.$$

Proof of the lower bound of Theorem 3.1.2(i) via Proposition 3.3.3-3.3.5. Fix $\theta > 0$ and $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$. Set

$$Z_n := \sum_{x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0})} 1_{\{x \text{ is successful}\}}, \quad W_{n,\ell} := \sum_{\substack{x, y \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0}) \\ \ell(x, y) = \ell}} 1_{\{x \text{ and } y \text{ are successful}\}}.$$

We have $E_0[(Z_n)^2] \leq \sum_{\ell=0}^n E_0[W_{n,\ell}]$. Recall (3.18). Taking b large enough, by Proposition 3.3.4 and 3.3.5, we have

$$E_0[W_{n,0}] \leq (1 + o(1_n))(K_n)^4 (q_n)^2, \quad \sum_{\ell=1}^n E_0[W_{n,\ell}] \leq o(1_n)(K_n)^4 (q_n)^2.$$

Thus, we have

$$E_0[(Z_n)^2] \leq (1 + o(1_n))(K_n)^4 (q_n)^2. \quad (3.19)$$

By (3.19), Proposition 3.3.4, and the Paley-Zygmund inequality, the following holds with high probability:

$$Z_n \geq e^{-n(\log \log n)^2} (K_n)^{2-2(\sqrt{\theta+2\eta\sqrt{\theta}}-\sqrt{\theta})^2}. \quad (3.20)$$

The lower bound of Theorem 3.1.2(i) follows from (3.20) and Proposition 3.3.3. $\square$

For the rest of this section, we will prove Proposition 3.3.3-3.3.5.

Proof of Proposition 3.3.3. We will prove the following:

$$P_0 \left[\frac{L_{\tau_{t_\theta}}^{K_n}(x) - t_\theta}{\sqrt{2t_\theta}} < (1 - 1/\log \log n)(1 - 1/n)^2 \eta \cdot 2\sqrt{2/\pi} \log K_n, \right. \\ \left. \text{and } x \text{ is successful for some } x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0}) \right] \\ \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.21)$$

The statement in Proposition 3.3.3 follows immediately from this. Note that if x is successful and $\tau_{t_\theta} \geq \sum_{j=0}^{n_0} \tau_x^{(j)}[r_{n,1}, r_{n,0}]$, we have $L_{\tau_{t_\theta}}^{K_n}(x) \geq \sum_{j=1}^{n_{n-1}-n} L_x^{(j)}[r_{n,n}, r_{n,n-1}]$. It is clear that if $\tau_{t_\theta} < \sum_{j=0}^{n_0} \tau_x^{(j)}[r_{n,1}, r_{n,0}]$, we have $\sum_{j=0}^{n_0} \tau_x^{(j)}[r_{n,1}, r_{n,0}] > \lambda$ or $\tau_{t_\theta} \leq \lambda$ for any $\lambda > 0$. By these observations, the probability in (3.21) is bounded above by $I_1 + I_2 + I_3$, where

$$I_1 := P_0 \left[\text{For some } x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0}), \sum_{j=1}^{n_{n-1}-n} L_x^{(j)}[r_{n,n}, r_{n,n-1}] \right. \\ \left. < (\theta + 2(1 - 1/\log \log n)(1 - 1/n)^2 \eta \sqrt{\theta}) \frac{4}{\pi} (\log K_n)^2 \right],$$

$$I_2 := P_0 \left[\sum_{j=0}^{n_0} \tau_x^{(j)}[r_{n,1}, r_{n,0}] > \lambda_n \text{ for some } x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0}) \right],$$

$$I_3 := P_0(\tau_{t_\theta} \leq \lambda_n),$$

where $\lambda_n := (1 + 1/n^{1/4})2/\pi(K_n)^2 \log(r_{n,0}/r_{n,1})n_0 \leq (1 - 1/\sqrt{n})(K_n)^2 t_\theta$. By Lemma 3.2.2 and 3.2.4, we have

$$I_2 \leq (K_n)^2 \exp \left(-\frac{c_1}{\sqrt{n}} \frac{\log(r_{n,0}/r_{n,1})}{\log(K_n/r_{n,1})} n_0 \right) \leq (K_n)^2 \exp(-c_2(\theta)n^{\frac{3}{2}} \log n) = o(1_n),$$

$$I_3 \leq P_0(\tau_{t_\theta} \leq (1 - 1/\sqrt{n})(K_n)^2 t_\theta) \leq 6 \exp(-c_3(\theta) \log K_n/n) = o(1_n).$$

From now, we will prove $I_1 = o(1_n)$. Fix $x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0})$. Set P_x to be

$$P_0 \left[\sum_{j=1}^{n_{n-1}-n} L_x^{(j)}[r_{n,n}, r_{n,n-1}] < \left(\theta + 2\left(1 - \frac{1}{\log \log n}\right)\left(1 - \frac{1}{n}\right)^2 \eta \sqrt{\theta} \right) \frac{4}{\pi} (\log K_n)^2 \right].$$

We have for $\varphi > 0$,

$$P_x \leq \exp \left\{ \frac{\varphi}{G_{D(x, r_{n, n-1})}(x, x)} (\theta + 2(1 - \frac{1}{\log \log n})(1 - \frac{1}{n})^2 \eta \sqrt{\theta}) \frac{4}{\pi} (\log K_n)^2 \right\} \\ \cdot E_0 \left[\exp \left\{ - \frac{\varphi}{G_{D(x, r_{n, n-1})}(x, x)} \sum_{j=1}^{n_{n-1}-n} L_x^{(j)}[r_{n, n}, r_{n, n-1}] \right\} \right] \quad (3.22)$$

$$\leq \exp \left\{ \frac{\varphi}{G_{D(x, r_{n, n-1})}(x, x)} (\theta + 2(1 - \frac{1}{\log \log n})(1 - \frac{1}{n})^2 \eta \sqrt{\theta}) \frac{4}{\pi} (\log K_n)^2 \right\} \\ \cdot \left\{ \max_{y \in \partial D(x, r_{n, n})} E_y \left[\exp \left\{ - \frac{\varphi}{G_{D(x, r_{n, n-1})}(x, x)} L_{T_{\partial D(x, r_{n, n-1})}}(x) \right\} \right] \right\}^{n_{n-1}-n} \quad (3.23)$$

$$\leq e^{c_4(\theta)(\varphi n \log n + \frac{\varphi}{1+\varphi}(\log n)^3)} \exp\{-18n(\log n)^2 f_n(\varphi)\}, \quad (3.24)$$

where

$$f_n(\varphi) := -\varphi(\theta + 2(1 - 1/\log \log n)(1 - 1/n)^2 \eta \sqrt{\theta}) \\ + \frac{\varphi}{1 + \varphi} (1 - 1/n^{1/4})(1 - 1/n)^2 (\theta + 2\eta \sqrt{\theta}).$$

We have used the exponential Chebyshev inequality in (3.22), the strong Markov property at times $\sum_{i=0}^{j-1} \tau_x^{(i)}[r_{n, n}, r_{n, n-1}]$, $1 \leq j \leq n_{n-1} - n$ in (3.23), and Lemma 3.2.1(i), 3.2.3 in (3.24). Taking φ at which $f_n(\varphi)$ attains the maximum, we have

$$P_x \leq e^{c_5(\theta) \frac{n \log n}{\log \log n}} \exp\{-c_6(\theta)n(\log n / \log \log n)^2\} = o((K_n)^{-2}).$$

Therefore, we have proved $I_1 = o(1_n)$ and (3.21). $\square$

Proof of Proposition 3.3.4. Fix $x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n, 0})$. By Lemma 3.2.1(ii) and the strong Markov property, we have

$$P_0(x \text{ is successful}) = \sum P(N_\ell^x = m_\ell \text{ for } 1 \leq \ell \leq n-1) = (1 + o(1_n))q_n, \quad (3.25)$$

where

$$q_n := \sum \prod_{\ell=1}^{n-1} \binom{m_\ell + m_{\ell-1} - 1}{m_\ell} \left(\frac{1}{2}\right)^{m_\ell + m_{\ell-1}}. \quad (3.26)$$

Here the summations in (3.25) and (3.26) are over all $m_1, \dots, m_{n-1}$ with $|m_i - n_i| \leq n$ for $1 \leq i \leq n-1$. By the Stirling formula, we have for all m_i with $|m_i - n_i| \leq n, 1 \leq i \leq n-1$,

$$\begin{aligned} & \binom{m_\ell + m_{\ell-1} - 1}{m_\ell} \left(\frac{1}{2}\right)^{m_\ell + m_{\ell-1}} \geq \frac{c_1}{\sqrt{m_\ell}} \cdot \frac{(m_\ell + m_{\ell-1})^{m_\ell + m_{\ell-1}}}{(m_\ell)^{m_\ell} (m_{\ell-1})^{m_{\ell-1}}} \left(\frac{1}{2}\right)^{m_\ell + m_{\ell-1}} \\ & = c_1 \cdot (m_\ell)^{-1/2} \cdot \exp \left\{ m_{\ell-1} f \left(\frac{m_\ell}{m_{\ell-1}} \right) \right\} \\ & \geq c_1 \cdot (m_\ell)^{-1/2} \cdot \exp \left\{ -m_{\ell-1} \left(\frac{1}{4} \left(\frac{m_\ell}{m_{\ell-1}} - 1 \right)^2 + c_2(\theta) \left| \frac{m_\ell}{m_{\ell-1}} - 1 \right|^3 \right) \right\} \end{aligned} \quad (3.27)$$

$$\geq c_3(\theta) n^{-1} (\log n)^{-1/2} \exp \left\{ -6 \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right)^2 \log n \right\}, \quad (3.28)$$

where $f(u) := (1+u) \log(1+u) - u \log u - (1+u) \log 2, u > 0$ and we have used the Taylor expansion of f around 1 in (3.27). Therefore, we have by (3.26) and (3.28)

$$\begin{aligned} q_n & \geq n^{n-1} (c_3(\theta) n^{-1} (\log n)^{-1/2})^{n-1} \exp \left\{ -6 \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right)^2 n \log n \right\} \\ & \geq (c_4(\theta) (\log n)^{-1/2})^{n-1} (K_n)^{-2(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta})^2}. \end{aligned}$$

By a similar argument, we can obtain the upper bound of q_n . $\square$

In order to prove Proposition 3.3.5, we make some preparations. Fix $x \in \mathbb{Z}_N^2$ and $0 \leq \ell \leq n-1$. Set $e^{(0)} := \{X_t : 0 \leq t \leq T_{\partial D(x, r_{n, \ell+1})}\}$, and

$$e^{(i)} := \left\{ X_{t + \sum_{j=0}^{i-1} \tau_x^{(j)}[r_{n, \ell+1}, r_{n, \ell}]} : \sigma_x^{(i)}[r_{n, \ell+1}, r_{n, \ell}] \leq t \leq \tau_x^{(i)}[r_{n, \ell+1}, r_{n, \ell}] \right\}, \quad i \geq 1.$$

Let $\mathcal{G}_\ell^x := \sigma(e^{(i)} : i \geq 0)$. We will use the following lemma iteratively.

Lemma 3.3.6 *Fix $\eta \in (0, 1 + \frac{1}{2\sqrt{\theta}})$ and $\theta > 0$. There exists ε_n with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that the following holds for all $0 \leq \ell \leq n-2$, m_ℓ with $|m_\ell - n_\ell| \leq n, m_{\ell+1}, \dots, m_{n-1} > 0$, and $x \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0})$:*

$$\begin{aligned} & P_0(N_i^x = m_i \text{ for all } i = \ell, \dots, n-1 | \mathcal{G}_\ell^x) \\ & = (1 + \varepsilon_n) P_0(N_i^x = m_i \text{ for all } i = \ell+1, \dots, n-1 | N_\ell^x = m_\ell) \cdot 1_{\{N_\ell^x = m_\ell\}}. \end{aligned}$$

Proof. The proof is almost the same as that of Corollary 5.1 in [38] since Lemma 2.4 in [38] holds even for the continuous-time simple random walk. $\square$

Proof of Proposition 3.3.5. Proposition 3.3.5(ii) follows immediately from Lemma 3.3.6. We will prove Proposition 3.3.5(i). Fix $x, y \in \mathbb{Z}_{K_n}^2 \setminus D(0, r_{n,0})$ with $1 \leq \ell(x, y) \leq n - 2$. We will write

$$\ell := \ell(x, y).$$

By Lemma 3.3.6,

$$\begin{aligned} & P_0(x \text{ and } y \text{ are successful}) \\ & \leq P_0(|N_i^x - n_i| \leq n \text{ for } i = 1, \dots, \ell - 3, \ell, \dots, n - 1, \\ & \quad \text{and } |N_i^y - n_i| \leq n \text{ for } i = \ell, \dots, n - 1) \\ & \leq (1 + o(1_n)) P_0(|N_i^x - n_i| \leq n \text{ for } i = 1, \dots, \ell - 3, \ell, \dots, n - 1) \\ & \cdot \sum_{m_\ell: |m_\ell - n_\ell| \leq n} P_0(|N_i^y - n_i| \leq n \text{ for } i = \ell + 1, \dots, n - 1 | N_\ell^y = m_\ell). \end{aligned} \quad (3.29)$$

Note that since $D(y, r_{n,\ell-1}) \subset D(x, r_{n,\ell-2})$ and $D(x, r_{n,\ell-1}) \cap D(y, r_{n,\ell-1}) \neq \emptyset$, excursions between $\partial D(x, r_{n,i+1})$ and $\partial D(x, r_{n,i})$, $i = \ell - 2, \ell - 1$ may enter in $D(y, r_{n,\ell+1})$. In particular, events $\{|N_i^x - n_i| \leq n\}$, $i = \ell - 2, \ell - 1$ are not in $\mathcal{G}_\ell^y$ and thus are excluded from (3.29). We will prove the following:

$$\begin{aligned} & \sum_{m_\ell: |m_\ell - n_\ell| \leq n} P_0(|N_i^y - n_i| \leq n \text{ for } i = \ell + 1, \dots, n - 1 | N_\ell^y = m_\ell) \\ & \leq c_1(\theta) n e^{c_2(\theta)\ell \log \log n} q_n \exp \left\{ 6 \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right)^2 \ell \log n \right\}, \end{aligned} \quad (3.30)$$

$$\begin{aligned} & P_0(|N_i^x - n_i| \leq n \text{ for } i = 1, \dots, \ell - 3, \ell, \dots, n - 1) \\ & \leq c_3(\theta) n^{c_4(\theta)} e^{c_5(\theta)\ell \log \log n} q_n. \end{aligned} \quad (3.31)$$

Proposition 3.3.5(i) follows from (3.29), (3.30) and (3.31). First, we prove (3.30). By Proposition 3.3.4 and Lemma 3.3.6,

$$(1 + o(1_n)) q_n = P_0(y \text{ is successful})$$

$$\begin{aligned}
&\geq (1 + o(1_n)) \sum_{m_\ell: |m_\ell - n_\ell| \leq n} P_0(|N_i^y - n_i| \leq n \text{ for } 1 \leq i \leq \ell - 1, \quad N_\ell^y = m_\ell) \\
&\quad \cdot P_0(|N_i^y - n_i| \leq n \text{ for } \ell + 1 \leq i \leq n - 1 | N_\ell^y = m_\ell). \tag{3.32}
\end{aligned}$$

By a similar argument to the proof of Proposition 3.3.4, we have for all m_ℓ with $|m_\ell - n_\ell| \leq n$,

$$\begin{aligned}
&P_0(|N_i^y - n_i| \leq n \text{ for } 1 \leq i \leq \ell - 1, \quad N_\ell^y = m_\ell) \\
&\geq n^{-1} e^{-c_6(\theta)\ell \log \log n} \exp \left\{ -6 \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right)^2 \ell \log n \right\}. \tag{3.33}
\end{aligned}$$

(3.30) follows from (3.32) and (3.33).

Next, we prove (3.31). By Lemma 3.3.6, (3.30), and a similar argument to the proof of Proposition 3.3.4, we have

$$\begin{aligned}
&P_0(|N_i^x - n_i| \leq n \text{ for } i = 1, \dots, \ell - 3, \ell, \dots, n - 1) \\
&\leq (1 + o(1_n)) P_0(|N_i^x - n_i| \leq n \text{ for } 1 \leq i \leq \ell - 3) \\
&\quad \cdot \sum_{m_\ell: |m_\ell - n_\ell| \leq n} P_0(|N_i^x - n_i| \leq n \text{ for } \ell + 1 \leq i \leq n - 1 | N_\ell^x = m_\ell) \\
&\leq e^{-c_7(\theta)(\ell-3) \log \log n} \exp \left\{ -6 \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right)^2 (\ell - 3) \log n \right\} \\
&\quad \cdot c_1(\theta) q_n n e^{c_2(\theta)\ell \log \log n} \exp \left\{ 6 \left(\sqrt{\theta + 2\eta\sqrt{\theta}} - \sqrt{\theta} \right)^2 \ell \log n \right\} \\
&\leq q_n n^{c_8(\theta)} e^{c_9(\theta)\ell \log \log n}.
\end{aligned}$$

Therefore, we have proved (3.31). $\square$

3.4 Proof of Theorem 3.1.2(ii)

In this section, we prove Theorem 3.1.2(ii). First, we show the lower bound.

Proof of the lower bound of Theorem 3.1.2(ii). Fix $\theta > 1$, $\varepsilon > 0$, and $\eta \in (0, 1 - \frac{1}{2\sqrt{\theta}})$. We have for all $\lambda > 0$,

$$P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \leq -\eta \cdot 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right)$$

$$\geq \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : h_x^N \leq - \left(\sqrt{\theta} - \sqrt{\theta - 2\eta\sqrt{\theta}} \right) 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \quad (3.34)$$

$$= \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : h_x^N \geq \left(\sqrt{\theta} - \sqrt{\theta - 2\eta\sqrt{\theta}} \right) 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \quad (3.35)$$

$$\begin{aligned} &= \mathbb{P} \left(\left| \left\{ x \in \mathbb{Z}_N^2 : \frac{1}{\sqrt{2}} \left(h_x^N + \sqrt{2t_{\theta-2\eta\sqrt{\theta}}} \right) \geq \sqrt{\theta} \cdot 2/\sqrt{\pi} \log N \right\} \right| \geq \lambda \right) \\ &\geq P_0 \left(\left| \mathcal{L}_N^+ \left(\eta\sqrt{\theta}/\sqrt{\theta - 2\eta\sqrt{\theta}}, \theta - 2\eta\sqrt{\theta} \right) \right| \geq \lambda \right), \end{aligned} \quad (3.36)$$

where we have used Theorem 3.2.6 in (3.34) and (3.36), and the symmetry of the GFF in (3.35). Take $\lambda = N^{2-2(\sqrt{\theta}-\sqrt{\theta-2\eta\sqrt{\theta}})^2-\varepsilon}$. By the lower bound of Theorem 3.1.2(i), the probability in (3.36) goes to 1 as $N \rightarrow \infty$. $\square$

Next, we prove the upper bound of Theorem 3.1.2(ii).

Proof of the upper bound of Theorem 3.1.2(ii). Fix $\theta > 1$, $\varepsilon > 0$, and $\eta > 0$. Set

$$\alpha_1 := 1 - \frac{1}{\sqrt{\log N}}, \quad \alpha_2 := 1 - \frac{2}{\sqrt{\log N}} + \frac{K}{(\log N)^{3/4}}, \quad m_N := \lceil 2\theta(\log N)^{3/2} \rceil,$$

$$\lambda_N := (1 + K/2(\log N)^{1/4}) \frac{2}{\pi} N^2 \log(N^{\alpha_1}/N^{\alpha_2}) m_N,$$

where K is a sufficiently large positive constant. We have for all $\lambda > 0$,

$$\begin{aligned} &P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : \frac{L_{\tau_{t_\theta}}^N(x) - t_\theta}{\sqrt{2t_\theta}} \leq -\eta \cdot 2\sqrt{2/\pi} \log N \right\} \right| \geq \lambda \right) \\ &\leq P_0 \left(\left| \left\{ x \in \mathbb{Z}_N^2 : \sum_{j=1}^{m_N} L_x^{(j)}[N^{\alpha_2}, N^{\alpha_1}] \leq (\theta - 2\eta\sqrt{\theta}) 4/\pi (\log N)^2 \right\} \right| \geq \lambda \right) \end{aligned} \quad (3.37)$$

$$+ P_0 \left(\sum_{j=0}^{m_N} \tau_x^{(j)}[N^{\alpha_2}, N^{\alpha_1}] \geq \lambda_N \text{ for some } x \in \mathbb{Z}_N^2 \right) \quad (3.38)$$

$$+ P_0(\tau_{t_\theta} < \lambda_N). \quad (3.39)$$

By Lemma 3.2.2 and 3.2.4, (3.38) and (3.39) go to 0 as $N \rightarrow \infty$. In analogy to the proof of Proposition 3.3.3, by Lemma 3.2.1 and 3.2.3, we have for all $x \in \mathbb{Z}_N^2$,

$$P_0 \left(\sum_{j=1}^{m_N} L_x^{(j)}[N^{\alpha_2}, N^{\alpha_1}] \leq (\theta - 2\eta\sqrt{\theta}) \frac{4}{\pi} (\log N)^2 \right) \leq N^{-2(\sqrt{\theta} - \sqrt{\theta - 2\eta\sqrt{\theta}})^2 + o(1_N)}.$$

Therefore, by taking $\lambda = N^{2-2(\sqrt{\theta} - \sqrt{\theta - 2\eta\sqrt{\theta}})^2 + \varepsilon}$, we can show that (3.37) goes to 0 as $N \rightarrow \infty$. $\square$

Chapter 4

Extremes of local times for simple random walks on symmetric trees

We consider local times of the simple random walk on the b -ary tree of depth n at time much larger than the maximal hitting time and study a point process which encodes the location of the vertex with the maximal local time and the properly centered maximum over leaves of each subtree of depth r_n rooted at the $(n - r_n)$ level, where $(r_n)_{n \geq 1}$ satisfies $\lim_{n \rightarrow \infty} r_n = \infty$ and $\lim_{n \rightarrow \infty} r_n/n \in [0, 1)$. We show that the point process weakly converges to a Cox process with intensity measure $\alpha Z_\infty(dx) \otimes e^{-2\sqrt{\log b} y} dy$, where $\alpha > 0$ is a constant and Z_∞ is a random Borel measure on $\mathbb{R}$ characterized by the so-called derivative martingale of the associated branching random walk. As a corollary, we establish the convergence in law of the maximum of the local times over leaves to a randomly shifted Gumble distribution.

4.1 Introduction

Much efforts have been made in the study of the so-called log-correlated random field such as the branching Brownian motion (BBM), the branching random walk (BRW), and the two-dimensional discrete Gaussian free field (DGFF). One of the remarkable features of these models is that laws of their maxima share the common property: each of the laws weakly converges to a randomly shifted Gumble distribution [66, 2, 26]. It is believed that

each of the limiting extremal processes of a wide class of log-correlated fields converges to a so-called randomly shifted decorated Poisson point process [91] and it is established for the BBM [3, 8] and the BRW [75], and partially for the two-dimensional DGFF [19, 20].

It is well-known that local times of random walks on graphs have close relationships with DGFFs thanks to “the generalized second Ray-Knight theorem” [47] which has many applications, for example, to cover times [40, 39, 95]. Since occupation time fields of simple random walks on trees or on the two-dimensional lattice are closely related to BRWs or the two-dimensional DGFF respectively, it is natural to expect that their maxima and cover times belong to the universal class mentioned above: it is known that cover times have the subleading terms similarly to other log-correlated fields [41, 14], but further details are still open.

In this paper, we consider local times of the simple random walk on the b -ary tree of depth n at time much larger than the maximal hitting time and study convergence of a point process encoding extreme local maxima of the local times as $n \rightarrow \infty$.

To state our result, we begin with some notation. We fix an arbitrary integer $b \geq 2$ throughout the paper. We will write T to denote the b -ary tree with root ρ : this is a rooted tree whose vertices have exactly b children. Let T_i be the i th generation of T . Set $T_{\leq n} := \cup_{i=0}^n T_i$. For $v \in T$, we will write $|v|$ to denote the depth of v . For $u \in T$, let T^u be the subtree of T rooted at u , and we define T_i^u and $T_{\leq n}^u$ similarly. For $v, u \in T$, let $v \wedge u$ be the most recent common ancestor of v and u . Let $X = (X_t, t \geq 0, P_v, v \in T_{\leq n})$ be the continuous-time simple random walk on $T_{\leq n}$ with exponential holding times of parameter 1. We define the local time of X by

$$L_t^n(v) := \frac{1}{\deg(v)} \int_0^t 1_{\{X_s=v\}} ds, \quad v \in T_{\leq n}, \quad t \geq 0,$$

where $\deg(v)$ is the degree of v , and the inverse local time by

$$\tau(t) := \inf\{s \geq 0 : L_s^n(\rho) > t\}, \quad t \geq 0.$$

Let $E(T)$ be the set of edges on T . Let $(Y_e)_{e \in E(T)}$ be independent and identically distributed random variables whose common law is the normal distribution with mean 0 and variance 1. To each $v \in T$, we assign $h_v := \sum_{i=1}^{|v|} Y_{e_i^v}$, where $e_1^v, \dots, e_{|v|}^v$ are edges on the unique path from ρ to v . We

will call $(h_v)_{v \in T}$ a BRW on T . It is well-known that the so-called derivative martingale

$$D_n := \sum_{v \in T_n} \left(\sqrt{\log b} \, n - \frac{1}{\sqrt{2}} h_v \right) e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, n - \frac{1}{\sqrt{2}} h_v \right)}$$

converges almost surely as $n \rightarrow \infty$, and the limit

$$D_\infty := \lim_{n \rightarrow \infty} D_n \quad (4.1)$$

is positive and finite almost surely (see, for example, [17, Theorem 5.1, 5.2] or [2, Proposition A.3]).

Fix $t > 0$ and $n \in \mathbb{N}$. To each $v \in T$, we assign a distinct label $(\bar{v}_1, \dots, \bar{v}_{|v|})$ with $\bar{v}_i \in \{0, \dots, b-1\}$ for all $1 \leq i \leq |v|$ so that the vertices with labels $(\bar{v}_1, \dots, \bar{v}_{|v|}, k)$, $k \in \{0, \dots, b-1\}$ are children of v . We define the location of $v \in T$ by

$$\sigma(v) := \sum_{i=1}^{|v|} \frac{\bar{v}_i}{b^i}. \quad (4.2)$$

For each $(x, y) \in \mathbb{R}^2$, we will write $\delta_{(x,y)}$ to denote the Dirac measure at (x, y) . For each $0 \leq m \leq n$, we define the point process by

$$\Xi_{n,t}^{(m)} := \sum_{u \in T_m} \delta_{\left(\sigma \left(\operatorname{argmax}_u L_{\tau(t)}^n \right), \max_{v \in T_{n-m}^u} \sqrt{L_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \right)}, \quad (4.3)$$

where the centering sequence $a_n(t)$ is given by

$$a_n(t) := \sqrt{\log b} \, n - \frac{3}{4\sqrt{\log b}} \log n - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t} + n}{\sqrt{t}} \right), \quad (4.4)$$

and for each $u \in T_m$, $\operatorname{argmax}_u L_{\tau(t)}^n$ is the vertex v_* on $T_{n-m}^u \subset T_n$ with $L_{\tau(t)}^n(v_*) = \max_{v \in T_{n-m}^u} L_{\tau(t)}^n(v)$. If two or more vertices on T_{n-m}^u attain the maximum, we take the vertex $\operatorname{argmax}_u L_{\tau(t)}^n$ so that $\sigma \left(\operatorname{argmax}_u L_{\tau(t)}^n \right)$ is the largest among such vertices. We regard $\Xi_{n,t}^{(m)}$ as an element of the space of all Radon measures on Borel sets of $\mathbb{R}^2$ topologized with the vague topology. Since this space is metrizable as a complete separable metric space, we can consider weak convergence of sequences of random measures. Given a random measure ν on $\mathbb{R}^2$, we will write $\text{PPP}(\nu)$ to denote a point process on $\mathbb{R}^2$ which, conditioned on ν , is a Poisson point process with intensity measure ν (that is $\text{PPP}(\nu)$ is a Cox process). We now state the main result of this paper:

Theorem 4.1.1 *There exist $c_1 > 0$ and a non-atomic random Borel measure Z_∞ on $\mathbb{R}$ with $Z_\infty(\mathbb{R}) = Z_\infty([0, 1]) = D_\infty$ in law such that the following holds: for any sequence $(t_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = \theta \in [0, \infty]$ and $t_n \geq c_1 n \log n$ for each $n \in \mathbb{N}$, and any sequence $(r_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} r_n = \infty$ and $\lim_{n \rightarrow \infty} r_n/n \in [0, 1)$, the point process $\Xi_{n, t_n}^{(n-r_n)}$ weakly converges to a Cox process*

$$PPP \left(\frac{4}{\sqrt{\pi}} \beta_* \gamma_* Z_\infty(dx) \otimes 2\sqrt{\log b} e^{-2\sqrt{\log b} y} dy \right) \quad (4.5)$$

as $n \rightarrow \infty$, where

$$\beta_* := \begin{cases} \sqrt{\frac{\theta+1}{\theta+\sqrt{\log b}}} & \text{if } \theta \in [0, \infty), \\ 1 & \text{if } \theta = \infty, \end{cases} \quad (4.6)$$

and

$$\gamma_* := \lim_{\ell \rightarrow \infty} \int_{\ell^{2/5}}^{\ell} z e^{2\sqrt{\log b} z} \mathbb{P} \left(\max_{v \in T_\ell} \frac{1}{\sqrt{2}} h_v > \sqrt{\log b} \ell + z \right) dz. \quad (4.7)$$

Remark 4.1.2 *The existence of the limit (4.7) is non-trivial. It is proved in the proof of Proposition 4.5.1. Results similar to Theorem 4.1.1 are known for the BBM [7] and for the two-dimensional DGFF [19]. Our setting is inspired by [24].*

By Theorem 4.1.1 and a tail estimate of the maximum of local times over leaves (Proposition 4.3.1(i) below), we have:

Corollary 4.1.3 *There exists $c_1 > 0$ such that for all $\lambda \in \mathbb{R}$ and any sequence $(t_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = \theta \in [0, \infty]$ and $t_n \geq c_1 n \log n$ for each $n \in \mathbb{N}$,*

$$\lim_{n \rightarrow \infty} P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} \leq \sqrt{t_n} + a_n(t_n) + \lambda \right) = \mathbb{E} \left[e^{-\frac{4}{\sqrt{\pi}} \beta_* \gamma_* D_\infty e^{-2\sqrt{\log b} \lambda}} \right], \quad (4.8)$$

where β_* and γ_* are given by (4.6) and (4.7), respectively.

Remark 4.1.4 Let $(h_v)_{v \in T}$ be a BRW on T . By Theorem 4.2.4(ii) and Lemma 4.7.7, one can show that for all $\lambda \in \mathbb{R}$ and any sequence $(t_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} \sqrt{t_n}/n^2 = \infty$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left(\max_{v \in T_n} \frac{1}{\sqrt{2}} h_v \leq m_n + \lambda \right) \\ &= \lim_{n \rightarrow \infty} P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} \leq \sqrt{t_n} + m_n + \lambda \right) = \mathbb{E} \left[e^{-\frac{4}{\sqrt{\pi}} \gamma_* D_\infty e^{-2\sqrt{\log b} \lambda}} \right], \end{aligned} \quad (4.9)$$

where γ_* is given in (4.7) and the centering sequence m_n is defined by

$$m_n := \sqrt{\log b} \, n - \frac{3}{4\sqrt{\log b}} \log n.$$

(Note that convergence of the maximum of the BRW has already been established in [9, 2].) By Corollary 4.1.3, to make the limit law of $\max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} - \sqrt{t_n} - \tilde{a}_n(t_n)$ the same as (4.9), we need to take $\tilde{a}_n(t_n)$ as follows :

$$\tilde{a}_n(t_n) = \begin{cases} m_n - \frac{1}{4\sqrt{\log b}} \log \left(\frac{n}{\sqrt{t_n}} \right) - \frac{\log(\log b)}{8\sqrt{\log b}}, & \text{if } \lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = 0 \\ m_n - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\theta + \sqrt{\log b}}{\theta} \right), & \text{if } \lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = \theta \in (0, \infty), \\ m_n, & \text{if } \lim_{n \rightarrow \infty} \frac{\sqrt{t_n}}{n} = \infty. \end{cases}$$

(We assumed that $t_n \geq c_1 n \log n$ for each $n \in \mathbb{N}$ and some sufficiently large $c_1 > 0$.) Thus, to obtain the limit law (4.9), we need non-trivial extra corrections in the case $\lim_{n \rightarrow \infty} \sqrt{t_n}/n \in [0, \infty)$ and exactly the same centering sequence as the one for the BRW in the case $\lim_{n \rightarrow \infty} \sqrt{t_n}/n = \infty$.

The organization of this paper is as follows. Section 4.2 gives preliminary lemmas which we use repeatedly throughout the paper. In Section 4.3, we obtain tail probabilities of the maximum of local times over leaves which are essential to next sections. In Section 4.4, we show that two leaves with local times near maxima are either very close or far away. This implies that $\Xi_{n,t_n}^{(q)}|_{\mathbb{R} \times [z, \infty)} = \Xi_{n,t_n}^{(n-r_n)}|_{\mathbb{R} \times [z, \infty)}$ with probability tending to 1 as $n \rightarrow \infty$ and then $q \rightarrow \infty$, which is one of the key steps in the proof of Theorem 4.1.1.

In Section 4.5, we obtain a limiting tail of the maximum of local times over leaves which is crucial to study the Laplace functional of $\Xi_{n,t_n}^{(q)}$. In Section 4.6, we give the proof of Theorem 4.1.1 and Corollary 4.1.3.

To prove Theorem 4.1.1, we use techniques developed in [42, 26] in the context of the two-dimensional DGFF. Since local times are not Gaussian random variables, we need some modifications in our setting. To overcome the difficulty, inspired by [41, 14], we apply the generalized second Ray-Knight theorem mentioned above. It implies that the local time process along the path from the root to a leaf behaves like a zero-dimensional squared Bessel process and this enables us to employ the argument of [42, 26]. We should emphasize that it is more convenient to study “continuous” version of local times rather than the original “discrete” ones especially when we estimate tail probabilities of the maximum of local times over leaves. To take the advantage, motivated by [72, 95], we consider the local time process of the Brownian motion on the associated metric tree as the “continuous” version.

We will write $c_1, c_2, \dots$ to denote positive universal constants whose values are fixed within each argument. We use $c_1(M), c_2(M) \dots$ for positive constants which depend on M . Given sequences $(c_n)_{n \geq 1}$ and $(c'_n)_{n \geq 1}$, we will write $c'_n = O(c_n)$ if there exists an universal constant C such that $|c'_n/c_n| \leq C$ for all $n \geq 1$, and we will write $c'_n = O_r(c_n)$ if the constant C depends on r . We write $|S|$ to denote the cardinality of a set S .

4.2 Preliminary lemmas

In this section, we collect some lemmas which we use repeatedly throughout this paper. We first recall metric trees and Brownian motions on them. In the study of local times of random walks on graphs, Lupu [72] and Zhai [95] found usefulness of the corresponding metric graphs and Brownian motions (or their discrete approximations). We follow the approach and find its advantages in obtaining precise tail probabilities of the maximum of local times over leaves on the b -ary tree. Let T be the b -ary tree. We regard each $e \in E(T)$ as an interval of length $1/2$ by setting $I_e := \{e\} \times (0, \frac{1}{2})$. Set $\bar{I}_e := I_e \cup \{e^-, e^+\}$, where $e^-, e^+ \in T$ be the endpoints of the edge e . Let π_e be the map from I_e to $(0, \frac{1}{2})$ defined by $\pi_e((e, x)) := x$. We extend π_e to the map from $\bar{I}_e$ to $[0, \frac{1}{2}]$ by setting $\pi_e(e^-) := 0, \pi_e(e^+) := \frac{1}{2}$. We define a metric tree of depth

n by

$$\tilde{T}_{\leq n} := T_{\leq n} \cup \bigcup_{e \in E(T_{\leq n})} I_e,$$

where $E(T_{\leq n})$ is the set of all the edges in $T_{\leq n}$. We define the metric $d(\cdot, \cdot)$ on $\tilde{T}_{\leq n}$ as follows: for $x, y \in \tilde{T}_{\leq n}$, let e_x and e_y be the edges with $x \in I_{e_x}$ and $y \in I_{e_y}$, respectively. In the case $I_{e_x} \neq I_{e_y}$, we set

$$d(x, y) := \min \left\{ |\pi_{e_x}(x) - \pi_{e_x}(v)| + \frac{1}{2}d_g(v, u) + |\pi_{e_y}(u) - \pi_{e_y}(y)| \right\},$$

where d_g is the graph distance on $T_{\leq n}$ and the minimum is taken over $v \in \{e_x^-, e_x^+\}$ and $u \in \{e_y^-, e_y^+\}$. In the case $I_{e_x} = I_{e_y}$, we set $d(x, y) := |\pi_{e_x}(x) - \pi_{e_x}(y)|$. We define a measure m on $\tilde{T}_{\leq n}$ by

$$m(dx) := \sum_{e \in E(T_{\leq n})} 1_{I_e}(x) \nu_e(dx),$$

where $\nu_e := \nu \circ \pi_e$, and ν is the Lebesgue measure on $(0, 1/2)$. We have a m -symmetric Hunt process on $\tilde{T}_{\leq n}$ with continuous sample paths such that on each I_e , it behaves like a standard Brownian motion on $(0, 1/2)$ until it hits $\{e^-, e^+\}$, and when it starts at a vertex v , it chooses one of the edges incident to v uniformly at random, and moves on it as described above. See, for example, [50, 63, 72] for the construction. We write $\tilde{X} = (\tilde{X}_t, t \geq 0, \tilde{P}_x, x \in \tilde{T}_{\leq n})$ to denote the process and call it a Brownian motion on $\tilde{T}_{\leq n}$. It is known that $\tilde{X}$ restricted on $T_{\leq n}$ behaves like a simple random walk on $T_{\leq n}$ in the following sense: for all $v \in T_{\leq n}$ and $1 \leq i \leq \deg(v)$,

$$P_v(\tilde{X}_{S_v} = v_i) = \frac{1}{\deg(v)}, \quad (4.10)$$

where $v_1, \dots, v_{\deg(v)}$ are vertices on $T_{\leq n}$ adjacent to v , and S_v is the hitting time of $\{v_1, \dots, v_{\deg(v)}\}$ by $\tilde{X}$. See, for example, [50, Theorem 2.1] or [72, Section 2]. By [72, Section 2], $\tilde{X}$ has a space-time continuous local time $\{\tilde{L}_t^n(x) : (t, x) \in [0, \infty) \times \tilde{T}_{\leq n}\}$ and the following holds for each $v \in T_{\leq n}$ under $\tilde{P}_v$:

$$\tilde{L}_{S_v}^n(v) \text{ has the same law as exponential distribution with mean } \frac{1}{\deg(v)}. \quad (4.11)$$

We define the inverse local time by

$$\tilde{\tau}(t) := \inf\{s \geq 0 : \tilde{L}_s^n(\rho) > t\}, \quad t > 0.$$

By (4.10) and (4.11), we have

$$\text{the law of } \left(\tilde{L}_{\tilde{\tau}(t)}^n(v) \right)_{v \in T_{\leq n}} \text{ under } \tilde{P}_\rho$$

is the same as

$$\text{that of } \left(L_{\tau(t)}^n(v) \right)_{v \in T_{\leq n}} \text{ under } P_\rho. \quad (4.12)$$

The following is a remarkable property of local times of the Brownian motion on $\tilde{T}_{\leq n}$. The discrete version has already been considered in [39, Lemma 2.6].

Lemma 4.2.1 *Fix $n \in \mathbb{N}, t > 0$, and $a \in T_{\leq n} \setminus T_n$. Let $\mathcal{F}^\uparrow$ be the σ -field generated by $\tilde{L}_{\tilde{\tau}(t)}^n(x)$, $x \in \{a\} \cup \tilde{T}_{\leq n} \setminus \tilde{T}^a$, where $\tilde{T}^a$ is the metric tree corresponding to the subtree $T_{\leq n-|a|}^a$. Then, the law of $\left\{ \tilde{L}_{\tilde{\tau}(t)}^n(x) : x \in \tilde{T}^a \setminus \{a\} \right\}$ under $\tilde{P}_\rho(\cdot | \mathcal{F}^\uparrow)$ is the same as that of $\left\{ \tilde{L}_{\tilde{\tau}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(a))}^\downarrow(x) : x \in \tilde{T}^a \setminus \{a\} \right\}$ under $\tilde{P}_a$, where $\left\{ \tilde{L}_t^\downarrow(x) : (t, x) \in [0, \infty) \times \tilde{T}^a \right\}$ is a local time of a Brownian motion on $\tilde{T}^a$ and $\tilde{\tau}^\downarrow(t) := \inf\{s : \tilde{L}_s^\downarrow(a) > t\}$.*

The proof of Lemma 4.2.1 is given in Section 4.7.1.

Let $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$ be a centered Gaussian process with $\mathbb{E}(\tilde{h}_x \tilde{h}_y) = \tilde{E}_x(\tilde{L}_{H_\rho}^n(y))$ for all $x, y \in \tilde{T}_{\leq n}$, where $H_x := \inf\{t \geq 0 : \tilde{X}_t = x\}$. We will call $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$ a Gaussian process associated with $\tilde{T}_{\leq n}$. We have an explicit representation of the covariance of $\tilde{h}$.

Lemma 4.2.2 *For all $x, y \in \tilde{T}_{\leq n}$,*

$$\mathbb{E}(\tilde{h}_x \tilde{h}_y) = d(\rho, x) + d(\rho, y) - d(x, y).$$

The proof of Lemma 4.2.2 is given in Section 4.7.2. Let $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$ be a Gaussian process associated with $\tilde{T}_{\leq n}$ and $(h_v)_{v \in T}$ be a BRW on T . By Lemma 4.2.2, we have for all $n \in \mathbb{N}$

$$\{\tilde{h}_v : v \in T_{\leq n}\} = \{h_v : v \in T_{\leq n}\}, \quad \text{in law.} \quad (4.13)$$

Lemma 4.2.2 implies the following.

Corollary 4.2.3 *Let $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$ be a Gaussian process associated with $\tilde{T}_{\leq n}$. For any leaf $v \in T_n$, the law of $\{\tilde{h}_{v_s} : 0 \leq s \leq n\}$ is the same as that of a standard Brownian motion $\{B_s : 0 \leq s \leq n\}$ on $\mathbb{R}$, where v_s is the point on the unique path in $\tilde{T}_{\leq n}$ from ρ to v with $d(\rho, v_s) = s/2$.*

The so-called generalized second Ray-Knight theorem connects local times and associated Gaussian processes:

Theorem 4.2.4 (i) ([47, Theorem 1]) *Fix $n \in \mathbb{N}$. Let $\{\tilde{h}_x : x \in \tilde{T}_{\leq n}\}$ be a Gaussian process associated with $\tilde{T}_{\leq n}$. For each $t > 0$, the law of $\{\tilde{L}_{\tilde{\tau}(t)}^n(x) + \frac{1}{2}\tilde{h}_x^2 : x \in \tilde{T}_{\leq n}\}$ under $\tilde{P}_\rho \times \mathbb{P}$ is the same as that of $\{\frac{1}{2}(\tilde{h}_x + \sqrt{2t})^2 : x \in \tilde{T}_{\leq n}\}$. (ii) ([95]) *For all $t > 0$ and $n \in \mathbb{N}$, on the same probability space, one can construct a local time $(L_{\tau(t)}^n(v))_{v \in T_{\leq n}}$ and two BRWs $(h_v)_{v \in T_{\leq n}}$, $(h'_v)_{v \in T_{\leq n}}$ on $T_{\leq n}$ satisfying the following:**

$$(L_{\tau(t)}^n(v))_{v \in T_{\leq n}} \text{ and } (h_v)_{v \in T_{\leq n}} \text{ are independent,} \quad (4.14)$$

$$L_{\tau(t)}^n(v) + \frac{1}{2}(h_v)^2 = \frac{1}{2} \left(h'_v + \sqrt{2t} \right)^2, \text{ for each } v \in T_{\leq n}, \text{ almost surely.} \quad (4.15)$$

The construction of the coupling in Theorem 4.2.4(ii) can be found in the proof of Theorem 3.1 of [95]. Let $C[0, \infty)$ be the space of real-valued continuous functions on $[0, \infty)$ and $\mathcal{B}(C[0, \infty))$ be the σ -field generated by cylinder sets in $C[0, \infty)$. We have a nice connection between the local time and the 0-dimensional squared Bessel process.

Lemma 4.2.5 *For all $t > 0$ and $v \in T_n$, the law of $\{\tilde{L}_{\tilde{\tau}(t)}^n(v_s) : 0 \leq s \leq n\}$ is the same as that of $\{\frac{1}{2}X_s : 0 \leq s \leq n\}$ under $\mathbb{Q}_{2t}^0$, where $\mathbb{Q}_x^d$ is a law on $(C[0, \infty), \mathcal{B}(C[0, \infty)))$ under which the coordinate process $\{X_s : s \geq 0\}$ is a d -dimensional squared Bessel process started at x .*

The proof of Lemma 4.2.5 is based on an additivity property of squared Bessel processes, Corollary 4.2.3 and Theorem 4.2.4(i). See [14, Proof of Lemma 7.7] for the details. It is known that the laws of 0-dimensional and 1-dimensional squared Bessel processes are related to each other by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}_x^0}{d\mathbb{Q}_x^1} \Big|_{\mathcal{F}_t \cap \{H_0 > t\}} = \left(\frac{x}{X_t} \right)^{1/4} \exp \left(-\frac{3}{8} \int_0^t \frac{1}{X_s} ds \right), \quad (4.16)$$

for all $t > 0$ and $x > 0$, where $H_0 := \inf\{t \geq 0 : X_t = 0\}$. See, for example, [14, (7.30)]. The transition semigroup $\{Q_t^0 : t \geq 0\}$ of a 0-dimensional squared Bessel process is given by

$$Q_t^0(x, \cdot) = \exp\left(-\frac{x}{2t}\right) \delta_0 + \tilde{Q}_t(x, \cdot), \quad x > 0, \quad (4.17)$$

where δ_0 is the Dirac measure at 0 and $\tilde{Q}_t(x, \cdot)$ has the density

$$q_t^0(x, y) = \frac{1}{2t} \sqrt{\frac{x}{y}} \exp\left(-\frac{x+y}{2t}\right) I_1\left(\frac{\sqrt{xy}}{t}\right), \quad x, y \in (0, \infty), \quad (4.18)$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind

$$I_1(x) = \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{2k+1} \frac{1}{k!(k+1)!}. \quad (4.19)$$

We will use the following asymptotic behavior of $I_1(\cdot)$:

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}} (1 + O(1/z)), \quad \text{as } z \rightarrow \infty. \quad (4.20)$$

See, for example, [86, Chapter XI, §1] for the details on 0-dimensional squared Bessel process. To estimate tail probabilities of the maximum of local times over leaves, we frequently use the following.

Lemma 4.2.6 ([26, Lemma 3.6]) *Fix a constant $C > 0$. For $z > 0$ and $s > 0$, set*

$$\begin{aligned} \mu_{s,z}(x)dx &:= P_0^B \left(\frac{B_s}{\sqrt{2}} \in dx, \frac{B_r}{\sqrt{2}} \leq z, \text{ for all } 0 \leq r \leq s \right) \\ &= \frac{1}{\sqrt{\pi s}} \left(e^{-\frac{x^2}{s}} - e^{-\frac{(2z-x)^2}{s}} \right) dx, \quad x \leq z. \end{aligned} \quad (4.21)$$

(i) *There exists $c_1 > 0$ such that for any $z \geq 1$ and nonnegative measurable function f on $(-\infty, z + z^{1/20}]$,*

$$\begin{aligned} &E_0^B \left[f\left(\frac{B_s}{\sqrt{2}}\right) 1_{\{B_r/\sqrt{2} \leq z + z^{1/20} + C(r \wedge (s-r))^{1/20}, \text{ for all } 0 \leq r \leq s\}} \right] \\ &\leq c_1 \int_{-\infty}^{z+z^{1/20}} f(x) z s^{-3/2} e^{-x^2/s} \left(z + z^{1/20} - x + 1_{\{x \geq z + z^{1/20} - 1\}} 1_{\{s \geq 2(z + z^{1/20} + 1)\}} \right) dx \end{aligned}$$

$$+ z^{1/20} 1_{\{x \geq z + \frac{1}{2} z^{1/20}\}} 1_{\{s \leq 2(z + z^{1/20} + 1)\}} \Big) dx. \quad (4.22)$$

(ii) There exist $c_2 > 0$ and a function δ_z with $\lim_{z \rightarrow \infty} \delta_z = 0$ such that for all $z \geq c_2$, $s > 4z^2$, and nonnegative measurable function f on $(z - \frac{s}{4z}, 0]$,

$$\begin{aligned} & E_0^B \left[f \left(B_s / \sqrt{2} \right) 1_{\{B_r / \sqrt{2} \leq z + z^{1/20} + C(r \wedge (s-r))^{1/20}, \text{ for all } 0 \leq r \leq s, B_s / \sqrt{2} \in (z - \frac{s}{4z}, 0]\}} \right] \\ & \leq (1 + \delta_z) \int_{z - \frac{s}{4z}}^0 f(x) \mu_{s,z}(x) dx. \end{aligned} \quad (4.23)$$

The statement of Lemma 4.2.6 is slightly different from [26, Lemma 3.6]. The proof of Lemma 4.2.6 is given in Section 4.7.5 for the sake of completeness.

4.3 Tail of maximum of local time over leaves

The aim of this section is to obtain the following tail estimates of the maximum of local times of the simple random walk on the b -ary tree over leaves. Recall (4.4).

Proposition 4.3.1 (i) There exist $c_1, c_2 \in (0, \infty)$ such that for all $t > 0, y \geq 0$, and $n \in \mathbb{N}$,

$$P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \leq c_1(1+y) e^{-2\sqrt{\log b} y} e^{-c_2 \frac{y^2}{n}}. \quad (4.24)$$

(ii) There exist $c_3 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $y \in [0, 2\sqrt{n}]$, and $t \geq n$,

$$P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq c_3(1+y) e^{-2\sqrt{\log b} y}. \quad (4.25)$$

We first prove Theorem 4.3.1(i). Fix $\kappa \in (\frac{1}{2\sqrt{\log b}}, \infty)$. For $y > 0$ and $n \in \mathbb{N}$, set

$$\begin{aligned} G_y^n(t) := & \left\{ \text{there exist } v \in T_n \text{ and } 0 \leq s \leq n \text{ such that} \right. \\ & \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \geq \sqrt{t} + \frac{a_n(t)}{n} s + \kappa(\log(s \wedge (n-s)))_+ + y + 1 \right\}, \end{aligned} \quad (4.26)$$

where $c_+ := \max\{c, 0\}$. We need the following.

Lemma 4.3.2 *There exist $c_1, c_2 \in (0, \infty)$ such that for all $t > 0$, $n \in \mathbb{N}$, and $y \geq 0$,*

$$\tilde{P}_\rho(G_y^m(t)) \leq c_1(1+y)e^{-2\sqrt{\log b} \ y} e^{-c_2 \frac{y^2}{n}}. \quad (4.27)$$

Proof. We first consider the case $y > M$, where M is sufficiently large constant. For a fixed $\delta \in (0, 1)$, set

$$g_{y,t,n}(s) := \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + y + 1, \quad 0 \leq s \leq n, \quad (4.28)$$

$$m_{y,t,n}(j) := \sqrt{t} + \frac{a_n(t)}{n}j + \kappa \min_{j \leq r \leq j+1} (\log(r \wedge (n-r)))_+ + y + 1, \quad 0 \leq j \leq n-1.$$

Recall the probability measure $\mathbb{Q}_x^d$ defined in Lemma 4.2.5 and set

$$\tau := \inf \left\{ s \geq 0 : \sqrt{X_s/2} \geq g_{y,t,n}(s) \right\},$$

where X is the coordinate process. By Lemma 4.2.5, we have

$$\begin{aligned} & \tilde{P}_\rho(G_y^n(t)) \\ & \leq \tilde{P}_\rho \left(\text{there exist } 0 \leq j \leq n-1, v \in T_{j+1}, \text{ and } s \in [j, j+1] \text{ such that} \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_r)} < g_{y,t,n}(r), \text{ for all } 0 \leq r \leq j, \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \geq g_{y,t,n}(s) \right) \\ & \leq \sum_{j=0}^{n-1} b^{j+1} \mathbb{Q}_{2t}^0 \left(\tau \in (j, j+1], \sqrt{X_{j+1}/2} \geq \delta m_{y,t,n}(j) \right) \\ & \quad + \sum_{j=0}^{n-1} b^{j+1} \mathbb{Q}_{2t}^0 \left(\tau \in (j, j+1], \sqrt{X_{j+1}/2} < \delta m_{y,t,n}(j) \right) \\ & =: \sum_{j=0}^{n-1} b^{j+1} I_j^{(1)} + \sum_{j=0}^{n-1} b^{j+1} I_j^{(2)}. \end{aligned} \quad (4.29)$$

Fix $0 \leq j \leq n-1$. We first estimate $I_j^{(2)}$. We have

$$I_j^{(2)} = \mathbb{Q}_{2t}^0 \left[1_{\{\tau \in (j, j+1)\}} \mathbb{Q}_{X_\tau}^0 \left(\sqrt{X_{j+1-\tau}/2} < \delta m_{y,t,n}(j) \right) \right]$$

$$\begin{aligned}
&= \mathbb{Q}_{2t}^0 \left[1_{\{\tau \in (j, j+1)\}} \exp \left(-\frac{X_\tau}{2(j+1-\tau)} \right) \right] \\
&+ \mathbb{Q}_{2t}^0 \left[1_{\{\tau \in (j, j+1)\}} \int_0^{2\delta^2(m_{y,t,n}(j))^2} \frac{\sqrt{X_\tau/z}}{2(j+1-\tau)} e^{-\frac{X_\tau+z}{2(j+1-\tau)}} I_1 \left(\frac{\sqrt{X_\tau z}}{j+1-\tau} \right) dz \right] \\
&=: J_1 + J_2,
\end{aligned} \tag{4.30}$$

where we have used the strong Markov property of a 0-dimensional squared Bessel process in the first equality and (4.17)-(4.18) in the second equality. By the definition of τ , we have

$$J_1 \leq \exp\{-(m_{y,t,n}(j))^2\}. \tag{4.31}$$

Assume that $\tau \in (j, j+1)$. Recall (4.19). If $z \leq \frac{M(j+1-\tau)^2}{X_\tau}$, then we have

$$I_1 \left(\frac{\sqrt{X_\tau z}}{j+1-\tau} \right) \leq \sum_{k=0}^{\infty} (\sqrt{M}/2)^{2k+1} \frac{1}{k!(k+1)!} \leq c_1(M). \tag{4.32}$$

If $z > \frac{M(j+1-\tau)^2}{X_\tau}$, then by (4.20) and the assumption that M is sufficiently large, we have

$$I_1 \left(\frac{\sqrt{X_\tau z}}{j+1-\tau} \right) \leq c_2 \frac{e^{\frac{\sqrt{X_\tau z}}{j+1-\tau}}}{\sqrt{\frac{\sqrt{X_\tau z}}{j+1-\tau}}}, \tag{4.33}$$

By (4.32) and (4.33), we have

$$J_2 \leq c_3(M) \left(\max_{j \leq r \leq j+1} g_{y,t,n}(r) \right) \exp\{-c_4(m_{y,t,n}(j))^2\}, \tag{4.34}$$

where we have used the inequality $(j+1-\tau)^{-1/2} e^{-\frac{c}{j+1-\tau}} \leq e^{-c}$ if $c > 1/2$. By (4.31) and (4.34), we have

$$\sum_{j=0}^{n-1} b^{j+1} I_j^{(2)} \leq c_5(M) \sum_{j=0}^{n-1} b^{j+1} e^{-c_6(\sqrt{t}+y+1)^2} e^{-c_7(\sqrt{t}+y+1)j} \leq c_8(M) e^{-c_6(\sqrt{t}+y+1)^2}, \tag{4.35}$$

where in the last inequality, we have used the assumption that $y > M$ and M is sufficiently large.

Next, we will estimate $I_j^{(1)}$. Fix $1 \leq j \leq n-1$. We have

$$I_j^{(1)} = \mathbb{Q}_{2t}^1 \left[\left(\frac{2t}{X_{j+1}} \right)^{\frac{1}{4}} \exp \left(-\frac{3}{8} \int_0^{j+1} \frac{ds}{X_s} \right) 1_{\left\{ \tau \in (j, j+1], \sqrt{\frac{X_{j+1}}{2}} \geq \delta m_{y,t,n}(j) \right\} \cap \{H_0 > j+1\}} \right]$$

$$\begin{aligned}
&\leq \sqrt{\frac{\sqrt{t}}{\delta m_{y,t,n}(j)}} P_{\sqrt{2t}}^B \left(B_r/\sqrt{2} < g_{y,t,n}(r), \text{ for all } 0 \leq r \leq j, \right. \\
&\quad \left. B_s/\sqrt{2} \geq g_{y,t,n}(s), \text{ for some } j \leq s \leq j+1 \right) \\
&\leq \sqrt{\frac{\sqrt{t}}{\delta m_{y,t,n}(j)}} E_0^B \left[1_{\{B_r/\sqrt{2} < g_{y,t,n}(r) - \sqrt{t}, \text{ for all } 0 \leq r \leq j\}} \right. \\
&\quad \left. P_{B_j}^B \left(B_s/\sqrt{2} \geq g_{y,t,n}(j+s) - \sqrt{t}, \text{ for some } s \in [0, 1] \right) \right], \tag{4.36}
\end{aligned}$$

where we have used (4.16) in the first equality, the fact that the law of a 1-dimensional squared Bessel process is the same as that of a square of a standard Brownian motion on $\mathbb{R}$ in the second inequality, and the translation invariance and Markov property of B in the last inequality.

Let $\tilde{P}_j^B$ be the probability measure defined by

$$\tilde{P}_j^B(A) = E_0^B \left[1_A \exp \left\{ \frac{\sqrt{2}a_n(t)}{n} B_j - \frac{(a_n(t))^2}{n^2} j \right\} \right], \quad A \in \sigma(B_s : s \leq j). \tag{4.37}$$

By the Girsanov theorem, under $\tilde{P}_j^B$, the process

$$\left\{ \tilde{B}_s := B_s - \frac{\sqrt{2}a_n(t)}{n} s : 0 \leq s \leq j \right\} \tag{4.38}$$

is a standard Brownian motion on $\mathbb{R}$ started at 0. Note that we have

$$g_{y,t,n}(r) - \sqrt{t} - \frac{a_n(t)}{n} r \leq y + c_9 + \kappa \log(j \wedge (n-j)) + c_{10}(r \wedge (j-r))^{\frac{1}{20}}, \quad 0 \leq r \leq j,$$

$$g_{y,t,n}(j+s) - \sqrt{t} - \frac{a_n(t)}{n} j \geq y - c_{11} + \kappa \log(j \wedge (n-j)), \quad 0 \leq s \leq 1.$$

By this and (4.36), we have

$$\sqrt{\frac{\delta m_{y,t,n}(j)}{\sqrt{t}}} I_j^{(1)}$$

$$\begin{aligned}
&\leq \tilde{E}_j^B \left[e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_j - \frac{(a_n(t))^2}{n^2}j} 1_{\left\{ \frac{\tilde{B}_r}{\sqrt{2}} < y + c_9 + \kappa \log(j \wedge (n-j)) + c_{10}(r \wedge (j-r))^{1/20}, \text{ for all } 0 \leq r \leq j \right\}} \right. \\
&\quad \left. P_{\tilde{B}_j}^B \left(\max_{0 \leq s \leq 1} \frac{B_s}{\sqrt{2}} \geq y - c_{11} + \kappa \log(j \wedge (n-j)) \right) \right] \\
&\leq c_{12} e^{-\frac{2a_n(t)}{n}y - \frac{(a_n(t))^2}{n^2}j} \left(1 + \frac{y}{j+1} \right) (y + c_9 + \kappa \log(j \wedge (n-j))) \\
&\quad \cdot j^{-\frac{3}{2}} (j \wedge (n-j))^{-2\sqrt{\log b}} \kappa e^{-c_{13}\frac{y^2}{j+1}}, \tag{4.39}
\end{aligned}$$

where we have used Lemma 4.2.6(i) and the estimate

$$P_0^B \left(\max_{0 \leq s \leq 1} \frac{B_s}{\sqrt{2}} \geq \lambda \right) \leq e^{-\lambda^2}, \quad \text{for each } \lambda > 0 \tag{4.40}$$

(see, for example, [61, Chapter 2, (8.4)]) in the last inequality. Similarly, in the case $j = 0$, by (4.16) and (4.40), we have

$$I_0^{(1)} \leq c_{14} P_0^B \left(\max_{0 \leq s \leq 1} \frac{B_s}{\sqrt{2}} \geq y + 1 \right) \leq c_{14} e^{-(y+1)^2}. \tag{4.41}$$

Thus, by (4.39) and (4.41), we have

$$\sum_{j=0}^{n-1} b^{j+1} I_j^{(1)} \leq c_{15} (1+y) e^{-2\sqrt{\log b} y} e^{-c_{16}\frac{y^2}{n}}. \tag{4.42}$$

Thus, by (4.29), (4.35) and (4.42), we have (4.27) for $y > M$. For $y \leq M$, (4.27) holds if we take c_1 in (4.27) sufficiently large depending on M . $\square$

Using Lemma 4.3.2, we now prove Proposition 4.3.1(i):

Proof of Proposition 4.3.1(i). Recall (4.26) and (4.28). We have

$$\begin{aligned}
&P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \\
&\leq \tilde{P}_\rho \left(\text{there exists } v \in T_n \text{ such that } \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \leq g_{y,t,n}(s), \text{ for all } s \leq n, \right. \\
&\quad \left. \sqrt{\tilde{L}_{\tau(t)}^n(v)} \in \left[\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1 \right] \right)
\end{aligned}$$

$$+ \tilde{P}_\rho(G_y^n(t)). \quad (4.43)$$

Fix $v \in T_n$. Recall the probability measure $\tilde{P}_n^B$ and the process $\tilde{B}$ defined in (4.37) and (4.38). By using Lemma 4.2.5 and (4.16) in a way similar to the derivation of (4.36), we have

$$\begin{aligned} & \sqrt{\frac{\sqrt{t} + a_n(t) + y}{\sqrt{t}}} \tilde{P}_\rho \left(\sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \leq g_{y,t,n}(s), \text{ for all } 0 \leq s \leq n, \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right) \\ & \leq \tilde{E}_n^B \left[e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_n - \frac{(a_n(t))^2}{n}} 1_{\left\{ \frac{\tilde{B}_s}{\sqrt{2}} \leq y+1+c_1\kappa(s \wedge (n-s))^{\frac{1}{20}}, \text{ for all } 0 \leq s \leq n, \frac{\tilde{B}_n}{\sqrt{2}} \in [y, y+1] \right\}} \right] \\ & \leq c_2 b^{-n} (1+y) e^{-2\sqrt{\log b} y} e^{-c_3 \frac{y^2}{n}} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}}, \end{aligned} \quad (4.44)$$

where we have used the inequality

$$g_{y,t,n}(s) - \sqrt{t} - \frac{a_n(t)}{n}s \leq y + 1 + c_1\kappa(s \wedge (n-s))^{1/20}, \text{ for each } 0 \leq s \leq n$$

in the first inequality, and Lemma 4.2.6(i) in the last inequality. Thus, by (4.43)-(4.44) and Lemma 4.3.2, we have (4.24). $\square$

Next, we prove Proposition 4.3.1(ii). Fix $\delta \in (0, 1)$. For $v \in T_n$, set the event

$$\begin{aligned} A_v^n(t) := & \left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1, \text{ for all } 0 \leq s \leq n, \right. \\ & \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right\}. \end{aligned} \quad (4.45)$$

To obtain Proposition 4.3.1(ii), we will apply the second moment method to $\sum_{v \in T_n} 1_{A_v^n(t)}$. We first need the following:

Lemma 4.3.3 *There exist $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $y \in [0, 2\sqrt{n}]$, $t \geq n$, and $v \in T_n$,*

$$\tilde{P}_\rho(A_v^n(t)) \geq c_1 b^{-n} (1+y) e^{-2\sqrt{\log b} y}. \quad (4.46)$$

Proof. By Lemma 4.2.5 and (4.16), we have

$$\begin{aligned}
& \tilde{P}_\rho(A_v^n(t)) \\
& \geq c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} P_0^B \left(-(1-\delta)\sqrt{t} \leq B_s/\sqrt{2} < \frac{a_n(t)}{n}s + y + 1, \right. \\
& \quad \left. \text{for all } 0 \leq s \leq n, B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1) \right) \\
& \geq c_1 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} (J_1 - J_2), \tag{4.47}
\end{aligned}$$

where we have used $t \geq n$ to show that $\exp\left(-\frac{3}{8} \int_0^n \frac{ds}{X_s}\right) \geq c_1$ under the event that $\sqrt{X_s/2} \geq \delta\sqrt{t}$ for all $0 \leq s \leq n$, and set

$$\begin{aligned}
J_1 &:= P_0^B \left(\frac{B_s}{\sqrt{2}} < \frac{a_n(t)}{n}s + y + 1, \text{ for all } 0 \leq s \leq n, \frac{B_n}{\sqrt{2}} \in a_n(t) + [y, y + 1) \right), \\
J_2 &:= P_0^B \left(\frac{B_s}{\sqrt{2}} < -(1-\delta)\sqrt{t}, \text{ for some } 0 \leq s \leq n, \frac{B_n}{\sqrt{2}} \in a_n(t) + [y, y + 1) \right).
\end{aligned}$$

We first obtain an upper bound of J_2 . By using the density

$$P_0^B \left(B_s \in dx, \max_{0 \leq r \leq s} B_r \in dz \right) = \frac{2(2z-x)}{\sqrt{2\pi}s^3} e^{-\frac{(2z-x)^2}{2s}} dx dz, \quad s > 0, x \leq z, z \geq 0, \tag{4.48}$$

(see, for example, [61, Chapter 2, Proposition 8.1]), for all $n \in \mathbb{N}$ and $y \in [0, 2\sqrt{n}]$, we have

$$\begin{aligned}
J_2 &= P_0^B \left(\max_{0 \leq s \leq n} B_s/\sqrt{2} > (1-\delta)\sqrt{t}, B_n/\sqrt{2} \in (-a_n(t) - y - 1, -a_n(t) - y] \right) \\
&\leq c_2 n^{-1/2} \exp \left\{ -\frac{(2(1-\delta)\sqrt{t} + a_n(t) + y)^2}{n} \right\} \\
&\leq c_3 b^{-n} n e^{-c_4 \sqrt{n}} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}} e^{-2\sqrt{\log b} y}, \tag{4.49}
\end{aligned}$$

where we have used the symmetry of B in the first equality.

Next, we obtain a lower bound of J_1 . Recall the probability measure $\tilde{P}_n^B$ and the process $\tilde{B}$ defined in (4.37) and (4.38). For all $n \geq n_0$ (n_0 is sufficiently large) and $y \in [0, 2\sqrt{n}]$, we have

$$\begin{aligned} J_1 &= \tilde{E}_n \left[e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_n - \frac{(a_n(t))^2}{n}} 1_{\{\tilde{B}_s/\sqrt{2} < y+1, \text{ for all } 0 \leq s \leq n, \tilde{B}_n/\sqrt{2} \in [y, y+1)\}} \right] \\ &\geq \int_{1/2}^1 e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}(y+1) - \frac{(a_n(t))^2}{n}} \frac{1}{\sqrt{\pi n}} \left(e^{-\frac{(y+1-z)^2}{n}} - e^{-\frac{(y+1+z)^2}{n}} \right) dz \\ &\geq c_5 b^{-n} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}} (y+1) e^{-2\sqrt{\log b} y}, \end{aligned} \quad (4.50)$$

where we used (4.21) in the second inequality and $e^{-\frac{(y+1-z)^2}{n}} - e^{-\frac{(y+1+z)^2}{n}} \geq c_6 \frac{y+1}{n}$ for all $z \in [1/2, 1]$ in the last inequality. Thus, by (4.47), (4.49)-(4.50), we have (4.46). $\square$

To obtain upper bounds of $\tilde{P}_\rho(A_u^n(t) \cap A_v^n(t))$, $u, v \in T_n$, we need the following:

Lemma 4.3.4 *There exists $c_1 > 0$ such that for all $n \in \mathbb{N}$, $t > 0$, $v \in T_n$, $0 \leq \ell \leq n-1$, $s < (\sqrt{t} + \frac{a_n(t)}{n}\ell + y+1)^2$, and $y \geq 0$,*

$$\begin{aligned} &\tilde{P}_{v_\ell} \left(\sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(s)}^\downarrow}(v_r) < \sqrt{t} + \frac{a_n(t)}{n}r + y+1, \text{ for all } \ell \leq r \leq n, \right. \\ &\quad \left. \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(s)}^\downarrow}(v) \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y+1] \right) \\ &\leq c_1 (n-\ell)^{-\frac{3}{2}} \sqrt{\frac{\sqrt{s}}{\sqrt{t} + a_n(t) + y}} \left(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n}\ell + y+1 \right) e^{-\frac{(\sqrt{t} - \sqrt{s} + a_n(t) + y)^2}{n-\ell}}, \end{aligned} \quad (4.51)$$

where $\{\tilde{L}_r^\downarrow(x) : (r, x) \in [0, \infty) \times \tilde{T}^{v_\ell}\}$ is a local time of a Brownian motion on $\tilde{T}^{v_\ell} \subset \tilde{T}_{\leq n}$, and $\tilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v_\ell) > s\}$.

Proof. Let P_ℓ be the probability on the left-hand side of (4.51). Recall the probability measure $\tilde{P}_{n-\ell}^B$ and the process $\tilde{B}$ defined in (4.37) and (4.38). By

using Lemma 4.2.5 and (4.16) in a way similar to the proof of Proposition 4.3.1(i), we have

$$\begin{aligned}
& \sqrt{\frac{\sqrt{t} + a_n(t) + y}{\sqrt{s}}} P_\ell \\
& \leq \tilde{E}_{n-\ell}^B \left[e^{-\frac{\sqrt{2}a_n(t)}{n} \tilde{B}_{n-\ell} - \frac{(a_n(t))^2}{n^2} (n-\ell)} 1_{\left\{ \frac{\tilde{B}_{n-\ell}}{\sqrt{2}} \in \sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + [0,1) \right\}} \right. \\
& \quad \left. 1_{\left\{ \frac{\tilde{B}_r}{\sqrt{2}} < \sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + 1, \text{ for all } 0 \leq r \leq n-\ell \right\}} \right] \\
& = \int_0^1 e^{\frac{2a_n(t)}{n} z - \frac{2a_n(t)}{n} \left(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + 1 \right) - \frac{(a_n(t))^2}{n^2} (n-\ell)} \\
& \quad \frac{1}{\sqrt{\pi(n-\ell)}} \left(e^{-\frac{\left(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + 1 - z \right)^2}{n-\ell}} - e^{-\frac{\left(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + 1 + z \right)^2}{n-\ell}} \right) dz \\
& \leq c_1 (n-\ell)^{-3/2} \left(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y + 1 \right) e^{-\frac{(\sqrt{t} - \sqrt{s} + \frac{a_n(t)}{n} \ell + y)^2}{n-\ell}}, \quad (4.52)
\end{aligned}$$

where we have used the density (4.21) in the second equality and the inequality $1 - e^{-x} \leq x$ for each $x \geq 0$ in the last inequality. Thus, we have obtained (4.51). $\square$

By Lemma 4.3.4 with $s = t$ and $\ell = 0$, we have the following.

Corollary 4.3.5 *There exists $c_1 > 0$ such that for all $n \in \mathbb{N}$, $t > 0$, $v \in T_n$, and $y \geq 0$,*

$$\tilde{P}_\rho(A_v^n(t)) \leq c_1 b^{-n} (y+1) e^{-2\sqrt{\log b} y}. \quad (4.53)$$

Proof of Proposition 4.3.1(ii). Fix any $n \geq n_0$, $t \geq n$ and $y \in [0, 2\sqrt{n}]$, where we take $n_0 \in \mathbb{N}$ large enough. Set

$$Z := \sum_{v \in T_n} 1_{A_v^n(t)}.$$

We have

$$P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq \tilde{P}_\rho(Z \geq 1) \geq \frac{\left(\tilde{E}_\rho[Z] \right)^2}{\tilde{E}_\rho[Z^2]}. \quad (4.54)$$

By Lemma 4.3.3, we have

$$\tilde{E}_\rho[Z] \geq c_1(1+y)e^{-2\sqrt{\log b} \ y}. \quad (4.55)$$

The rest of the proof focuses on obtaining an upper bound of $\tilde{E}_\rho[Z^2]$. We have

$$\tilde{E}_\rho[Z^2] = \tilde{E}_\rho[Z] + \sum_{\ell=0}^{n-1} \sum_{\substack{v,u \in T_n, \\ |v \wedge u| = \ell}} \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)). \quad (4.56)$$

By Corollary 4.3.5, we have

$$\tilde{E}_\rho[Z] \leq c_2(y+1)e^{-2\sqrt{\log b} \ y}. \quad (4.57)$$

Fix $1 \leq \ell \leq n-1$ and $v, u \in T_n$ with $|v \wedge u| = \ell$. Let $\tilde{L}_s^\downarrow(x)$, $(s, x) \in [0, \infty) \times \tilde{T}^{v_\ell}$ be a local time of a Brownian motion on $\tilde{T}^{v_\ell} \subset \tilde{T}_{\leq n}$. Set $\tilde{\tau}(t) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(v_\ell) > t\}$. For $w \in \{v, u\}$ and $s \geq 0$, we define the event $C_w^\downarrow(s)$ by

$$C_w^\downarrow(s) := \left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(s)}^\downarrow(w_r)} < \sqrt{t} + \frac{a_n(t)}{n}r + y + 1, \text{ for all } \ell \leq r \leq n, \right. \\ \left. \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(s)}^\downarrow(w)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right\}.$$

By Lemma 4.2.1, we have

$$\begin{aligned} & \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) \\ &= \tilde{E}_\rho \left[1_{\left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1, \text{ for all } s \leq \ell \right\}} \tilde{P}_{v_\ell} \left(\bigcap_{w \in \{v, u\}} C_w^\downarrow \left(\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \right] \\ &= \tilde{E}_\rho \left[1_{\left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1, \text{ for all } s \leq \ell \right\}} \prod_{w \in \{v, u\}} \tilde{P}_{v_\ell} \left(C_w^\downarrow \left(\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \right] \\ &\leq \sum_{i=0}^{\lceil (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}\ell + y \rceil} \tilde{E}_\rho \left[1_{\left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1, \text{ for all } 0 \leq s \leq \ell \right\}} \right. \\ &\quad \left. 1_{\left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n}\ell + y + 1 + [-i-1, -i] \right\}} \prod_{w \in \{v, u\}} \tilde{P}_{v_\ell} \left(C_w^\downarrow \left(\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \right], \end{aligned} \quad (4.58)$$

where we have used the independence of $C_v^\downarrow(s)$ and $C_u^\downarrow(s)$ for each $s \geq 0$. The independence follows from that of two types of excursions of a Brownian motion around v_ℓ restricted on $\tilde{T}^{v_{\ell+1}} \cup I_{\{v_\ell, v_{\ell+1}\}}$ or on $\tilde{T}^{u_{\ell+1}} \cup I_{\{v_\ell, u_{\ell+1}\}}$. Fix $i \leq \left\lceil (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}\ell + y \right\rceil$. By Lemma 4.3.4, under the event $\left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n}\ell + y + 1 + [-i-1, -i] \right\}$, we have for all $w \in \{v, u\}$

$$\begin{aligned} & \tilde{P}_{v_\ell} \left(C_w^\downarrow \left(\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell) \right) \right) \\ & \leq c_3(n-\ell)^{-\frac{3}{2}}(i+1) \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}\ell + y + 1 - i}{\sqrt{t} + a_n(t) + y}} e^{-\frac{\left(\frac{a_n(t)}{n}(n-\ell) + i - 1\right)^2}{n-\ell}}. \end{aligned} \quad (4.59)$$

By almost the same argument as the proof of Lemma 4.3.4, we have

$$\begin{aligned} & \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}\ell + y - i}{\sqrt{t}}} \tilde{P}_\rho \left(\sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \frac{a_n(t)}{n}s + y + 1, \text{ for all } s \leq \ell, \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_\ell)} \in \sqrt{t} + \frac{a_n(t)}{n}\ell + y + 1 + [-i-1, -i] \right) \\ & \leq c_4 b^{-\ell} \ell^{-3/2} (y+1) e^{-2\sqrt{\log b} \ y} e^{\frac{2a_n(t)}{n}i} e^{\frac{3 \log n}{2n}\ell} e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}\ell}. \end{aligned} \quad (4.60)$$

By (4.58)-(4.60), we have

$$\begin{aligned} & \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}\ell + y}} \cdot \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) \\ & \leq c_5 b^{-2n+\ell} \ell^{-\frac{3}{2}} (n-\ell)^{-3} n^3 (y+1) e^{-2\sqrt{\log b} \ y} e^{-\frac{3 \log n}{2n}\ell} e^{-\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}\ell}. \end{aligned} \quad (4.61)$$

By (4.61), we have

$$\sum_{\ell=1}^{n-1} \sum_{\substack{v, u \in T_n, \\ |v \wedge u| = \ell}} \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) \leq c_6 (y+1) e^{-2\sqrt{\log b} \ y}. \quad (4.62)$$

In the case $\ell = 0$, by Corollary 4.3.5, we have

$$\sum_{\substack{v, u \in T_n, \\ |v \wedge u| = 0}} \tilde{P}_\rho(A_v^n(t) \cap A_u^n(t)) = \sum_{\substack{v, u \in T_n, \\ |v \wedge u| = 0}} \tilde{P}_\rho(A_v^n(t)) \tilde{P}_\rho(A_u^n(t)) \leq c_7 (y+1) e^{-2\sqrt{\log b} \ y}, \quad (4.63)$$

where we have used the independence of $A_v^n(t)$ and $A_u^n(t)$ for each $v, u \in T_n$ with $|v \wedge u| = 0$ in the first equality. The independence follows from that of two types of excursions of a Brownian motion on $\tilde{T}_{\leq n}$ around ρ restricted on $\tilde{T}^{v_1} \cup I_{\{\rho, v_1\}}$ or on $\tilde{T}^{u_1} \cup I_{\{\rho, u_1\}}$.

Thus, by (4.54)-(4.57) and (4.62)-(4.63), we have (4.25). $\square$

4.4 Geometry of near maxima

In this section, we will prove that two leaves with local times near maxima are either very close or far away. More specifically, the following is the aim of this section.

Proposition 4.4.1 *There exist $c_1, c_2 \in (0, \infty)$, $n_0, r_0 \in \mathbb{N}$, and $t_0 > 0$ such that for all $n \geq n_0$, $t \geq t_0$, and $r_0 \leq r \leq n/4$,*

$$P_\rho \left(\text{there exist } v, u \in T_n \text{ with } r \leq |v \wedge u| \leq n - r \text{ such that} \right. \\ \left. \sqrt{L_{\tau(t)}^n(v)}, \sqrt{L_{\tau(t)}^n(u)} \in [\sqrt{t} + a_n(t) - c_1 \log r, \infty) \right) \leq c_2 r^{-1/8}.$$

Remark 4.4.2 *Results similar to Proposition 4.4.1 are known for the BBM [6] and for the two-dimensional DGFF [42].*

For $n \in \mathbb{N}$, $t > 0$, and $k \in \mathbb{Z}$, set

$$\Gamma_k^n(t) := \left\{ v \in T_n : \sqrt{L_{\tau(t)}^n(v)} \in [\sqrt{t} + a_n(t) - k - 1, \sqrt{t} + a_n(t) - k] \right\}.$$

Remark 4.4.3 *Fix $n' > n \geq 1$, $t > 0$, and $k \in \mathbb{Z}$. Set*

$$\Gamma_k^{n',n}(t) := \left\{ v \in T_n : \sqrt{L_{\tau(t)}^{n'}(v)} \in [\sqrt{t} + a_n(t) - k - 1, \sqrt{t} + a_n(t) - k] \right\}.$$

Since the law of the simple random walk on $T_{\leq n'}$ watched only on $T_{\leq n}$ is the same as that of the simple random walk on $T_{\leq n}$, we have $|\Gamma_k^{n',n}(t)| = |\Gamma_k^n(t)|$ in law.

In the proof of Proposition 4.4.1, we will use the following repeatedly.

Lemma 4.4.4 (i) *There exist $c_1 > 0$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$, $t \geq t_0$, $k \leq -1$, and $\lambda \in \mathbb{R}$ with $k + \lambda \geq 0$,*

$$\begin{aligned} & P_\rho \left(|\Gamma_k^n(t)| \geq e^{2\sqrt{\log b} (k+\lambda)} \right) \\ & \leq c_1(\lambda + 1)e^{-2\sqrt{\log b} \left(\lambda - \frac{3}{4\sqrt{\log b}} \log(\lceil (k+\lambda)^2 \rceil \vee 1) - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t} + a_n(t) - k - 1 + \lceil (k+\lambda)^2 \rceil \vee 1}{\sqrt{t} + a_n(t) - k - 1} \right) \right)}. \end{aligned} \quad (4.64)$$

(ii) *There exist $c_2, c_3 \in (0, \infty)$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$, $t \geq t_0$, $\lambda > 0$, and $k \geq 0$ with $\sqrt{t} + a_n(t) - k - 1 \geq c_2$,*

$$\begin{aligned} & P_\rho \left(|\Gamma_k^n(t)| \geq e^{2\sqrt{\log b} (k+\lambda)} \right) \\ & \leq c_3(\lambda + 1)e^{-2\sqrt{\log b} \left(\lambda - \frac{3}{4\sqrt{\log b}} \log(\lceil (k+\lambda)^2 \rceil \vee 1) - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t} + a_n(t) - k - 1 + \lceil (k+\lambda)^2 \rceil \vee 1}{\sqrt{t} + a_n(t) - k - 1} \right) \right)}. \end{aligned}$$

Proof. (i) Fix $n \in \mathbb{N}$, $t > 0$, $k \leq -1$, and $\lambda \in \mathbb{R}$ with $k + \lambda \geq 0$. Set $r := \lceil (k + \lambda)^2 \rceil \vee 1$ and $y := \lambda - \frac{3}{4\sqrt{\log b}} \log r - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t} + a_n(t) - k - 1 + r}{\sqrt{t} + a_n(t) - k - 1} \right)$. If $y < 0$, then it is clear that (4.64) holds because $P_\rho \left(|\Gamma_k^n(t)| \geq e^{2\sqrt{\log b} (k+\lambda)} \right) \leq 1 \leq (\lambda + 1)e^{-2\sqrt{\log b} y}$. So, we may assume that $y \geq 0$. Fix any $K > 0$. We have

$$\begin{aligned} & P_\rho \left(\max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} \geq \sqrt{t} + a_{n+r}(t) + y \right) \\ & \geq P_\rho \left(|\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} \geq \sqrt{t} + a_{n+r}(t) + y \right) \\ & \geq P_\rho \left(|\Gamma_k^{n+r,n}(t)| \geq K \right) \\ & - P_\rho \left(|\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} < \sqrt{t} + a_{n+r}(t) + y \right). \end{aligned} \quad (4.65)$$

We estimate the second term on the right-hand side of (4.65). By Lemma 4.2.1, we have

$$\begin{aligned} & P_\rho \left(|\Gamma_k^{n+r,n}(t)| \geq K, \max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} < \sqrt{t} + a_{n+r}(t) + y \right) \\ & \leq \sum_{\substack{S \subset T_n, \\ |S| \geq K}} P_\rho \left(\Gamma_k^{n+r,n}(t) = S, \max_{v \in T_\tau^u} \sqrt{L_{\tau(t)}^{n+r}(v)} < \sqrt{t} + a_{n+r}(t) + y, \text{ for all } u \in S \right) \end{aligned}$$

$$= \sum_{\substack{S \subset T_n, \\ |S| \geq K}} E_\rho \left[1_{\{\Gamma_k^{n+r,n}(t)=S\}} \prod_{u \in S} \tilde{P}_u \left(\max_{v \in T_r^u} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^{n+r}(u))}(v) < \sqrt{t} + a_{n+r}(t) + y \right) \right], \quad (4.66)$$

where for each $u \in T_n$, $\{\tilde{L}_s^\downarrow(x) : (s, x) \in [0, \infty) \times \tilde{T}^u\}$ is a local time of a Brownian motion on $\tilde{T}^u \subset \tilde{T}_{\leq n+r}$ and $\tilde{\tau}^\downarrow(q) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(u) > q\}$. We omit the subscript u in $\tilde{L}_s^\downarrow(x)$ and $\tilde{\tau}^\downarrow$ to simplify the notation.

We estimate each probability on the right-hand side of (4.66). Fix $S \subset T_n$ with $|S| \geq K$ and $u \in S$. Note that under the event that $\Gamma_k^{n+r,n}(t) = S$, we have

$$\sqrt{t} + a_{n+r}(t) + y \leq \sqrt{L_{\tau(t)}^{n+r}(u) + a_r(L_{\tau(t)}^{n+r}(u))} + k + \lambda + 1,$$

$L_{\tau(t)}^{n+r}(u) \geq \sqrt{t} + a_n(t) - k - 1 \geq \sqrt{t_0}$ if $t \geq t_0$, and $k + \lambda + 1 \leq 2\sqrt{r}$ by the definition of r . By these, we can apply Proposition 4.7.2 for $t \geq t_0$, where t_0 is sufficiently large, and have

$$\begin{aligned} & \tilde{P}_u \left(\max_{v \in T_r^u} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^{n+r}(u))}(v) < \sqrt{t} + a_{n+r}(t) + y \right) \\ & \leq 1 - \tilde{P}_u \left(\max_{v \in T_r^u} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^{n+r}(u))}(v) \geq \sqrt{L_{\tau(t)}^{n+r}(u) + a_r(L_{\tau(t)}^{n+r}(u))} + k + \lambda + 1 \right) \\ & \leq 1 - c_1 e^{-2\sqrt{\log b} (k+\lambda+1)}. \end{aligned} \quad (4.67)$$

By (4.65)-(4.67), we have

$$\begin{aligned} & P_\rho \left(\max_{v \in T_{n+r}} \sqrt{L_{\tau(t)}^{n+r}(v)} \geq \sqrt{t} + a_{n+r}(t) + y \right) \\ & \geq \left(1 - \exp \left\{ -c_1 K e^{-2\sqrt{\log b} (k+\lambda+1)} \right\} \right) P_\rho (|\Gamma_k^{n+r,n}(t)| \geq K). \end{aligned} \quad (4.68)$$

By Remark 4.4.3, (4.68) with $K := e^{2\sqrt{\log b} (k+\lambda)}$, and Proposition 4.3.1(i), we have (4.64).

(ii) The proof of (ii) is almost the same as that of (i), so we omit the detail. $\square$

For the rest of this section, we focus on proving the following.

Lemma 4.4.5 Fix $0 < \underline{c} < \bar{c} < \frac{3}{4\sqrt{\log b}}$.

(i) There exist $c_1 > 0$, $n_0, s_0 \in \mathbb{N}$, and $t_0 > 0$ such that for all $n \geq n_0$, $t \geq t_0$, and $s_0 \leq s \leq n - s_0$,

$$\begin{aligned}
& P_\rho \left[\text{there exist } v, u \in T_n \text{ with } |v \wedge u| = s \text{ such that} \right. \\
& \quad \left. \sqrt{L_{\tau(t)}^n(v)}, \sqrt{L_{\tau(t)}^n(u)} \in \left[\sqrt{t} + a_n(t) - (\bar{c} - \underline{c}) \log(s \wedge (n - s)), \infty \right) \right] \\
& \leq c_1 (\log(s \wedge (n - s)))^8 (s \wedge (n - s))^{-3+4\bar{c}\sqrt{\log b}-2\underline{c}\sqrt{\log b}} \\
& \quad + c_1 (\log(s \wedge (n - s)))^6 (s \wedge (n - s))^{-2\underline{c}\sqrt{\log b}}. \tag{4.69}
\end{aligned}$$

Before we prove this, let us show that Lemma 4.4.5 implies Proposition 4.4.1. *Proof of Proposition 4.4.1 via Lemma 4.4.5.* Fix any $n \geq n_0$, $t \geq t_0$, and $r_0 \leq r \leq n/4$, where we take $n_0, r_0 \in \mathbb{N}$ and $t_0 > 0$ sufficiently large. By Lemma 4.4.5 with $\underline{c} = \frac{5}{8\sqrt{\log b}}$ and $\bar{c} = \frac{11}{16\sqrt{\log b}}$, we have

$$\begin{aligned}
& P_\rho \left(\text{there exist } v, u \in T_n \text{ with } r \leq |v \wedge u| \leq n - r \text{ such that} \right. \\
& \quad \left. \sqrt{L_{\tau(t)}^n(v)}, \sqrt{L_{\tau(t)}^n(u)} \in \left[\sqrt{t} + a_n(t) - (\bar{c} - \underline{c}) \log r, \infty \right) \right) \\
& \leq c_1 \sum_{s=r}^{n-r} \left\{ (\log(s \wedge (n - s)))^8 (s \wedge (n - s))^{-\frac{3}{2}} \right. \\
& \quad \left. + (\log(s \wedge (n - s)))^6 (s \wedge (n - s))^{-\frac{5}{4}} \right\} \\
& \leq c_2 \sum_{s=r}^{\infty} s^{-9/8} \leq c_3 r^{-1/8}. \quad \square
\end{aligned}$$

Proof of Lemma 4.4.5. Fix any $n \geq n_0$, $t \geq t_0$, $s_0 \leq s \leq n - s_0$, where we take $n_0, s_0 \in \mathbb{N}$ and $t_0 > 0$ sufficiently large. Set $z := \underline{c} \log(s \wedge (n - s))$. For $k \in \mathbb{Z}$, we set $j^*(k, z) := \lceil \max\{|k|, z\} \rceil$. Fix $t_* > 0$. We have

$$P_\rho \left[\text{there exist } v, u \in T_n \text{ with } |v \wedge u| = s \text{ such that} \right.$$

$$\begin{aligned}
& \sqrt{L_{\tau(t)}^n(v)}, \sqrt{L_{\tau(t)}^n(u)} \in \left[\sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \infty \right) \\
& \leq P_\rho \left[\text{there exists } k \in \mathbb{Z} \text{ with } \sqrt{t} + a_s(t) - k > 0 \text{ and } j \geq -k \text{ such that} \right. \\
& \quad \left| \Gamma_k^{n,s}(t) \right| \in \left[e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)} \right), \text{ and} \\
& \quad \text{there exist } w \in \Gamma_k^{n,s}(t) \text{ and } w_1, w_2 \in T_1^w \text{ with } w_1 \neq w_2 \text{ such that} \\
& \quad \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \left. \right] \\
& \leq \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} P_\rho \left[\left| \Gamma_k^{n,s}(t) \right| \in \left[e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)} \right), \text{ and} \right. \\
& \quad \text{there exist } w \in \Gamma_k^{n,s}(t) \text{ and } w_1, w_2 \in T_1^w \text{ with } w_1 \neq w_2 \text{ such that} \\
& \quad \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \left. \right] \\
& \quad + \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} P_\rho \left(\left| \Gamma_k^{n,s}(t) \right| \geq e^{2\sqrt{\log b}(k+j^*(k,z))} \right) \\
& \quad + \sum_{\substack{k \in \mathbb{Z}, \\ 0 < \sqrt{t} + a_s(t) - k \leq t_* + 1}} P_\rho \left[\text{there exist } w \in T_s \text{ with } \sqrt{L_{\tau(t)}^n(w)} \leq t_* + 1, \right. \\
& \quad \text{and } w_1, w_2 \in T_1^w \text{ with } w_1 \neq w_2 \text{ such that} \\
& \quad \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \left. \right] \\
& =: J_1 + J_2 + J_3. \tag{4.70}
\end{aligned}$$

For each $w \in T_s$, let $\tilde{L}^\downarrow$ be a local time of a Brownian motion on $\tilde{T}^w \subset \tilde{T}_{\leq n}$ and set $\tilde{\tau}^\downarrow(q) := \inf\{p \geq 0 : \tilde{L}_p^\downarrow(w) > q\}$. We omit the subscript w in $\tilde{L}^\downarrow$ and

$\tilde{\tau}^\downarrow(q)$. We first estimate J_1 .

$$\begin{aligned}
J_1 &\leq \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} \sum_{\substack{S \subset T_s, \\ |S| \in [e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)})}} \sum_{w \in S} \sum_{\substack{w_1, w_2 \in T_1^w, \\ w_1 \neq w_2}} \\
&P_\rho \left[\Gamma_k^{n,s}(t) = S, \right. \\
&\quad \left. \max_{v \in T_{n-s-1}^{w_i}} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z, \text{ for all } i \in \{1, 2\} \right] \\
&= \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} \sum_{\substack{S \subset T_s, \\ |S| \in [e^{2\sqrt{\log b}(k+j)}, e^{2\sqrt{\log b}(k+j+1)})}} \sum_{w \in S} \sum_{\substack{w_1, w_2 \in T_1^w, \\ w_1 \neq w_2}} E_\rho \left[1_{\{\Gamma_k^{n,s}(t) = S\}} \right. \\
&\quad \left. \prod_{i \in \{1, 2\}} \tilde{P}_w \left(\max_{v \in T_{n-s-1}^{w_i}} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(L_{\tau(t)}^n(w))}^\downarrow} (v) \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \right) \right] \\
&\leq c_1 \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^{j^*(k,z)} e^{2\sqrt{\log b}(k+j)} P_\rho \left(|\Gamma_k^{n,s}(t)| \geq e^{2\sqrt{\log b}(k+j)} \right) \\
&\quad \cdot \tilde{P}_w \left(\max_{v \in T_{n-s}^w} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow((\sqrt{t} + a_s(t) - k)^2)}^\downarrow} (v) \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \right)^2, \tag{4.71}
\end{aligned}$$

where we have used Lemma 4.2.1 similarly to the argument in (4.58) in the second equality. Note that for $k \in \mathbb{Z}$ with $k + z \geq 0$, we have

$$\begin{aligned}
&\sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \\
&\geq (\sqrt{t} + a_s(t) - k) + a_{n-s}((\sqrt{t} + a_s(t) - k)^2) \\
&\quad + \left(\frac{3}{4\sqrt{\log b}} - \bar{c} \right) \log(s \wedge (n-s)) + k + z - c_2.
\end{aligned}$$

Thus, by Proposition 4.3.1(i), we have for each $w \in T_s$

$$\tilde{P}_w \left(\max_{v \in T_{n-s}^w} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow((\sqrt{t} + a_s(t) - k)^2)}^\downarrow} (v) \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \right)$$

$$\leq c_3 (\log(s \wedge (n-s)) + k) (s \wedge (n-s))^{-\frac{3}{2} + 2\bar{c}\sqrt{\log b}} e^{-2\sqrt{\log b}(k+z)}. \quad (4.72)$$

Set $\mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$ and $\mathbb{Z}_- := \mathbb{Z} \cap (-\infty, -1]$. By (4.71)-(4.72), Remark 4.4.3, and Lemma 4.4.4, we have for sufficiently large t_*

$$\begin{aligned} J_1 &\leq c_4 \sum_{\substack{k \in \mathbb{Z}_+, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=1}^{j^*(k,z)} e^{2\sqrt{\log b}(k+j)} \\ &\quad \cdot (1+j) e^{-2\sqrt{\log b} \left(j - \frac{3}{4\sqrt{\log b}} \log(k+j)^2 - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t} + a_s(t) - k - 1 + (k+j)^2}{\sqrt{t} + a_s(t) - k - 1} \right) \right)} \\ &\quad \cdot (\log(s \wedge (n-s)) + k)^2 (s \wedge (n-s))^{-3+4\bar{c}\sqrt{\log b}} e^{-4\sqrt{\log b}(k+z)} \\ &+ c_4 \sum_{\substack{k \in \mathbb{Z}_+, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} \sum_{j=-k}^0 e^{2\sqrt{\log b}(k+j)} \\ &\quad \cdot (\log(s \wedge (n-s)) + k)^2 (s \wedge (n-s))^{-3+4\bar{c}\sqrt{\log b}} e^{-4\sqrt{\log b}(k+z)} \\ &+ c_4 \sum_{\substack{k \in \mathbb{Z}_-, j=|k| \\ |k| \leq z}}^{\lceil z \rceil} e^{2\sqrt{\log b}(k+j)} \\ &\quad \cdot (1+j) e^{-2\sqrt{\log b} \left(j - \frac{3}{4\sqrt{\log b}} \log(\lceil (k+j)^2 \vee 1 \rceil) - \frac{1}{4\sqrt{\log b}} \log \left(\frac{\sqrt{t} + a_s(t) - k - 1 + \lceil (k+j)^2 \vee 1 \rceil}{\sqrt{t} + a_s(t) - k - 1} \right) \right)} \\ &\quad \cdot (\log(s \wedge (n-s)) + k)^2 (s \wedge (n-s))^{-3+4\bar{c}\sqrt{\log b}} e^{-4\sqrt{\log b}(k+z)} \\ &+ c_4 \sum_{\substack{k \in \mathbb{Z}_-, \\ |k| > z}} e^{2\sqrt{\log b}(k+|k|)} \cdot (1+|k|) e^{-2\sqrt{\log b} |k|} \\ &\leq c_5 (\log(s \wedge (n-s)))^8 (s \wedge (n-s))^{-3+4\bar{c}\sqrt{\log b} - 2\bar{c}\sqrt{\log b}} \\ &\quad + c_5 (\log(s \wedge (n-s))) (s \wedge (n-s))^{-2\bar{c}\sqrt{\log b}}. \end{aligned} \quad (4.73)$$

Next, we estimate J_2 . By Lemma 4.4.4 and Remark 4.4.3, we have

$$\begin{aligned} J_2 &\leq \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k - 1 \geq t_*}} c_6 (1 + j^*(k, z)) e^{-2\sqrt{\log b} j^*(k, z)} (\lceil (k + j^*(k, z))^2 \rceil \vee 1)^{3/2} \\ &\quad \cdot \sqrt{\frac{\sqrt{t} + a_s(t) - k - 1 + \lceil (k + j^*(k, z))^2 \rceil \vee 1}{\sqrt{t} + a_s(t) - k - 1}} \\ &\leq c_7 (\log(s \wedge (n-s)))^6 (s \wedge (n-s))^{-2\bar{c}\sqrt{\log b}}. \end{aligned} \quad (4.74)$$

Next, we estimate J_3 . By using Lemma 4.2.1 in a way similar to the argument in (4.71), we have

$$J_3 \leq c_8 \cdot \sum_{\substack{k \in \mathbb{Z}, \\ \sqrt{t} + a_s(t) - k \in (0, t_* + 1]}} \sum_{w \in T_s} \tilde{P}_w \left(\max_{v \in T_{n-s}^w} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow((t_*)^2)}^\downarrow(v)} \geq \sqrt{t} + a_n(t) - \bar{c} \log(s \wedge (n-s)) + z \right)^2. \quad (4.75)$$

By Proposition 4.3.1(i) and (4.75), we have

$$J_3 \leq c_9 s^6 b^{-3s} (s \wedge (n-s))^{4(\bar{c}-\underline{c})\sqrt{\log b}}. \quad (4.76)$$

Thus, by (4.70), (4.73), (4.74), and (4.76), we have (4.69). $\square$

4.5 Limiting tail of the maximum of local times

The aim of this section is to prove the following.

Proposition 4.5.1 *Fix positive sequences $(y_j^+)_{j \geq 1}, (y_j^-)_{j \geq 1}$ with $y_j^- \leq y_j^+$ for each $j \geq 1$ and $\lim_{j \rightarrow \infty} y_j^- = \infty$. For each $j \geq 1$, fix $(t_n^+(j))_{n \geq 1}, (t_n^-(j))_{n \geq 1}$ with $t_n^-(j) \leq t_n^+(j)$ for each $n \geq 1$ and $\lim_{n \rightarrow \infty} \sqrt{t_n^+(j)}/n = \lim_{n \rightarrow \infty} \sqrt{t_n^-(j)}/n = \theta \in [0, \infty]$. There exists $c_* > 0$ such that the following holds: for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that for each $j \geq j_0$, there exists $n_0(j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$,*

$$\frac{P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y_j \right)}{y_j e^{-2\sqrt{\log b} y_j}} \in \left[\frac{4}{\sqrt{\pi}} \beta_* \gamma_* - \varepsilon, \frac{4}{\sqrt{\pi}} \beta_* \gamma_* + \varepsilon \right], \quad (4.77)$$

uniformly in y_j and t satisfying that

$$y_j^- \leq y_j \leq y_j^+, \quad t \geq c_* n \log n \text{ and } t_n^-(j) \leq t \leq t_n^+(j), \quad (4.78)$$

where β_* and γ_* are the constants defined in (4.6) and (4.7).

We begin with some notation. Fix $\delta \in (0, 1)$ and $\kappa \in \left(\frac{1}{2\sqrt{\log b}}, \infty \right)$. For all $t > 0$, $y > 0$, $\ell(y) \in \mathbb{N}$, $n > \ell(y)$, and $v \in T_{n-\ell(y)}$, set

$$F_{v, y, \ell(y)}^n(t) := \begin{cases} \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y, & \text{for all } 0 \leq s \leq n - \ell(y), \end{cases}$$

$$\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \Big\},$$

$$\begin{aligned} & \tilde{F}_{v,y,\ell(y)}^n(t) \\ &:= \left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y + y^{\frac{1}{20}} + \kappa(\log(s \wedge (n - \ell(y) - s)))_+, \right. \\ & \quad \left. \text{for all } 0 \leq s \leq n - \ell(y), \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right\}, \end{aligned}$$

$$\Lambda_{y,\ell(y)}^n(t) := \sum_{v \in T_{n-\ell(y)}} 1_{F_{v,y,\ell(y)}^n(t)}, \quad \tilde{\Lambda}_{y,\ell(y)}^n(t) := \sum_{v \in T_{n-\ell(y)}} 1_{\tilde{F}_{v,y,\ell(y)}^n(t)}.$$

In Lemma 4.5.4, we will show that $P_\rho(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y) / \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)]$ is close to 1. To obtain an upper bound of the ration, we need the following:

Lemma 4.5.2 *There exist $c_1, c_2 \in (0, \infty)$, $y_0 > 0$, and $\{\delta_{y'} : y' > 0\}$ with $\lim_{y' \rightarrow \infty} \delta_{y'} = 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n \log n$,*

$$\begin{aligned} & \frac{\tilde{E}_\rho \left[\Lambda_{y,\ell(y)}^n(t) \right]}{\tilde{E}_\rho \left[\tilde{\Lambda}_{y,\ell(y)}^n(t) \right]} \\ & \geq (1 - \delta_y) \left(1 - c_2(\ell(y))^{-\frac{3}{2}} (\log \ell(y))^{60} - \delta_y y^{-1} (\ell(y))^{-\frac{3}{2}} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2}{n-\ell(y)} t} \right). \end{aligned} \tag{4.79}$$

Proof. Fix any $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, where we take $y_0 > 0$ large enough. Throughout the proof, given $n \in \mathbb{N}$, we assume that $t \geq c_* n \log n$, where c_* is a sufficiently large positive constant. Fix $v \in T_{n-\ell(y)}$. Let $\tilde{L}^\downarrow$ be a local time of a Brownian motion on $\tilde{T}^v \subset \tilde{T}_{\leq n}$ and set $\tilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v) > s\}$. Recall $\tilde{P}_{n-\ell(y)}^B$, $\tilde{B}$ defined in (4.37) and (4.38), and $\mu_{n-\ell(y),y}$, δ_y

in Lemma 4.2.6. Let $\mu_{n-\ell(y),y}^*$ be the density on the right-hand side of (4.22) with $s = n - \ell(y)$ and $z = y$. We have

$$\begin{aligned}
& \tilde{P}_\rho \left(\tilde{F}_{v,y,\ell(y)}^n(t) \right) - \tilde{P}_\rho \left(F_{v,y,\ell(y)}^n(t) \right) \\
&= \tilde{E}_\rho \left[1_{\left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + y + y^{1/20} + \kappa(\log(s \wedge (n-\ell(y)-s)))_+, \text{ for all } 0 \leq s \leq n-\ell(y) \right\}} \right. \\
&\quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^n(v))}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
&- \tilde{E}_\rho \left[1_{\left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + y, \text{ for all } 0 \leq s \leq n-\ell(y) \right\}} \right. \\
&\quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^n(v))}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
&\leq \tilde{E}_{n-\ell(y)}^B \left[1_{\left\{ -(1-\delta)\sqrt{t} - \frac{a_n(t)}{n}(n-\ell(y)) \leq \tilde{B}_{n-\ell(y)}/\sqrt{2} \leq y + y^{1/20} \right\}} \right. \\
&\quad 1_{\left\{ \tilde{B}_s/\sqrt{2} \leq y + y^{1/20} + \kappa(\log(s \wedge (n-\ell(y)-s)))_+, \text{ for all } 0 \leq s \leq n-\ell(y) \right\}} \\
&\quad e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_{n-\ell(y)} - \frac{(a_n(t))^2}{n^2}(n-\ell(y))} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + \frac{\tilde{B}_{n-\ell(y)}}{\sqrt{2}}}} \\
&\quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + \tilde{B}_{n-\ell(y)}/\sqrt{2})^2)}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
&- \tilde{E}_{n-\ell(y)}^B \left[1_{\left\{ -(1-\delta)\sqrt{t} - \frac{a_n(t)}{n}(n-\ell(y)) \leq \tilde{B}_{n-\ell(y)}/\sqrt{2} \leq y \right\}} 1_{\left\{ \tilde{B}_s/\sqrt{2} \leq y, \text{ for all } 0 \leq s \leq n-\ell(y) \right\}} \right. \\
&\quad e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_{n-\ell(y)} - \frac{(a_n(t))^2}{n^2}(n-\ell(y))} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + \frac{\tilde{B}_{n-\ell(y)}}{\sqrt{2}}}} \\
&\quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + \tilde{B}_{n-\ell(y)}/\sqrt{2})^2)}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
&\leq \int_{[0, y+y^{1/20}] \cup [-(1-\delta)\sqrt{t} - \frac{a_n(t)}{n}(n-\ell(y)), y - \frac{n-\ell(y)}{4y}]} \mu_{n-\ell(y),y}^*(x)
\end{aligned}$$

$$\begin{aligned}
& \cdot \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + x}} \\
& \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + x)^2)}(u) > \sqrt{t} + a_n(t) + y \right) \\
& \cdot e^{-\frac{2a_n(t)}{n}x - \frac{(a_n(t))^2}{n^2}(n - \ell(y))} dx \\
& + \int_{y - \frac{n - \ell(y)}{4y}}^0 \delta_y \mu_{n - \ell(y), y}(x) \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + x}} \\
& \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + x)^2)}(u) > \sqrt{t} + a_n(t) + y \right) \\
& \cdot e^{-\frac{2a_n(t)}{n}x - \frac{(a_n(t))^2}{n^2}(n - \ell(y))} dx \\
& =: J_1 + J_2, \tag{4.80}
\end{aligned}$$

where we have used Lemma 4.2.1 in the first equality, Lemma 4.2.5 and (4.16) in the second inequality, and Lemma 4.2.6 in the third inequality. By Proposition 4.3.1(i) and the assumption $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}y^{1/20}}$, taking $n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0(y, \ell(y))$

$$J_1 \leq c_1 b^{-(n - \ell(y))} y e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} (\log \ell(y))^{60}. \tag{4.81}$$

Next, we estimate J_2 . We have

$$\begin{aligned}
J_2 & \leq \delta_y E_0^B \left[1_{\left\{ -(1-\delta)\sqrt{t} \leq B_s/\sqrt{2} \leq \frac{a_n(t)}{n}s + y, \text{ for all } 0 \leq s \leq n - \ell(y) \right\}} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + B_{n - \ell(y)}/\sqrt{2}}} \right. \\
& \quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + B_{n - \ell(y)}/\sqrt{2})^2)}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
& \quad + \delta_y E_0^B \left[1_{\left\{ -(1-\delta)\sqrt{t} \leq B_{n - \ell(y)}/\sqrt{2} \leq \frac{a_n(t)}{n}(n - \ell(y)), \min_{0 \leq s \leq n - \ell(y)} B_s/\sqrt{2} < -(1-\delta)\sqrt{t} \right\}} \right. \\
& \quad \cdot \sqrt{\frac{\sqrt{t}}{\sqrt{t} + B_{n - \ell(y)}/\sqrt{2}}} \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + B_{n - \ell(y)}/\sqrt{2})^2)}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
& =: J_{2,1} + J_{2,2}. \tag{4.82}
\end{aligned}$$

By Lemma 4.2.5 and (4.16), we have for each $n \geq 1$

$$J_{2,1} \leq c_2 \delta_y \tilde{P}_\rho(F_{v,y,\ell(y)}^n(t)), \tag{4.83}$$

where we have used that under the event that $\sqrt{X_s/2} \geq \delta\sqrt{t}$ for all $0 \leq s \leq n - \ell(y)$, we have $\exp\left(\frac{3}{8} \int_0^{n-\ell(y)} \frac{ds}{X_s}\right) \leq c_2$ under the assumption $t \geq c_* n \log n$.

By (4.48), the symmetry of B and Proposition 4.3.1(i), taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$

$$\begin{aligned} J_{2,2} &\leq c_3 \delta_y \int_{-\frac{a_n(t)}{n}(n-\ell(y))}^{(1-\delta)\sqrt{t}} \int_{(1-\delta)\sqrt{t}}^{\infty} \frac{2q-p}{(n-\ell(y))^{3/2}} e^{-\frac{(2q-p)^2}{n-\ell(y)}} \\ &\quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow((\sqrt{t}-p)^2)}^\downarrow}(u) \geq \sqrt{t} + a_n(t) + y \right) dp dq \\ &\leq c_4 \delta_y b^{-(n-\ell(y))} e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2 t}{n-\ell(y)}}. \end{aligned} \quad (4.84)$$

Thus, by (4.80)-(4.84), we have

$$\begin{aligned} (1 + c_2 \delta_y) \tilde{E}_\rho(\Lambda_{y,\ell(y)}^n(t)) &\geq \tilde{E}_\rho(\tilde{\Lambda}_{y,\ell(y)}^n(t)) - c_1 y e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} (\log \ell(y))^{60} \\ &\quad - c_4 \delta_y e^{-2\sqrt{\log b} y} (\ell(y))^{-3/2} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2 t}{n-\ell(y)}}. \end{aligned} \quad (4.85)$$

For the rest of the proof, we obtain a lower bound of $\tilde{E}_\rho(\tilde{\Lambda}_{y,\ell(y)}^n(t))$. Recall (4.26). We have

$$\begin{aligned} &\tilde{E}_\rho(\tilde{\Lambda}_{y,\ell(y)}^n(t)) \\ &\geq \tilde{P}_\rho \left[\left\{ \text{there exists } v \in T_{n-\ell(y)} \text{ with } \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \geq \delta\sqrt{t}, \text{ for all } s \leq n - \ell(y), \right. \right. \\ &\quad \left. \left. \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right\} \cap \left(G_{y+y^{1/20}-1}^{n-\ell(y)}(t) \right)^c \right] \\ &\geq P_\rho \left(\max_{u \in T_n} \sqrt{L_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right) - \tilde{P}_\rho \left(G_{y+y^{1/20}-1}^{n-\ell(y)}(t) \right) \\ &\quad - \tilde{P}_\rho \left[\text{there exists } v \in T_{n-\ell(y)} \text{ with } \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \delta\sqrt{t}, \text{ for some } s \leq n - \ell(y), \right. \\ &\quad \left. \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right]. \end{aligned} \quad (4.86)$$

By Proposition 4.3.1(ii), taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$

$$P_\rho \left(\max_{u \in T_n} \sqrt{L_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right) \geq c_5 y e^{-2\sqrt{\log b} y}. \quad (4.87)$$

By Lemma 4.3.2, we have

$$\tilde{P}_\rho \left(G_{y+y^{1/20}-1}^{n-\ell(y)}(t) \right) \leq c_6 e^{-2\sqrt{\log b} y^{1/20}} y e^{-2\sqrt{\log b} y}. \quad (4.88)$$

We estimate the third term on the right-hand side of (4.86). Fix $v^* \in T_{n-\ell(y)}$. By Lemma 4.2.1, 4.2.5, and (4.16), taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$

$$\begin{aligned} & \tilde{P}_\rho \left[\text{there exists } v \in T_{n-\ell(y)} \text{ such that for some } s \leq n - \ell(y), \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} < \delta \sqrt{t}, \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right] \\ & \leq b^{n-\ell(y)} \tilde{E}_\rho \left[1_{\left\{ \sqrt{\tilde{L}_{\tau(t)}^n(v^*)} < \delta \sqrt{t}, \text{ for some } 0 \leq s \leq n - \ell(y), \sqrt{\tilde{L}_{\tau(t)}^n(v^*)} > 0 \right\}} \right. \\ & \quad \left. \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell(y)}^{v^*}} \sqrt{\tilde{L}_{\tau(t)}^{\downarrow}(\tilde{L}_{\tau(t)}^n(v^*))}(u)} > \sqrt{t} + a_n(t) + y \right) \right] \\ & \leq b^{n-\ell(y)} E_0^B \left[1_{\left\{ \frac{B_s}{\sqrt{2}} < -(1-\delta)\sqrt{t}, \text{ for some } s \leq n - \ell(y), \frac{B_{n-\ell(y)}}{\sqrt{2}} > -\sqrt{t} \right\}} \sqrt{\frac{\sqrt{2t}}{\sqrt{2t} + B_{n-\ell(y)}}} \right. \\ & \quad \left. \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell(y)}^{v^*}} \sqrt{\tilde{L}_{\tau(t)}^{\downarrow}((\sqrt{t} + B_{n-\ell(y)}/\sqrt{2})^2)}(u)} > \sqrt{t} + a_n(t) + y \right) \right] \\ & \leq c_7 (\ell(y))^{-3/2} (n + \sqrt{t})^3 e^{-2(1-\delta)\sqrt{\log b} \sqrt{t}} e^{-2\sqrt{\log b} y}. \quad (4.89) \end{aligned}$$

where we have used Proposition 4.3.1(i), (4.48), and the Gaussian density of $B_{n-\ell(y)}$ in the last inequality.

By (4.86)-(4.89), taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$

$$\tilde{E}_\rho(\tilde{\Lambda}_{y, \ell(y)}^n(t)) \geq c_8 y e^{-2\sqrt{\log b} y}. \quad (4.90)$$

By (4.85) and (4.90), we have (4.79). $\square$

To obtain a lower bound of $P_\rho(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y) / \tilde{E}_\rho[\Lambda_{y, \ell(y)}^n(t)]$, we need the following:

Lemma 4.5.3 *There exist $c_1, c_2, c_3 \in (0, \infty)$, $y_0 > 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n \log n$,*

$$\frac{\tilde{E}_\rho \left[\left(\Lambda_{y, \ell(y)}^n(t) \right)^2 \right]}{\tilde{E}_\rho \left[\Lambda_{y, \ell(y)}^n(t) \right]} \leq 1 + c_2 y^{-1/2} + c_3 y e^{-2\sqrt{\log b} y}. \quad (4.91)$$

Proof. Fix any $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, where we take $y_0 > 0$ large enough. Throughout the proof, given $n \in \mathbb{N}$, we assume $t \geq c_* n \log n$ for some sufficiently large $c_* > 0$. We have

$$\begin{aligned} \tilde{E}_\rho \left[\left(\Lambda_{y, \ell(y)}^n(t) \right)^2 \right] &= \tilde{E}_\rho \left[\Lambda_{y, \ell(y)}^n(t) \right] \\ &\quad + \sum_{k=1}^{n-\ell(y)} \sum_{\substack{v, w \in T_{n-\ell(y)}, \\ |v \wedge w| = n-\ell(y)-k}} \tilde{P}_\rho \left(F_{v, y, \ell(y)}^n(t) \cap F_{w, y, \ell(y)}^n(t) \right). \end{aligned} \quad (4.92)$$

Fix $1 \leq k \leq n - \ell(y) - 1$ and $v, w \in T_{n-\ell(y)}$ with $|v \wedge w| = n - \ell(y) - k$. Let $\tilde{L}^\downarrow$ be a local time of a Brownian motion on $\tilde{T}^{v \wedge w} \subset \tilde{T}_{\leq n}$ and set $\tilde{\tau}^\downarrow(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v \wedge w) > s\}$. We have

$$\begin{aligned} &\tilde{P}_\rho \left(F_{v, y, \ell(y)}^n(t) \cap F_{w, y, \ell(y)}^n(t) \right) \\ &= \tilde{E}_\rho \left[1_{\left\{ \delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y, \text{ for all } 0 \leq s \leq n - \ell(y) - k \right\}} \right. \\ &\quad \prod_{x \in \{v, w\}} \tilde{P}_{v \wedge w} \left(\delta \sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^n(v \wedge w))(x_s)} \leq \sqrt{t} + \frac{a_n(t)}{n} s + y, \right. \\ &\quad \left. \left. \text{for all } n - \ell(y) - k \leq s \leq n - \ell(y), \right. \right. \\ &\quad \left. \left. \max_{u \in T_{\ell(y)}^x} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t)}^n(v \wedge w))(u)} > \sqrt{t} + a_n(t) + y \right) \right], \end{aligned} \quad (4.93)$$

where we have used Lemma 4.2.1 in a way similar to the argument in (4.58).

Fix $0 \leq i \leq \lfloor (1 - \delta) \sqrt{t} + \frac{a_n(t)}{n} (n - \ell(y) - k) + y \rfloor$. Assume that

$$\sqrt{L_{\tau(t)}^n(v \wedge w)} \in \sqrt{t} + \frac{a_n(t)}{n} (n - \ell(y) - k) + y - i + (-1, 0]. \quad (4.94)$$

Under the assumption (4.94), we estimate the probabilities in (4.93). Let $\tilde{L}^{\downarrow\downarrow}$ be a local time of a Brownian motion on $\tilde{T}^v \subset \tilde{T}_{\leq n}$ and set $\tilde{\tau}^{\downarrow\downarrow}(s) := \inf\{r \geq 0 : \tilde{L}_r^{\downarrow\downarrow}(v) > s\}$. Recall $\tilde{P}_k^B$ and $\tilde{B}$ defined in (4.37) and (4.38). We have

$$\begin{aligned}
& \tilde{P}_{v \wedge w} \left(\delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^{\downarrow}(L_{\tau(t)}^n(v \wedge w))}^{\downarrow}}(v_s) \leq \sqrt{t} + \frac{a_n(t)}{n}s + y, \right. \\
& \quad \text{for all } n - \ell(y) - k \leq s \leq n - \ell(y), \\
& \quad \left. \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^{\downarrow}(L_{\tau(t)}^n(v \wedge w))}^{\downarrow}}(u) > \sqrt{t} + a_n(t) + y \right) \\
&= \tilde{E}_{v \wedge w} \left[1_{\left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}^{\downarrow}(L_{\tau(t)}^n(v \wedge w))}^{\downarrow}}(v_s) \leq \sqrt{t} + \frac{a_n(t)}{n}s + y, \text{ for all } n - \ell(y) - k \leq s \leq n - \ell(y) \right\}} \right. \\
& \quad \left. \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^{\downarrow\downarrow}(\tilde{L}_{\tilde{\tau}^{\downarrow}(L_{\tau(t)}^n(v \wedge w))}^{\downarrow})}^{\downarrow\downarrow}}(u) > \sqrt{t} + a_n(t) + y \right) \right] \\
&\leq \tilde{E}_k^B \left[1_{\left\{ -(1-\delta)\sqrt{t} - \frac{a_n(t)}{n}(n - \ell(y)) - y + i \leq \tilde{B}_k/\sqrt{2} \leq i+1, \tilde{B}_s/\sqrt{2} \leq i+1, \text{ for all } 0 \leq s \leq k \right\}} \right. \\
& \quad e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_k - \frac{(a_n(t))^2}{n^2}k} \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y - i - 1 + \tilde{B}_k/\sqrt{2}}} \\
& \quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^{\downarrow\downarrow}((\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y - i + \tilde{B}_k/\sqrt{2})^2)}^{\downarrow\downarrow}}(u) > \sqrt{t} + a_n(t) + y \right) \Big] \\
&= \int_0^{(1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y + 1} \frac{1}{\sqrt{\pi k}} \left(e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \right) \\
& \quad e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}(i+1) - \frac{(a_n(t))^2}{n^2}k} \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y - z}} \\
& \quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^{\downarrow\downarrow}((\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y + 1 - z)^2)}^{\downarrow\downarrow}}(u) > \sqrt{t} + a_n(t) + y \right) dz, \\
& \tag{4.95}
\end{aligned}$$

where we have used Lemma 4.2.1 in the first equality, Lemma 4.2.5, (4.16), and (4.94) in the second inequality, and the density (4.21) in the last equality. By Proposition 4.3.1(i), taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $0 \leq z \leq (1 - \delta)\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y + 1$ and $n \geq n_0$

$$\begin{aligned} & \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau} \downarrow \downarrow}^{\downarrow \downarrow} \left(\left(\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y + 1 - z \right)^2 \right)}(u) > \sqrt{t} + a_n(t) + y \right) \\ & \leq c_1 (\log \ell(y) + z) (\ell(y))^{-3/2} e^{-2\sqrt{\log b}} z e^{-c_2 \frac{z^2}{\ell(y)}} \\ & \quad \cdot \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y + 1 - z}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y + 1 - z + \ell(y)}}. \quad (4.96) \end{aligned}$$

We also use the following estimate: $e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \leq 1$ for each $1 \leq k \leq \lfloor y \rfloor$, and $e^{-\frac{(i+1-z)^2}{k}} - e^{-\frac{(i+1+z)^2}{k}} \leq \frac{4(i+1)z}{k}$ for each $\lfloor y \rfloor \leq k \leq n - \ell(y) - 1$. By this and (4.95)-(4.96), the right-hand side of (4.95) is bounded from above by

$$c_3 A b^{-k} e^{-\frac{2a_n(t)}{n}i} e^{\frac{3k \log n}{2n}} e^{\frac{k \log \left(\frac{\sqrt{t}+n}{\sqrt{t}} \right)}{2n}} \sqrt{\frac{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i}{\sqrt{t} + n}}, \quad (4.97)$$

where $A := k^{-1/2}(\ell(y))^{-1/2}$ if $1 \leq k \leq \lfloor y \rfloor$, and $A := k^{-3/2}(i+1)$ if $\lfloor y \rfloor \leq k \leq n - \ell(y) - 1$.

Recall the events in the indicator function in (4.93) and in (4.94). We estimate the probability of the intersection of these events. Using Lemma 4.2.5, (4.16), and the density (4.21) similarly to the proof of Lemma 4.3.4 and taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for $n \geq n_0$

$$\begin{aligned} & \tilde{P}_\rho \left(\delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n}(v_s) \leq \sqrt{t} + \frac{a_n(t)}{n}s + y, \text{ for all } 0 \leq s \leq n - \ell(y) - k, \right. \\ & \quad \left. \sqrt{L_{\tau(t)}^n}(v \wedge w) \in \sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - i + (-1, 0] \right) \\ & \leq \int_i^{i+1} \frac{1}{\sqrt{\pi(n - \ell(y) - k)}} \left(e^{-\frac{(y-z)^2}{n - \ell(y) - k}} - e^{-\frac{(y+z)^2}{n - \ell(y) - k}} \right) \\ & \quad e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}y - \frac{(a_n(t))^2}{n^2}(n - \ell(y) - k)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y) - k) + y - z}} dz. \quad (4.98) \end{aligned}$$

We will use the following: $e^{-\frac{(y-z)^2}{n-\ell(y)-k}} - e^{-\frac{(y+z)^2}{n-\ell(y)-k}} \leq \frac{4yz}{n-\ell(y)-k}$ for each $1 \leq k \leq n - \ell(y) - \lfloor y \rfloor$, and $e^{-\frac{(y-z)^2}{n-\ell(y)-k}} - e^{-\frac{(y+z)^2}{n-\ell(y)-k}} \leq 1$ for each $n - \ell(y) - \lfloor y \rfloor \leq k \leq n - \ell(y) - 1$. By this, the right-hand side of (4.98) is bounded from above by

$$c_4 A' b^{-(n-\ell(y)-k)} e^{\frac{2a_n(t)}{n} i} e^{-2\sqrt{\log b} y} \cdot e^{\frac{3(n-\ell(y)-k)}{2n} \log n} e^{\frac{(n-\ell(y)-k) \log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)-k) + y - i - 1}}, \quad (4.99)$$

where $A' := (n - \ell(y) - k)^{-3/2}(i+1)y$ if $1 \leq k \leq n - \ell(y) - \lfloor y \rfloor$, and $A' := (n - \ell(y) - k)^{-1/2}$ if $n - \ell(y) - \lfloor y \rfloor \leq k \leq n - \ell(y) - 1$.

By (4.93), (4.97), and (4.99), we have

$$\sum_{k=1}^{n-\ell(y)-1} \sum_{\substack{v,w \in T_{n-\ell(y)}, \\ |v \wedge w| = n-\ell(y)-k}} \tilde{P}_\rho(F_{v,y,\ell(y)}^n(t) \cap F_{w,y,\ell(y)}^n(t)) \leq c_5 y^{1/2} e^{-2\sqrt{\log b} y}. \quad (4.100)$$

For the case $k = n - \ell(y)$, by the independence of excursions of a Brownian motion around ρ , we have

$$\begin{aligned} & \sum_{\substack{v,w \in T_{n-\ell(y)}, \\ |v \wedge w| = 0}} \tilde{P}_\rho(F_{v,y,\ell(y)}^n(t) \cap F_{w,y,\ell(y)}^n(t)) \\ &= \sum_{\substack{v,w \in T_{n-\ell(y)}, \\ |v \wedge w| = 0}} \tilde{P}_\rho(F_{v,y,\ell(y)}^n(t)) \tilde{P}_\rho(F_{w,y,\ell(y)}^n(t)) \\ &\leq \left(\tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] \right)^2. \end{aligned} \quad (4.101)$$

Thus, by (4.92), (4.100), and (4.101), we have

$$\tilde{E}_\rho \left[\left(\Lambda_{y,\ell(y)}^n(t) \right)^2 \right] \leq \tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] + c_5 y^{1/2} e^{-2\sqrt{\log b} y} + \left(\tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)] \right)^2. \quad (4.102)$$

We will obtain upper and lower bounds of $\tilde{E}_\rho[\Lambda_{y,\ell(y)}^n(t)]$. By (4.85) and (4.90),

taking $n_0 = n_0(y, \ell(y))$ large enough, we have for all $n \geq n_0$ (recall that we assumed $t \geq c_* n \log n$ and c_* is sufficiently large positive constant)

$$\tilde{E}_\rho[\Lambda_{y, \ell(y)}^n(t)] \geq c_6 y e^{-2\sqrt{\log b} y}. \quad (4.103)$$

Fix $v^* \in T_{n-\ell(y)}$. Let $\hat{L}^\downarrow$ be a local time of a Brownian motion on $\tilde{T}^{v^*} \subset \tilde{T}_{\leq n}$ and set $\hat{\tau}^\downarrow(s) := \inf\{r \geq 0 : \hat{L}_r^\downarrow(v^*) > s\}$. By using Lemma 4.2.1, 4.2.5, (4.16), and the density (4.21) similarly to the argument in (4.95), we have

$$\begin{aligned} \tilde{E}_\rho[\Lambda_{y, \ell(y)}^n(t)] &= b^{(n-\ell(y))} \tilde{P}_\rho(F_{v^*, y, \ell(y)}^n(t)) \\ &\leq b^{(n-\ell(y))} \int_0^{(1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y))+y} \frac{1}{\sqrt{\pi(n-\ell(y))}} \left(e^{-\frac{(y-z)^2}{n-\ell(y)}} - e^{-\frac{(y+z)^2}{n-\ell(y)}} \right) \\ &\quad \cdot e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}y - \frac{(a_n(t))^2}{n^2}(n-\ell(y))} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + y - z}} \\ &\quad \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell(y)}^{v^*}} \sqrt{\hat{L}_{\hat{\tau}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y))+y-z))^2}}^\downarrow(u) > \sqrt{t} + a_n(t) + y \right) dz \\ &\leq c_7 y e^{-2\sqrt{\log b} y}, \end{aligned} \quad (4.104)$$

where we have used Proposition 4.3.1(i) and the inequality $e^{-\frac{(y-z)^2}{n-\ell(y)}} - e^{-\frac{(y+z)^2}{n-\ell(y)}} \leq \frac{4yz}{n-\ell(y)}$ in the last inequality. Thus, by (4.102)-(4.104), we have (4.91). $\square$
Using Lemma 4.5.2 and 4.5.3, we prove the following:

Lemma 4.5.4 *There exist $c_1, c_2, c_3 \in (0, \infty)$, $y_0 > 0$, and $\{\delta_{y'} : y' > 0\}$ with $\lim_{y' \rightarrow \infty} \delta_{y'} = 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n \log n$,*

$$\begin{aligned} &\frac{P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right)}{\tilde{E}_\rho \left[\Lambda_{y, \ell(y)}^n(t) \right]} \\ &\leq (1 + \delta_y) \left(1 + c_2 e^{-2\sqrt{\log b} y^{1/20}} + c_2 (\ell(y))^{-3/2} (n + \sqrt{t})^3 e^{-2\sqrt{\log b} (1-\delta)\sqrt{t}} \right) \\ &\quad \cdot \left(1 + c_2 (\ell(y))^{-3/2} (\log \ell(y))^{60} + c_2 \delta_y y^{-1} (\ell(y))^{-3/2} n^3 (n + \sqrt{t}) e^{-\frac{(1-\delta)^2}{n-\ell(y)} t} \right), \end{aligned} \quad (4.105)$$

$$\frac{P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right)}{\tilde{E}_\rho \left[\Lambda_{y, \ell(y)}^n(t) \right]} \geq 1 - c_3 y^{-1/2} - c_3 y e^{-2\sqrt{\log b} y}. \quad (4.106)$$

Proof. Fix any $y \geq y_0$, $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, $n \geq n_0$, and $t \geq c_* n \log n$, where we take $y_0 > 0$, $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$, $c_* > 0$ large enough. We first obtain the upper bound. We have

$$\begin{aligned} & P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right) \\ &= P_\rho \left(\text{there exists } v \in T_{n-\ell(y)} \text{ with } \max_{u \in T_{\ell(y)}^v} \sqrt{L_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y \right) \\ &\leq \tilde{E}_\rho[\tilde{\Lambda}_{y, \ell(y)}^n(t)] + \tilde{P}_\rho \left(G_{y+y^{1/20}-1}^{n-\ell(y)} \right) \\ &\quad + \tilde{P}_\rho \left(\text{there exists } v \in T_{n-\ell(y)} \text{ with } \max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tau(t)}^n(u)} > \sqrt{t} + a_n(t) + y, \right. \\ &\quad \left. \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} < \delta \sqrt{t}, \text{ for some } 0 \leq s \leq n - \ell(y) \right) \\ &\leq \left(1 + c_1 e^{-2\sqrt{\log b} y^{\frac{1}{20}}} + c_1 (\ell(y))^{-\frac{3}{2}} (n + \sqrt{t})^3 e^{-2\sqrt{\log b} (1-\delta)\sqrt{t}} \right) \tilde{E}_\rho[\tilde{\Lambda}_{y, \ell(y)}^n(t)], \end{aligned} \quad (4.107)$$

where we have used Lemma 4.3.2, (4.89), and (4.90) in the last inequality. By (4.107) and Lemma 4.5.2, we have (4.105). By Lemma 4.5.3, we have

$$\begin{aligned} & P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y \right) \\ &\geq \tilde{P}_\rho \left(\Lambda_{y, \ell(y)}^n(t) \geq 1 \right) \\ &\geq \frac{\left(\tilde{E}_\rho[\Lambda_{y, \ell(y)}^n(t)] \right)^2}{\tilde{E}_\rho \left[\left(\Lambda_{y, \ell(y)}^n(t) \right)^2 \right]} \\ &\geq \left(1 - c_2 y^{-1/2} - c_2 y e^{-2\sqrt{\log b} y} \right) \tilde{E}_\rho[\Lambda_{y, \ell(y)}^n(t)], \end{aligned}$$

which proves (4.106). $\square$

For each interval $I \subset \mathbb{R}$, set

$$\Lambda_{y,\ell(y),I}^n(t) := \sum_{v \in T_{n-\ell(y)}} 1_{F_{v,y,\ell(y)}^n(t) \cap \left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in \sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + I \right\}}. \quad (4.108)$$

Set the interval

$$J_{\ell(y)} := \left(-\ell(y) + y, -(\ell(y))^{2/5} + y \right]. \quad (4.109)$$

Then the following holds:

Lemma 4.5.5 *There exist $c_1, c_2 \in (0, \infty)$ and $y_0 > 0$ such that the following holds: for all $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, there exists $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ such that for all $n \geq n_0$ and $t \geq c_1 n \log n$,*

$$\frac{\tilde{E}_\rho \left[\Lambda_{y,\ell(y),J_{\ell(y)}}^n(t) \right]}{\tilde{E}_\rho \left[\Lambda_{y,\ell(y)}^n(t) \right]} \geq 1 - c_2(\ell(y))^{-3/10}. \quad (4.110)$$

Proof. Fix any $y \geq y_0$ and $\ell(y) > e^{\frac{8\sqrt{\log b}}{3}} y^{1/20}$, where we take $y_0 > 0$ large enough. Throughout the proof, given $n \in \mathbb{N}$, we assume that $t \geq c_* n \log n$ for some sufficiently large $c_* > 0$. Fix $v \in T_{n-\ell(y)}$ and any interval $I \subset \mathbb{R}$. Let $\tilde{L}^\downarrow$ be a local time of a Brownian motion on $\tilde{T}^v \subset \tilde{T}_{\leq n}$ and set $\tau^\downarrow(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v) > s\}$. By using Lemma 4.2.1, 4.2.5, (4.16), and the density (4.21) similarly to the argument in (4.104), taking $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough, we have for all $n \geq n_0$

$$\begin{aligned} & \tilde{P}_\rho \left(F_{v,y,\ell(y)}^n(t) \cap \left\{ \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in \sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + I \right\} \right) \\ & \leq \int_{(y-I) \cap [0, (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + y]} \frac{1}{\sqrt{\pi(n-\ell(y))}} \left(e^{-\frac{(y-z)^2}{n-\ell(y)}} - e^{-\frac{(y+z)^2}{n-\ell(y)}} \right) \\ & \quad \cdot e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}y - \frac{(a_n(t))^2}{n^2}(n-\ell(y))} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + y - z}} \\ & \quad \cdot \tilde{P}_v \left(\max_{u \in T_{\ell(y)}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + y - z)^2)}^\downarrow}(u) > \sqrt{t} + a_n(t) + y \right) dz \\ & \leq c_1 b^{-(n-\ell(y))} (\ell(y))^{-3/2} y e^{-2\sqrt{\log b} y} \\ & \quad \cdot \int_{(y-I) \cap [0, (1-\delta)\sqrt{t} + \frac{a_n(t)}{n}(n-\ell(y)) + y]} z (\log \ell(y) + z) e^{-c_2 \frac{z^2}{\ell(y)}} \end{aligned}$$

$$\cdot \sqrt{\frac{\sqrt{t} + n}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell(y)) + y - z + \ell(y)}} dz, \quad (4.111)$$

where we have used Proposition 4.3.1(i) and the inequality $e^{-\frac{(y-z)^2}{n-\ell(y)}} - e^{-\frac{(y+z)^2}{n-\ell(y)}} \leq \frac{4yz}{n-\ell(y)}$ in the last inequality. In the cases $I = (-\infty, y - \ell(y)]$ and $I = (y - (\ell(y))^{2/5}, \infty)$, the right-hand side of (4.111) is bounded from above by

$$c_3 b^{-(n-\ell(y))} (\ell(y))^{-3/10} y e^{-2\sqrt{\log b} y}$$

for all $n \geq n_0$, where $n_0 = n_0(y, \ell(y)) \in \mathbb{N}$ large enough. By this and (4.103), we have

$$\begin{aligned} & \tilde{E}_\rho [\Lambda_{y, \ell(y)}^n(t)] \\ &= \tilde{E}_\rho [\Lambda_{y, \ell(y), J_{\ell(y)}}^n(t)] + \tilde{E}_\rho [\Lambda_{y, \ell(y), (-\infty, y - \ell(y)]}^n(t)] + \tilde{E}_\rho [\Lambda_{y, \ell(y), (y - (\ell(y))^{2/5}, \infty)}^n(t)] \\ &\leq \tilde{E}_\rho [\Lambda_{y, \ell(y), J_{\ell(y)}}^n(t)] + c_4 (\ell(y))^{-3/10} \tilde{E}_\rho [\Lambda_{y, \ell(y)}^n(t)], \end{aligned}$$

which implies (4.110). $\square$

Proof of Proposition 4.5.1. Let $(h_v)_{v \in T}$ be a BRW on T defined in Section 4.1. By Lemma 4.7.7, one can show that the sequence

$$\left(\int_{\ell^{2/5}}^{\ell} z e^{2\sqrt{\log b} z} \mathbb{P} \left[\max_{u \in T_\ell} h_u / \sqrt{2} > \sqrt{\log b} \ell + z \right] dz \right)_{\ell \geq 1} \quad (4.112)$$

is bounded from above and away from 0. Fix a nondecreasing sequence $(\ell_0(y_k^+))_{k \geq 1}$ with $\ell_0(y_k^+) > e^{\frac{8\sqrt{\log b}}{3}(y_k^+)^{1/20}}$ for each $k \geq 1$. By the boundedness of the sequence (4.112), there exists a subsequence $(\ell_0(y_{k_j}^+))_{j \geq 1}$ of $(\ell_0(y_k^+))_{k \geq 1}$ such that the finite positive limit

$$\tilde{\gamma}_* := \lim_{j \rightarrow \infty} \int_{(\ell_0(y_{k_j}^+))^{2/5}}^{\ell_0(y_{k_j}^+)} z e^{2\sqrt{\log b} z} \mathbb{P} \left[\max_{u \in T_{\ell_0(y_{k_j}^+)}} h_u / \sqrt{2} > \sqrt{\log b} \ell_0(y_{k_j}^+) + z \right] dz \quad (4.113)$$

exists. We set

$$\ell_j := \ell_0(y_{k_j}^+).$$

Note that by the definition of $\ell_0(y_j^+)$, for any y_j with $y_j \leq y_j^+$, we have

$$\ell_j \geq \ell_0(y_j^+) > e^{\frac{8\sqrt{\log b}}{3}(y_j^+)^{1/20}} \geq e^{\frac{8\sqrt{\log b}}{3}(y_j)^{1/20}}.$$

Fix $j \geq 1$ and y_j with $y_j^- \leq y_j \leq y_j^+$. Recall (4.108) and the interval (4.109). Fix $v^* \in T_{n-\ell_j}$. Let $\tilde{L}^\downarrow$ be a local time of a Brownian motion on $\tilde{T}^{v^*} \subset \tilde{T}_{\leq n}$ and set $\tilde{\tau}(s) := \inf\{r \geq 0 : \tilde{L}_r^\downarrow(v^*) > s\}$. Recall $\tilde{P}_{n-\ell_j}^B$ and $\tilde{B}$ defined in (4.37) and (4.38). We have for $t \geq n \log n$

$$\begin{aligned} & b^{-n+\ell_j} \tilde{E}_\rho \left[\Lambda_{y_j, \ell_j, J_{\ell_j}}^n(t) \right] \\ &= \tilde{E}_\rho \left[1_{\left\{ \delta\sqrt{t} \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v^*)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + y_j, \text{ for all } 0 \leq s \leq n-\ell_j, \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v^*)} \in \sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + J_{\ell_j} \right\}} \right. \\ & \quad \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(v^*))}(u) > \sqrt{t} + a_n(t) + y_j \right) \Big] \\ &= \left(1 + O\left(\frac{1}{\log n}\right) \right) E_0^B \left[1_{\left\{ -(1-\delta)\sqrt{t} \leq B_s/\sqrt{2} \leq \frac{a_n(t)}{n}s + y_j, \text{ for all } 0 \leq s \leq n-\ell_j \right\}} \right. \\ & \quad 1_{\left\{ B_{n-\ell_j}/\sqrt{2} \in \frac{a_n(t)}{n}(n-\ell_j) + J_{\ell_j} \right\}} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + B_{n-\ell_j}/\sqrt{2}}} \\ & \quad \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + B_{n-\ell_j}/\sqrt{2})^2)}(u) > \sqrt{t} + a_n(t) + y_j \right) \Big] \\ &= \left(1 + O\left(\frac{1}{\log n}\right) \right) \tilde{E}_{n-\ell_j}^B \left[1_{\left\{ \tilde{B}_s/\sqrt{2} \leq y_j, \text{ for all } 0 \leq s \leq n-\ell_j, \tilde{B}_{n-\ell_j}/\sqrt{2} \in J_{\ell_j} \right\}} \right. \\ & \quad e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_{n-\ell_j} - \frac{(a_n(t))^2}{n^2}(n-\ell_j)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + \tilde{B}_{n-\ell_j}/\sqrt{2}}} \\ & \quad \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow((\sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + \tilde{B}_{n-\ell_j}/\sqrt{2})^2)}(u) > \sqrt{t} + a_n(t) + y_j \right) \Big] \\ & \quad - \left(1 + O\left(\frac{1}{\log n}\right) \right) \tilde{E}_{n-\ell_j}^B \left[1_{\left\{ \tilde{B}_s/\sqrt{2} \leq y_j, \text{ for all } 0 \leq s \leq n-\ell_j, \tilde{B}_{n-\ell_j}/\sqrt{2} \in J_{\ell_j} \right\}} \right. \\ & \quad 1_{\left\{ \tilde{B}_s/\sqrt{2} < -(1-\delta)\sqrt{t} - \frac{a_n(t)}{n}s, \text{ for some } 0 \leq s \leq n-\ell_j \right\}} \\ & \quad e^{-\frac{\sqrt{2}a_n(t)}{n}\tilde{B}_{n-\ell_j} - \frac{(a_n(t))^2}{n^2}(n-\ell_j)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n}(n-\ell_j) + \tilde{B}_{n-\ell_j}/\sqrt{2}}} \Big] \end{aligned}$$

$$\begin{aligned}
& \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}^\downarrow_{\tilde{\tau}^\downarrow} \left(\left(\sqrt{t} + \frac{a_n(t)}{n} (n - \ell_j) + \tilde{B}_{n - \ell_j} / \sqrt{2} \right)^2 \right)} (u) > \sqrt{t} + a_n(t) + y_j \right) \Bigg] \\
& =: K_1 - K_2, \tag{4.114}
\end{aligned}$$

where we have used Lemma 4.2.1 in the first equality, and in the second equality, Lemma 4.2.5, (4.16), and the fact that for $t \geq n \log n$, $\exp \left(-\frac{3}{8} \int_0^{n - \ell_j} \frac{ds}{X_s} \right) = 1 + O((\log n)^{-1})$ under the event that $\sqrt{X_s/2} \geq \delta \sqrt{t}$ for all $0 \leq s \leq n - \ell_j$.

We first estimate K_2 . For all $\varepsilon > 0$ and $j \geq 1$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$, we have

$$\begin{aligned}
K_2 & \leq c_1 \tilde{E}_{n - \ell_j}^B \left[1_{\left\{ \tilde{B}_{n - \ell_j} / \sqrt{2} \in [(\ell_j)^{2/5} - y_j, \ell_j - y_j], \max_{0 \leq s \leq n - \ell_j} \tilde{B}_s / \sqrt{2} > (1 - \delta) \sqrt{t} \right\}} \right. \\
& \quad e^{\frac{\sqrt{2} a_n(t)}{n} \tilde{B}_{n - \ell_j} - \frac{(a_n(t))^2}{n^2} (n - \ell_j)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n} (n - \ell_j) - \tilde{B}_{n - \ell_j} / \sqrt{2}}} \\
& \quad \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}^\downarrow_{\tilde{\tau}^\downarrow} \left(\left(\sqrt{t} + \frac{a_n(t)}{n} (n - \ell_j) - \tilde{B}_{n - \ell_j} / \sqrt{2} \right)^2 \right)} (u) > \sqrt{t} + a_n(t) + y_j \right) \Bigg] \\
& \leq c_2 b^{-(n - \ell_j)} \sqrt{\ell_j} n^{1 - (1 - \delta)^2 c_*} y_j e^{-2\sqrt{\log b} y_j} \leq \varepsilon b^{-(n - \ell_j)} y_j e^{-2\sqrt{\log b} y_j}, \tag{4.115}
\end{aligned}$$

uniformly in y_j and t satisfying (4.78) (we take c_* large enough), where we have used the symmetry of $\tilde{B}$ in the first inequality, (4.48) and Proposition 4.3.1(i) in the second inequality.

Next, we estimate K_1 . By the density (4.21), we have

$$\begin{aligned}
K_1 & = \left(1 + O \left(\frac{1}{\log n} \right) \right) \\
& \quad \cdot \int_{(\ell_j)^{2/5}}^{\ell_j} \frac{1}{\sqrt{\pi(n - \ell_j)}} \left(e^{-\frac{(z - y_j)^2}{n - \ell_j}} - e^{-\frac{(z + y_j)^2}{n - \ell_j}} \right) \\
& \quad e^{\frac{2a_n(t)}{n} z - \frac{2a_n(t)}{n} y_j - \frac{(a_n(t))^2}{n^2} (n - \ell_j)} \sqrt{\frac{\sqrt{t}}{\sqrt{t} + \frac{a_n(t)}{n} (n - \ell_j) + y_j - z}} \\
& \quad \cdot \tilde{P}_{v^*} \left(\max_{u \in T_{\ell_j}^{v^*}} \sqrt{\tilde{L}^\downarrow_{\tilde{\tau}^\downarrow} \left(\left(\sqrt{t} + \frac{a_n(t)}{n} (n - \ell_j) + y_j - z \right)^2 \right)} (u) > \sqrt{t} + a_n(t) + y_j \right) dz \\
& = b^{-(n - \ell_j)} \left(1 + O \left(\frac{1}{\log n} \right) \right) \frac{4}{\sqrt{\pi}} y_j e^{-2\sqrt{\log b} y_j}
\end{aligned}$$

$$\begin{aligned}
& \cdot \int_{(\ell_j)^{2/5}}^{\ell_j} z e^{2\sqrt{\log b} z} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell_j) + y_j - z}} \\
& \cdot P_\rho \left(\frac{\max_{u \in T_{\ell_j}} L_{\tau(s_{z,y_j,\ell_j}^n(t))}^{\ell_j}(u) - s_{z,y_j,\ell_j}^n(t)}{\sqrt{2s_{z,y_j,\ell_j}^n(t)}} \right) \\
& > \sqrt{2} \left(\sqrt{\log b} \ell_j + z \right) + \Delta_{z,y_j,\ell_j}^n(t) \, dz, \quad (4.116)
\end{aligned}$$

where we have set

$$s_{z,y_j,\ell_j}^n(t) := \left(\sqrt{t} + \frac{a_n(t)}{n}(n - \ell_j) + y_j - z \right)^2$$

and $\Delta_{z,y_j,\ell_j}^n(t)$ as the remainder term, and used the fact that the law of the Brownian motion on $\tilde{T}^{v^*}$ starting at v^* is the same as that of the Brownian motion on $\tilde{T}_{\leq \ell_j}$ starting at ρ and the following in the second equality: for all $z \in [(\ell_j)^{2/5}, \ell_j]$ and $n \geq n_0(j)$, where we take $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ large enough,

$$\begin{aligned}
e^{-\frac{(z-y_j)^2}{n-\ell_j}} - e^{-\frac{(z+y_j)^2}{n-\ell_j}} &= \left(1 + O\left(\frac{1}{\log n}\right) \right) \frac{4y_j z}{n - \ell_j}, \\
e^{\frac{2a_n(t)}{n}z - \frac{2a_n(t)}{n}y_j - \frac{(a_n(t))^2}{n^2}(n - \ell_j)} &= \left(1 + O\left(\frac{1}{\log n}\right) \right) b^{-(n-\ell_j)} n^{3/2} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}} e^{2\sqrt{\log b} (z - y_j)}.
\end{aligned}$$

One can show that for each $j \geq 1$,

$$\lim_{n \rightarrow \infty} \Delta_{z,y_j,\ell_j}^n(t) = 0, \quad \text{uniformly in } z \in [(\ell_j)^{2/5}, \ell_j] \text{ and } y_j, t \text{ satisfying (4.78).} \quad (4.117)$$

By the assumption of $(t_n^+(j))_{n \geq 1}$ and $(t_n^-(j))_{n \geq 1}$, and the definition of β_* defined in (4.6), we have the following: for all $\varepsilon > 0$ and $j \geq 1$, there exists $n_0 = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\left| \sqrt{\frac{\sqrt{t} + n}{\sqrt{t} + \frac{a_n(t)}{n}(n - \ell_j) + y_j - z}} - \beta_* \right| < \varepsilon, \quad (4.118)$$

uniformly in $z \in [(\ell_j)^{2/5}, \ell_j]$ and y_j, t satisfying (4.78). By Theorem 4.2.4(i), (4.12), and (4.13), we have for any fixed $m \in \mathbb{N}$

$$\left(\frac{L_{\tau(s)}^m(v) - s}{\sqrt{2s}} \right)_{v \in T_m} \longrightarrow (h_v)_{v \in T_m} \quad \text{in law as } s \rightarrow \infty.$$

By this, the definition of (4.113), and (4.117), for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that the following holds: for each $j \geq j_0$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$,

$$\begin{aligned} \left| \int_{(\ell_j)^{\frac{2}{5}}}^{\ell_j} P_\rho \left(\frac{\max_{u \in T_{\ell_j}} L_{\tau(s_{z, y_j, \ell_j}^n(t))}^{\ell_j}(u) - s_{z, y_j, \ell_j}^n(t)}{\sqrt{2s_{z, y_j, \ell_j}^n(t)}} \right) \right. \\ \left. > \sqrt{2} \left(\sqrt{\log b} \ell_j + z \right) + \Delta_{z, y_j, \ell_j}^n(t) \right. \\ \left. \cdot z e^{2\sqrt{\log b} z} dz - \tilde{\gamma}_* \right| < \varepsilon, \end{aligned} \quad (4.119)$$

uniformly in y_j and t satisfying (4.78). Thus, by (4.114)-(4.116), (4.118), and (4.119), for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that the following holds: for each $j \geq j_0$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$,

$$\left| \frac{\tilde{E}_\rho \left[\Lambda_{y_j, \ell_j, J_{\ell_j}}^n(t) \right]}{y_j e^{-2\sqrt{\log b} y_j}} - \frac{4}{\sqrt{\pi}} \beta_* \tilde{\gamma}_* \right| < \varepsilon, \quad (4.120)$$

uniformly in y_j and t satisfying (4.78). By Lemma 4.5.4, 4.5.5, and (4.120), for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that the following holds: for each $j \geq j_0$, there exists $n_0(j) = n_0(y_j^-, y_j^+, \ell_j) \in \mathbb{N}$ such that for all $n \geq n_0(j)$,

$$\left| \frac{P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} > \sqrt{t} + a_n(t) + y_j \right)}{y_j e^{-2\sqrt{\log b} y_j}} - \frac{4}{\sqrt{\pi}} \beta_* \tilde{\gamma}_* \right| < \varepsilon, \quad (4.121)$$

uniformly in y_j and t satisfying (4.78) (we take $c_* > 0$ large enough).

Let $\hat{\gamma}_*$ be the limit of any convergent subsequence of (4.112). By taking a sub-subsequence, if necessary, and repeating the above argument, we have (4.121) if we replace $\tilde{\gamma}_*$ with $\hat{\gamma}_*$. Thus, the full sequence (4.112) converges to a finite positive constant, and we write γ_* to denote the limit. Therefore, we have (4.77). $\square$

4.6 Proof of Theorem 4.1.1 and Corollary 4.1.3

In this section, we prove Theorem 4.1.1 and Corollary 4.1.3. Recall (4.2). Let $(h_v)_{v \in T}$ be a BRW on T defined in Section 4.1. For $n \in \mathbb{N}$, $1 \leq r < n$, and $\lambda \in \mathbb{R}$, we set

$$F_r^n(\lambda) := \sum_{\substack{v \in T_n, \\ \sigma(v_r) \leq \lambda}} \left(\sqrt{\log b} \, n - \frac{1}{\sqrt{2}} h_v \right) e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, n - \frac{1}{\sqrt{2}} h_v \right)}. \quad (4.122)$$

Recall that v_r is the ancestor of v on T_r . Note that since $\sigma(v_r)$ falls in $[0, 1]$, we have $F_r^n(\lambda) = 0$ for all $\lambda < 0$ and $F_r^n(\lambda) = F_r^n(1)$ for all $\lambda \geq 1$. We begin with some lemmas.

Lemma 4.6.1 *For each $\lambda \in \mathbb{R}$, the limit $\lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} F_r^n(\lambda)$ exists almost surely.*

Proof. For each $v \in T_n$, set $h_v^{(n-r)} := h_v - h_{v_r}$. Then, we have

$$\begin{aligned} F_r^n(\lambda) &= \sum_{\substack{u \in T_r, \\ \sigma(u) \leq \lambda}} \sum_{v \in T_{n-r}^u} \left(\sqrt{\log b} \, r - h_u/\sqrt{2} + \sqrt{\log b} \, (n-r) - h_v^{(n-r)}/\sqrt{2} \right) \\ &\quad \cdot e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, r - h_u/\sqrt{2} \right)} e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, (n-r) - h_v^{(n-r)}/\sqrt{2} \right)} \\ &= \sum_{\substack{u \in T_r, \\ \sigma(u) \leq \lambda}} D_{n-r}^u e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, r - h_u/\sqrt{2} \right)} \\ &\quad + \sum_{\substack{u \in T_r, \\ \sigma(u) \leq \lambda}} W_{n-r}^u \left(\sqrt{\log b} \, r - h_u/\sqrt{2} \right) e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, r - h_u/\sqrt{2} \right)}, \end{aligned} \quad (4.123)$$

where for each $u \in T_r$, we have set

$$D_{n-r}^u := \sum_{v \in T_{n-r}^u} \left(\sqrt{\log b} \, (n-r) - h_v^{(n-r)}/\sqrt{2} \right) e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, (n-r) - h_v^{(n-r)}/\sqrt{2} \right)},$$

,

$$W_{n-r}^u := \sum_{v \in T_{n-r}^u} e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, (n-r) - h_v^{(n-r)}/\sqrt{2} \right)}.$$

By [73], and [17, Theorem 5.1, 5.2] or [2, Proposition A.3], the following holds for each $u \in T_r$:

$$D_\infty^u := \lim_{n \rightarrow \infty} D_{n-r}^u \in (0, \infty), \quad \lim_{n \rightarrow \infty} W_{n-r}^u = 0, \quad \text{almost surely.} \quad (4.124)$$

Thus, by (4.123) and (4.124), for each $r \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} F_r^n(\lambda) = \sum_{\substack{u \in T_r, \\ \sigma(u) \leq \lambda}} D_\infty^u e^{-2\sqrt{\log b} \left(\sqrt{\log b} r - h_u/\sqrt{2} \right)}, \quad \text{almost surely.} \quad (4.125)$$

Note that by Lemma 4.7.7(i), we have for each r ,

$$\sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v > 0, \quad \text{for all sufficiently large } n \text{ and } v \in T_n, \text{ almost surely.}$$

By this, the definition (4.122) of $F_r^n(\lambda)$ and the fact that $\{v \in T_n : \sigma(v_{r'}) \leq \lambda\} \subset \{v \in T_n : \sigma(v_r) \leq \lambda\}$ for all $r \leq r'$, the sequence $\{\lim_{n \rightarrow \infty} F_r^n(\lambda)\}_{r \geq 1}$ is nonincreasing for each $\lambda \in \mathbb{R}$. Thus, the limit $\lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} F_r^n(\lambda)$ exists almost surely for each $\lambda \in \mathbb{R}$. $\square$

For each $n \in \mathbb{N}$, we set

$$D_n^{(2)} := \sum_{v \in T_n} \left(\sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v \right)^2 e^{-4\sqrt{\log b} \left(\sqrt{\log b} n - \frac{1}{\sqrt{2}} h_v \right)}.$$

Then the following holds:

Lemma 4.6.2 *For all $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(D_n^{(2)} \geq \varepsilon \right) = 0. \quad (4.126)$$

Proof. Set $m_n := \sqrt{\log b} n - \frac{3}{4\sqrt{\log b}} \log n$. We have for any $y > 0$

$$\begin{aligned} & \mathbb{P} \left(D_n^{(2)} \geq \varepsilon \right) \\ & \leq \mathbb{P} \left(\sum_{v \in T_n} \left(\sqrt{\log b} n - h_v/\sqrt{2} \right)^2 e^{-4\sqrt{\log b} \left(\sqrt{\log b} n - h_v/\sqrt{2} \right)} 1_{\{h_v/\sqrt{2} \leq m_n + y\}} \geq \varepsilon \right) \\ & \quad + \mathbb{P} \left(\max_{v \in T_n} h_v/\sqrt{2} > m_n + y \right). \end{aligned} \quad (4.127)$$

Since h_v is a Gaussian random variable with mean 0 and variance n for each $v \in T_n$, a simple calculation implies that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{v \in T_n} \left(\sqrt{\log b} n - h_v/\sqrt{2} \right)^2 e^{-4\sqrt{\log b} \left(\sqrt{\log b} n - h_v/\sqrt{2} \right)} 1_{\{h_v/\sqrt{2} \leq m_n + y\}} \right] = 0.$$

By this and applying the Chebyshev inequality to the first term of (4.127) and Lemma 4.7.7(i) to the second term of (4.127), we have

$$\limsup_{n \rightarrow \infty} \mathbb{P} (D_n^{(2)} \geq \varepsilon) \leq c_1(1+y)e^{-2\sqrt{\log b} y}.$$

Since we can take arbitrary $y \in (0, \infty)$, this implies (4.126). $\square$

For $a \in \mathbb{R}$, an interval $I \subset \mathbb{R}$, $t > 0$, and $r, n \in \mathbb{N}$ with $r < n$, we set

$$\begin{aligned} \mathcal{B}_{r,t,n}^\sigma(I; a) := & \left\{ \sqrt{L_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \notin I, \right. \\ & \left. \text{for all } v \in T_n \text{ with } \sigma(v_r) \in [a - b^{-r}, a] \right\}. \end{aligned} \quad (4.128)$$

Then we have the following:

Lemma 4.6.3 *There exists $c_1 > 0$ such that the following holds: for any finite interval $I \subset \mathbb{R}$, there exist $c_2(I) > 0$ and $r_0(I) \in \mathbb{N}$ such that for all $t > 0$, $a \in \mathbb{R}$, $r \geq r_0(I)$, and $n > r$,*

$$P_\rho [(\mathcal{B}_{r,t,n}^\sigma(I; a))^c] \leq c_2(I)r^3e^{-c_1r}. \quad (4.129)$$

Proof. Fix any finite interval $I \subset \mathbb{R}$. Let $\{\bar{I}, \underline{I}\}$ be the boundary of I with $\underline{I} < \bar{I}$. Fix any $t > 0$, $a \in \mathbb{R}$, $r \geq r_0(I)$, and $n > r$, where we take $r_0(I) > |\bar{I}|$ large enough. Recall the event $G_r^n(t)$ defined in (4.26). We have

$$\begin{aligned} & P_\rho [(\mathcal{B}_{r,t,n}^\sigma(I; a))^c] \\ & \leq \tilde{P}_\rho \left(\text{there exists } v \in T_n \text{ with } \sigma(v_r) \in [a - b^{-r}, a] \text{ such that} \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + r + 1, \right. \\ & \quad \left. \text{for all } 0 \leq s \leq n, \sqrt{\tilde{L}_{\tau(t)}^n(v)} - \sqrt{t} - a_n(t) \in I \right) \\ & \quad + \tilde{P}_\rho (G_r^n(t)). \end{aligned} \quad (4.130)$$

We estimate the first term on the right-hand side of (4.130). Fix $v \in T_n$. Recall $\tilde{P}_n^B$ and $\tilde{B}$ defined in (4.37) and (4.38). We have

$$\sqrt{\frac{\sqrt{t} + a_n(t) + \underline{I}}{\sqrt{t}}} \tilde{P}_\rho \left(\sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \leq \sqrt{t} + \frac{a_n(t)}{n}s + \kappa(\log(s \wedge (n-s)))_+ + r + 1, \right.$$

$$\begin{aligned}
& \text{for all } 0 \leq s \leq n, \quad \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} - \sqrt{t} - a_n(t) \in I \\
& \leq \tilde{E}_n^B \left[1_{\left\{ \frac{\tilde{B}_s}{\sqrt{2}} \leq \kappa(\log(s \wedge (n-s)))_+ + r + 1, \text{ for all } 0 \leq s \leq n, \quad \frac{\tilde{B}_n}{\sqrt{2}} \in I \right\}} e^{-\frac{\sqrt{2}a_n(t)}{n} \tilde{B}_n - \frac{(a_n(t))^2}{n}} \right] \\
& \leq c_1(I) b^{-n} (r+1)(r+1-\underline{I}) \sqrt{\frac{\sqrt{t}+n}{\sqrt{t}}}, \tag{4.131}
\end{aligned}$$

where we have used Lemma 4.2.5 and (4.16) in the first inequality and Lemma 4.2.6(i) in the last inequality.

Next, we estimate the number of $v \in T_n$ satisfying $\sigma(v_r) \in [a - b^{-r}, a]$. We first consider the case $a \in [0, 1]$. In this case, we have a sequence $(x_i)_{i \geq 1}$ with $x_i \in \{0, \dots, b-1\}$ such that $a = \sum_{i=1}^{\infty} \frac{x_i}{b^i}$. In particular, we have $\sum_{i=1}^r \frac{\bar{v}_i - x_i}{b^i} \in [-b^{-r}, b^{-r}]$ for all $v \in T_n$ with $\sigma(v_r) \in [a - b^{-r}, a]$ and the label $(\bar{v}_1, \dots, \bar{v}_n)$. By this and a simple observation, we have

$$\begin{aligned}
& \{v \in T_n : \sigma(v_r) \in [a - b^{-r}, a]\} \\
& \subset \{v \in T_n : \bar{v}_i = x_i, \text{ for all } 1 \leq i \leq r\} \\
& \cup \bigcup_{i=1}^r \{v \in T_n : \bar{v}_j = x_j, \text{ for all } 1 \leq j \leq i-1, \bar{v}_i - x_i > 0, \\
& \quad \bar{v}_{j'} = x_{j'} - (b-1), \text{ for all } i < j' \leq r\} \\
& \cup \bigcup_{i=1}^r \{v \in T_n : \bar{v}_j = x_j, \text{ for all } 1 \leq j \leq i-1, \bar{v}_i - x_i < 0, \\
& \quad \bar{v}_{j'} = x_{j'} + (b-1), \text{ for all } i < j' \leq r\},
\end{aligned}$$

which implies

$$|\{v \in T_n : \sigma(v_r) \in [a - b^{-r}, a]\}| \leq c_2 r b^{n-r}. \tag{4.132}$$

By (4.130)-(4.132) and Lemma 4.3.2, we have (4.129). $\square$

Proof of Theorem 4.1.1. Recall (4.122) and Lemma 4.6.1. For each $r \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we set

$$F_r(\lambda) := \lim_{n \rightarrow \infty} F_r^n(\lambda).$$

By (4.125), for each $r \in \mathbb{N}$, the function $\lambda \mapsto F_r(\lambda)$ is nondecreasing and right-continuous, almost surely. Recall (4.1). By the sentence below (4.122),

we have for each $r \in \mathbb{N}$ $\lim_{\lambda \rightarrow \infty} F_r(\lambda) = D_\infty$ and $\lim_{\lambda \rightarrow -\infty} F_r(\lambda) = 0$. Thus, for each $r \in \mathbb{N}$, F_r determines the Borel measure Z_r on $\mathbb{R}$ satisfying $Z_r((\lambda_1, \lambda_2]) = F_r(\lambda_2) - F_r(\lambda_1)$ for all $\lambda_1 < \lambda_2$.

Recall the notion of weak convergence of random measures described above Theorem 4.1.1. Let $C_c^+(\mathbb{R})$ be the space of continuous, nonnegative functions on $\mathbb{R}$ with compact support. Since $Z_r(\mathbb{R}) = D_\infty < \infty$ almost surely for each $r \in \mathbb{N}$, we have the following: for all $\varepsilon > 0$ and $f \in C_c^+(\mathbb{R})$, there exists $M > 0$ such that for all $r \in \mathbb{R}$,

$$\mathbb{P}\left(\int_{\mathbb{R}} f(x) Z_r(dx) > M\right) \leq \mathbb{P}(\|f\|_\infty D_\infty > M) < \varepsilon,$$

where $\|f\|_\infty := \max_{x \in \mathbb{R}} f(x)$. Thus, by [85, Lemma 3.20], $(Z_r)_{r \geq 1}$ is tight. Thus, there exists a weakly convergent subsequence $(Z_{r_k})_{k \geq 1}$ of $(Z_r)_{r \geq 1}$ and we write Z_∞ to denote the weak limit.

Recall (4.6), (4.7), and (4.3). For each interval $I \subset \mathbb{R}$, we will write ∂I to denote its boundary. Fix any sequence of positive integers $(q_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} q_n = \infty$ and $\lim_{n \rightarrow \infty} q_n/n \in [0, 1]$. Fix any $(t_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} \sqrt{t_n}/n = \theta \in [0, \infty]$ and $t_n \geq c_* n \log n$, where c_* is a sufficiently large positive constant. We will prove the following: for all finite disjoint intervals $A_i := (\underline{A}_i, \overline{A}_i]$, $1 \leq i \leq m$ with $\mathbb{P}(Z_\infty(\partial A_i) = 0) = 1$ for all $1 \leq i \leq m$, finite intervals $B_i := (\underline{B}_i, \overline{B}_i]$, $1 \leq i \leq m$, and positive values $a_1, \dots, a_m$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} E_\rho \left[\exp \left\{ - \sum_{i=1}^m a_i \Xi_{n, t_n}^{(n-q_n)} (A_i \times B_i) \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ - \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m (1 - e^{-a_i}) Z_\infty(A_i) \left(e^{-2\sqrt{\log b} \underline{B}_i} - e^{-2\sqrt{\log b} \overline{B}_i} \right) \right\} \right]. \end{aligned} \tag{4.133}$$

Let $r_k < q < n$ be a positive integer. Recall the events (4.128). By the definition (4.3) of $\Xi_{n, t_n}^{(q)}$, we have

$$\begin{aligned} & E_\rho \left[\exp \left\{ - \sum_{i=1}^m a_i \Xi_{n, t_n}^{(q)} (A_i \times B_i) \right\} \right] \\ &= E_\rho \left[\exp \left\{ - \sum_{v \in T_q} \sum_{i=1}^m a_i 1_{\{\sigma(\arg \max_v L_{\tau(t_n)}^n) \in A_i\}} \right\} \right] \end{aligned}$$

$$\begin{aligned}
& 1_{\left\{\max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i\right\}} \left\} 1_{\bigcap_{i=1}^m \mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \underline{A}_i) \cap \mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \overline{A}_i)} \right] \\
& + E_\rho \left[\exp \left\{ - \sum_{v \in T_q} \sum_{i=1}^m a_i 1_{\left\{\sigma(\arg \max_v L_{\tau(t_n)}^n) \in A_i\right\}} \right. \right. \\
& \left. \left. 1_{\left\{\max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i\right\}} \right\} 1_{\bigcup_{i=1}^m (\mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \underline{A}_i))^c \cup (\mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \overline{A}_i))^c} \right] \\
& =: J_1 + J_2, \tag{4.134}
\end{aligned}$$

By Lemma 4.6.3, we have

$$J_2 \leq \sum_{i=1}^m c_1(B_i)(r_k)^3 e^{-c_2 r_k}. \tag{4.135}$$

We estimate J_1 in (4.134). Under $\bigcap_{i=1}^m \mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \overline{A}_i) \cap \mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \underline{A}_i)$, we have for all $v \in T_q$ and $1 \leq i \leq m$,

$$\begin{aligned}
& 1_{\left\{\sigma(\arg \max_v L_{\tau(t_n)}^n) \in A_i, \quad \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i\right\}} \\
& = 1_{\left\{\sigma(v_{r_k}) \in A_i, \quad \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i\right\}}. \tag{4.136}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
J_1 & = E_\rho \left[\exp \left\{ - \sum_{v \in T_q} \sum_{i=1}^m a_i 1_{\left\{\sigma(v_{r_k}) \in A_i, \quad \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i\right\}} \right\} \right] \\
& - E_\rho \left[\exp \left\{ - \sum_{v \in T_q} \sum_{i=1}^m a_i 1_{\left\{\sigma(v_{r_k}) \in A_i, \quad \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} - \sqrt{t_n} - a_n(t_n) \in B_i\right\}} \right\} \right. \\
& \quad \left. 1_{\bigcup_{i=1}^m (\mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \underline{A}_i))^c \cup (\mathcal{B}_{r_k, t_n, n}^\sigma(B_i; \overline{A}_i))^c} \right] \\
& =: J_{1,1} - J_{1,2}. \tag{4.137}
\end{aligned}$$

By Lemma 4.6.3, we have

$$J_{1,2} \leq \sum_{i=1}^m c_1(B_i)(r_k)^3 e^{-c_2 r_k}. \tag{4.138}$$

We estimate $J_{1,1}$. By Theorem 4.2.4(ii), on the same probability space (we will write P to denote the probability measure), we can construct a local time $(L_{\tau(t_n)}^n(v))_{v \in T_{\leq n}}$ and two BRWs $(h_v)_{v \in T_{\leq n}}, (h'_v)_{v \in T_{\leq n}}$ satisfying (4.14) and (4.15). Fix $\delta \in (0, 1)$. We set

$$C_q := \left\{ \frac{1}{\sqrt{2}}|h_v|, \frac{1}{\sqrt{2}}|h'_v| \leq \sqrt{\log b} \, q - \frac{3}{4\sqrt{\log b}}(1 - \delta) \log q, \text{ for all } v \in T_q \right\}. \quad (4.139)$$

For each $v \in T_q$, let $\tilde{L}^\downarrow$ be a local time of a Brownian motion on $\tilde{T}^v \subset \tilde{T}_{\leq n}$ and set $\tilde{\tau}^\downarrow(p) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(v) > p\}$. We omit the subscript v in $\tilde{L}^\downarrow$ and $\tilde{\tau}^\downarrow$. By Lemma 4.2.1, we have

$$J_{1,1} = E \left[1_{C_q} \prod_{v \in T_q} K_v \right] + E \left[1_{(C_q)^c} \prod_{v \in T_q} K_v \right] =: J_{1,1,1} + J_{1,1,2}, \quad (4.140)$$

where for each $v \in T_q$, we have set K_v by

$$\tilde{E}_v \left[\exp \left\{ - \sum_{i=1}^m a_i 1_{\left\{ \sigma(v_{r_k}) \in A_i, \max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(L_{\tau(t_n)}^n(v))}^\downarrow(u) - \sqrt{t_n} - a_n(t_n)} \in B_i \right\}} \right\} \right]. \quad (4.141)$$

By Lemma 4.7.7, we have

$$J_{1,1,2} \leq c_3(\log q) \cdot (q)^{-\frac{3}{2}\delta}. \quad (4.142)$$

We estimate $J_{1,1,1}$. Since $A_1, \dots, A_m$ are disjoint, we have for all $v \in T_q$,

$$\begin{aligned} & K_v \\ &= 1 - \tilde{E}_v \left[1 - e^{-\sum_{i=1}^m a_i 1_{\left\{ \sigma(v_{r_k}) \in A_i, \max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(L_{\tau(t_n)}^n(v))}^\downarrow(u) - \sqrt{t_n} - a_n(t_n)} \in B_i \right\}}} \right] \\ &= 1 - \sum_{i=1}^m (1 - e^{-a_i}) 1_{\{\sigma(v_{r_k}) \in A_i\}} \\ & \quad \cdot \tilde{P}_v \left(\max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(L_{\tau(t_n)}^n(v))}^\downarrow(u) - \sqrt{t_n} - a_n(t_n)} \in B_i \right) \end{aligned}$$

$$= \exp \left\{ \log \left(1 - \sum_{i=1}^m (1 - e^{-a_i}) 1_{\{\sigma(v_{r_k}) \in A_i\}} \right. \right. \\ \left. \left. \cdot \tilde{P}_v \left(\max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}(u) - \sqrt{t_n} - a_n(t_n) \in B_i \right) \right) \right\}. \quad (4.143)$$

On the event C_q , by (4.15), we have for all $v \in T_q$

$$\sqrt{t_n} + a_n(t_n) = \sqrt{L_{\tau(t_n)}^n(v) + a_{n-q}(L_{\tau(t_n)}^n(v))} + \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \delta_{v,q}^n, \quad (4.144)$$

where

$$\delta_{v,q}^n := \frac{3 \log(1 - q/n)}{4\sqrt{\log b}} + \frac{\log(1 + O_q(1/\sqrt{t_n}))}{4\sqrt{\log b}} + O_q(1/\sqrt{t_n}). \quad (4.145)$$

(Recall the notation O_q described in Section 4.1.) We take any sufficiently small $\varepsilon > 0$ and sufficiently large $q_0 \in \mathbb{N}$ which depends on $B_1, \dots, B_m$ and ε . We assume that $q \geq q_0$ and $n \geq n_0$, where we take sufficiently large $n_0 = n_0(q, \varepsilon) \in \mathbb{N}$. By Proposition 4.5.1 and (4.144)-(4.145), under the event C_q , we have for all $1 \leq i \leq m$ and $v \in T_q$,

$$\begin{aligned} & \tilde{P}_v \left(\max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}(u) - \sqrt{t_n} - a_n(t_n) \in B_i \right) \\ &= \tilde{P}_v \left(\max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}(u) - \sqrt{L_{\tau(t_n)}^n(v) + a_{n-q}(L_{\tau(t_n)}^n(v))} \right. \\ & \quad \left. \geq \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \underline{B}_i + \delta_{v,q}^n \right) \\ &= \tilde{P}_v \left(\max_{u \in T_{n-q}^v} \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(L_{\tau(t_n)}^n(v))}(u) - \sqrt{L_{\tau(t_n)}^n(v) + a_{n-q}(L_{\tau(t_n)}^n(v))} \right. \\ & \quad \left. \geq \sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \bar{B}_i + \delta_{v,q}^n \right) \\ &\leq (1 + c_4 \varepsilon) \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \left(\sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \underline{B}_i \right) e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \underline{B}_i \right)} \\ & \quad - (1 - c_4 \varepsilon) \frac{4}{\sqrt{\pi}} \beta_* \gamma_* \left(\sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \bar{B}_i \right) e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v + \bar{B}_i \right)}. \end{aligned} \quad (4.146)$$

We set

$$D_q^{(2)} := \sum_{v \in T_q} \left(\sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v \right)^2 e^{-4\sqrt{\log b} \left(\sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v \right)},$$

$$W_q := \sum_{v \in T_q} e^{-2\sqrt{\log b} \left(\sqrt{\log b} \, q - \frac{1}{\sqrt{2}} h'_v \right)}.$$

By (4.143), (4.146) and Taylor's expansion of the function $x \mapsto \log(1 - x)$, under the event $C_q \cap \left\{ D_q^{(2)} < \varepsilon \right\} \cap \{W_q < \varepsilon\}$, we have

$$\prod_{v \in T_q} K_v \leq e^{c_5 \left(1 + \sum_{i=1}^m |B_i| e^{-2\sqrt{\log b} \, B_i} + \sum_{i=1}^m |\bar{B}_i| e^{-2\sqrt{\log b} \, \bar{B}_i} \right) \varepsilon}$$

$$\cdot \exp \left\{ -\frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m (1 - e^{-a_i}) \left(F_{r_k}^q(\bar{A}_i) - F_{r_k}^q(\underline{A}_i) \right) \right.$$

$$\cdot \left. \left(e^{-2\sqrt{\log b} \, B_i} - e^{-2\sqrt{\log b} \, \bar{B}_i} - c_4 \varepsilon \left(e^{-2\sqrt{\log b} \, B_i} + e^{-2\sqrt{\log b} \, \bar{B}_i} \right) \right) \right\}, \quad (4.147)$$

where we abused the notation $F_{r_k}^q$ by replacing h with h' in (4.122). We can obtain a similar lower bound of $\prod_{v \in T_q} K_v$. By Lemma 4.7.7, Lemma 4.6.2, and the fact that $\lim_{q \rightarrow \infty} W_q = 0$, almost surely (see [73]), we have

$$\lim_{q \rightarrow \infty} P \left((C_q)^c \cup \{D_q^{(2)} \geq \varepsilon\} \cup \{W_q \geq \varepsilon\} \right) = 0. \quad (4.148)$$

Thus, by (4.134)-(4.135), (4.137)-(4.138), (4.140), (4.142) (4.147) and a similar lower bound, and (4.148), taking $n \rightarrow \infty$, then $q \rightarrow \infty$, then $k \rightarrow \infty$, and finally $\varepsilon \rightarrow 0$, we have

$$\lim_{q \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| E_\rho \left[\exp \left\{ -\sum_{i=1}^m a_i \Xi_{n, t_n}^{(q)}(A_i \times B_i) \right\} \right] \right.$$

$$\left. - \mathbb{E} \left[\exp \left\{ -\frac{4}{\sqrt{\pi}} \beta_* \gamma_* \sum_{i=1}^m (1 - e^{-a_i}) Z_\infty(A_i) \left(e^{-2\sqrt{\log b} \, B_i} - e^{-2\sqrt{\log b} \, \bar{B}_i} \right) \right\} \right] \right|$$

$$= 0. \quad (4.149)$$

Next, by using (4.149), we will prove (4.133). Let z_* be a real number with $z_* < \min_{1 \leq i \leq m} \underline{B}_i$. Take $q_0 = q_0(z_*) \in \mathbb{N}$ large enough and fix any $q \geq q_0$. Take $n \in \mathbb{N}$ large enough so that $q < n - q_n < n - q$ and $q < n/4$. We set

$$U_{z_*,q}^n(t_n) := \left\{ \text{there exist } v, u \in T_n \text{ with } q \leq |v \wedge u| \leq n - q \text{ such that} \right. \\ \left. \sqrt{L_{\tau(t_n)}^n(v)}, \sqrt{L_{\tau(t_n)}^n(u)} \in [\sqrt{t_n} + a_n(t_n) + z_*, \infty) \right\}. \quad (4.150)$$

Under the event $(U_{z_*,q}^n(t_n))^c$, by the definition of $\arg \max_v L_{\tau(t_n)}^n$, we have

$$\left\{ \arg \max_v L_{\tau(t_n)}^n : v \in T_{n-q_n}, \max_{u \in T_{q_n}^v} \sqrt{L_{\tau(t_n)}^n(u)} \geq \sqrt{t_n} + a_n(t_n) + z_* \right\} \\ = \left\{ \arg \max_v L_{\tau(t_n)}^n : v \in T_q, \max_{u \in T_{n-q}^v} \sqrt{L_{\tau(t_n)}^n(u)} \geq \sqrt{t_n} + a_n(t_n) + z_* \right\}. \quad (4.151)$$

By (4.151) and $z_* < \min_{1 \leq i \leq m} \underline{B}_i$, we have

$$E_\rho \left[\exp \left\{ - \sum_{i=1}^m a_i \Xi_{n,t_n}^{(n-q_n)}(A_i \times B_i) \right\} \right] \\ = E_\rho \left[e^{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(q)}(A_i \times B_i)} \right] - E_\rho \left[e^{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(q)}(A_i \times B_i)} 1_{U_{z_*,q}^n(t_n)} \right] \\ + E_\rho \left[e^{-\sum_{i=1}^m a_i \Xi_{n,t_n}^{(n-q_n)}(A_i \times B_i)} 1_{U_{z_*,q}^n(t_n)} \right] \\ =: L_1 - L_2 + L_3. \quad (4.152)$$

By Proposition 4.4.1, we have

$$L_2 \leq c_6 q^{-1/8}, \quad L_3 \leq c_6 q^{-1/8}. \quad (4.153)$$

Thus, by (4.149) and (4.152)-(4.153), we have (4.133).

Recall the random measures Z_r defined in the first part of the proof. Let $\tilde{Z}_\infty$ be any subsequential weak limit of $(Z_r)_{r \geq 1}$. Repeating the above argument (we additionally assume that $\mathbb{P}(\tilde{Z}_\infty(\partial A_i) = 0) = 1$ for all $1 \leq$

$i \leq m$), we have (4.133) if we replace Z_∞ with $\tilde{Z}_\infty$. Thus, random measures Z_∞ and $\tilde{Z}_\infty$ have the same finite dimensional distributions, which implies $Z_\infty = \tilde{Z}_\infty$ in law. Thus, the full sequence $(Z_r)_{r \geq 1}$ weakly converges to Z_∞ . By the sentence below (4.122) and the definition of $(Z_r)_{r \geq 1}$, we have $Z_\infty(\mathbb{R}) = Z_\infty([0, 1])$ almost surely and $Z_\infty([0, 1]) = D_\infty$ in law.

Note that the right-hand side of (4.133) is the Laplace transform of

$$\left(\text{PPP} \left(\frac{4}{\sqrt{\pi}} \beta_* \gamma_* Z_\infty(dx) \otimes 2\sqrt{\log b} e^{-2\sqrt{\log b} y} dy \right) (A_i \times B_i) : 1 \leq i \leq m \right).$$

Thus, (4.133) implies that the point process $\Xi_{n,t_n}^{(n-q_n)}$ weakly converges to the Cox process (4.5) as $n \rightarrow \infty$ (see, for example, [34, Proposition 11.1.VIII]).

Finally, we will prove that Z_∞ is non-atomic almost surely. Since the set of points $x \in \mathbb{R}$ with $\mathbb{P}(Z_\infty(\{x\}) > \delta) > \varepsilon$ for some $\delta, \varepsilon \in (0, \infty)$ is countable (see, for example, [34, Lemma 11.1.II]), it is enough to show that $\mathbb{P}(Z_\infty(\{a\}) = 0) = 1$ for each $a \in [0, 1]$. Fix $a \in [0, 1]$. Fix $I_m = [\underline{I}_m, \bar{I}_m] \subset \mathbb{R}$ be finite intervals with $I_m \downarrow \{a\}$ as $m \rightarrow \infty$ and $\mathbb{P}(Z_\infty(\partial I_m) = 0) = 1$. Since $\Xi_{n,t_n}^{(n-q_n)}$ weakly converges to the Cox process as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} P_\rho \left(\Xi_{n,t_n}^{(n-q_n)}(I_m \times [1, 2]) > 0 \right) = 1 - \mathbb{E} \left[e^{-\frac{4}{\sqrt{\pi}} \beta_* \gamma_* Z_\infty(I_m) (e^{-2\sqrt{\log b}} - e^{-4\sqrt{\log b}})} \right]. \quad (4.154)$$

On the other hand, by Proposition 4.7.1, we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} P_\rho \left(\Xi_{n,t_n}^{(n-q_n)}(I_m \times [1, 2]) > 0 \right) \\ & \leq \limsup_{n \rightarrow \infty} P_\rho \left(\max_{\substack{v \in T_n, \\ \sigma(v) \in I_m}} \sqrt{L_{\tau(t_n)}^n(v)} \geq \sqrt{t_n} + a_n(t_n) + 1 \right) \\ & \leq c_7 \limsup_{n \rightarrow \infty} (b^{-n} |\{v \in T_n : \sigma(v) \in I_m\}|)^{5/8} \leq c_7 (\bar{I}_m - \underline{I}_m)^{5/8}. \end{aligned} \quad (4.155)$$

By (4.154)-(4.155) and $I_m \downarrow \{a\}$ as $m \rightarrow \infty$, we have $Z_\infty(\{a\}) = 0$ almost surely. Therefore, we have proved that Z_∞ is non-atomic almost surely. $\square$

Proof of Corollary 4.1.3. Fix $(q_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} q_n = \infty$ and $\lim_{n \rightarrow \infty} q_n/n \in [0, 1)$. For all $\lambda \in \mathbb{R}$, $\tilde{\lambda} > \lambda \vee 0$, and $\varepsilon > 0$, we have

$$P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} \leq \sqrt{t_n} + a_n(t_n) + \lambda \right)$$

$$\begin{aligned}
&\geq P_\rho \left(\Xi_{n,t_n}^{(n-q_n)} \left((-\varepsilon, 1] \times (\lambda, \tilde{\lambda}) \right) = 0 \right) \\
&- P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} \geq \sqrt{t_n} + a_n(t_n) + \tilde{\lambda} \right) \\
&\geq P_\rho \left(\Xi_{n,t_n}^{(n-q_n)} \left((-\varepsilon, 1] \times (\lambda, \tilde{\lambda}) \right) = 0 \right) - c_1(1 + \tilde{\lambda})e^{-2\sqrt{\log b} \tilde{\lambda}}, \quad (4.156)
\end{aligned}$$

where we have used Proposition 4.3.1(i) in the last inequality, and obviously,

$$\begin{aligned}
&P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t_n)}^n(v)} \leq \sqrt{t_n} + a_n(t_n) + \lambda \right) \\
&\leq P_\rho \left(\Xi_{n,t_n}^{(n-q_n)} \left((-\varepsilon, 1] \times (\lambda, \tilde{\lambda}) \right) = 0 \right). \quad (4.157)
\end{aligned}$$

By Theorem 4.1.1, taking $n \rightarrow \infty$, and then $\tilde{\lambda} \rightarrow \infty$ in (4.156)-(4.157), we have (4.8). $\square$

4.7 Appendix

In this section, we give proof of some technical estimates.

4.7.1 Proof of Lemma 4.2.1

Proof of Lemma 4.2.1. Following [51], we will construct a version of the Brownian motion on $\tilde{T}_{\leq n}$ via piecing together excursions around a . Let $\tilde{X}^0 = (\tilde{X}_t^0, t \geq 0, \tilde{P}_x^0, x \in \tilde{T}_{\leq n} \setminus \{a\})$ is a Brownian motion on $\tilde{T}_{\leq n} \setminus \{a\}$ killed upon the hitting time $H_a^0 := \inf\{t \geq 0 : \tilde{X}_t^0 = a\}$. We define a space of excursions around a by

$$\begin{aligned}
W_a := \{w : \text{there exists } \xi(w) \in (0, \infty), w : [0, \xi(w)] \rightarrow \tilde{T}_{\leq n} \text{ is continuous,} \\
w(t) \in \tilde{T}_{\leq n} \setminus \{a\} \text{ for all } t \in (0, \xi(w)), \text{ and } w(0) = w(\xi(w)) = a\}.
\end{aligned}$$

Let $\sum_{s \in D} \delta_{(s, \mathbf{e}_s)}$ be a Poisson point process on $(0, \infty) \times W_a$ with intensity measure $ds \times \mathbf{n}$ defined on a probability space (Ω, P) , where the set $D \subset (0, \infty)$ is countable and $\mathbf{n}$ is the excursion law corresponding to $\tilde{X}^0$. See [51, Section 4] for the detail. Let ∂ be a new point and we extend $\mathbf{n}$ to a measure on $W_a \cup \{\partial\}$ by setting $\mathbf{n}(\{\partial\}) = 0$. For $s \notin D$, we set $\mathbf{e}_s := \partial$ and $\xi(\mathbf{e}_s) = 0$.

Set $J(s) := \sum_{r \leq s} \xi(\mathbf{e}_r)$. For $t \geq 0$, we can find s with $J(s-) \leq t \leq J(s)$ and set

$$Y_t := \begin{cases} \mathbf{e}_s(t - J(s-)) & \text{if } J(s-) < J(s), \\ a & \text{if } J(s-) = J(s). \end{cases} \quad (4.158)$$

Let Ω^0 be the sample space on which $\tilde{X}^0$ is defined. We assume that Ω^0 has an extra point ω^a with $\tilde{P}_x(\{\omega^a\}) = 0$ for all $x \in \tilde{T}_{\leq n} \setminus \{a\}$. Set $\tilde{P}_a^0 := \delta_{\omega^a}$. We define the product probability space $(\tilde{\Omega}, \tilde{P}_x)$ by

$$\tilde{\Omega} := \Omega^0 \times \Omega, \quad \tilde{P}_x := \tilde{P}_x^0 \times P, \quad x \in \tilde{T}_{\leq n}.$$

We define a new process $\tilde{X} = (\tilde{X}_t, t \geq 0, \tilde{P}_x, x \in \tilde{T}_{\leq n})$ as follows: for $\tilde{\omega} = (\omega^0, \omega) \in \tilde{\Omega} \setminus \{\omega^a\} \times \Omega$, we set

$$\tilde{X}_t(\tilde{\omega}) := \begin{cases} \tilde{X}_t^0(\omega^0) & \text{if } 0 \leq t < H_a^0(\omega^0), \\ Y_{t-H_a^0(\omega^0)}(\omega) & \text{if } t \geq H_a^0(\omega^0). \end{cases} \quad (4.159)$$

For $\tilde{\omega} = (\omega^a, \omega)$ with $\omega \in \Omega$, we set $\tilde{X}_t(\tilde{\omega}) := Y_t(\omega)$ for all $t \geq 0$. By [51, Theorem 4.1 and Corollary 5.1], $\tilde{X}$ is a Brownian motion on $\tilde{T}_{\leq n}$. We consider the space of excursions $W_a^\uparrow$ restricted on $\tilde{T}_{\leq n} \setminus \tilde{T}^a$ defined by

$$W_a^\uparrow := \{w \in W_a : w(t) \in \tilde{T}_{\leq n} \setminus \tilde{T}^a \text{ for all } t \in (0, \xi(w))\}.$$

Similarly, we define the space of excursions $W_a^\downarrow$ restricted on $\tilde{T}^a$. For $s \geq 0$, set

$$\mathbf{e}_s^\uparrow := \begin{cases} \mathbf{e}_s & \text{if } \mathbf{e}_s \in W_a^\uparrow, \\ \partial & \text{otherwise,} \end{cases} \quad \mathbf{e}_s^\downarrow := \begin{cases} \mathbf{e}_s & \text{if } \mathbf{e}_s \in W_a^\downarrow, \\ \partial & \text{otherwise.} \end{cases}$$

Since $W_a^\uparrow$ and $W_a^\downarrow$ are disjoint, two Poisson point processes $\sum_s \delta_{(s, \mathbf{e}_s^\uparrow)}$ and $\sum_s \delta_{(s, \mathbf{e}_s^\downarrow)}$ are independent. For $s \geq 0$, set $J^\uparrow(s) := \sum_{r \leq s} \xi(\mathbf{e}_r^\uparrow)$ and $J^\downarrow(s) := \sum_{r \leq s} \xi(\mathbf{e}_r^\downarrow)$. Let $\{\tilde{L}_t^n(x) : (t, x) \in [0, \infty) \times \tilde{T}_{\leq n}\}$ be a local time of $\tilde{X}$. For $t > 0$ and $\tilde{\omega} = (\omega^0, \omega) \in \tilde{\Omega}$, we set

$$\hat{L}_t^n(a)(\tilde{\omega}) := \inf\{s : H_a^0(\omega^0) + J(s)(\omega) > t\}.$$

One can check that $\{\widehat{L}_t^n(a) : t \geq 0\}$ is a local time of $\widetilde{X}$ at a . Thus, by [80, Lemma 3.6.8], $\widehat{L}_t^n(a) = \widetilde{L}_t^n(a)$ for all $t \geq 0$, $\widetilde{P}_\rho$ -almost surely. By [56, Theorem 2.6] and [79, Theorem 3.7], $\widetilde{X}$ has a continuous 1-potential density. Thus, by [80, Theorem 3.6.3], for each $y \in \widetilde{T}_{\leq n}$, there exists a positive sequence $(\varepsilon_m^y)_{m \geq 0}$ with $\lim_{m \rightarrow \infty} \varepsilon_m^y = 0$ such that $\widetilde{P}_\rho$ -almost surely,

$$\widetilde{L}_t^n(y) = \lim_{m \rightarrow \infty} \int_0^t f_{\varepsilon_m^y, y}(\widetilde{X}_s) ds, \quad \text{locally uniformly in } t, \quad (4.160)$$

where $f_{\varepsilon, y}$ is an approximate δ -function at y with respect to m : that is, $\{f_{\varepsilon, y} : \varepsilon > 0\}$ is a family of positive continuous functions on $\widetilde{T}_{\leq n}$ satisfying $\int_{\widetilde{T}_{\leq n}} f_{\varepsilon, y}(x) m(dx) = 1$ and $\text{supp}(f_{\varepsilon, y})$ is a compact neighborhood of y with $\text{supp}(f_{\varepsilon, y}) \downarrow \{y\}$ as $\varepsilon \rightarrow 0$. By (4.160), we have

$$\begin{aligned} \widetilde{L}_{\widetilde{\tau}(t)}^n(a) &= \inf\{s : \widetilde{L}_{H_a^0 + J(s)}^n(\rho) > t\} \\ &= \inf\left\{s : \lim_{m \rightarrow \infty} \left(\int_0^{H_a^0} f_{\varepsilon_m^{\rho}, \rho}(\widetilde{X}_u^0) du + \sum_{r \leq s} \int_0^{\xi(\mathbf{e}_r^\uparrow)} f_{\varepsilon_m^{\rho}, \rho}(\mathbf{e}_r^\uparrow(u)) du \right) > t \right\} \\ &=: s_*^n(t), \quad \text{a.s.}, \end{aligned} \quad (4.161)$$

$$\begin{aligned} \widetilde{\tau}(t) &= \inf\{s : \widetilde{L}_s^n(\rho) > t\} \\ &= \inf\left\{s : \lim_{m \rightarrow \infty} \left(\int_0^{H_a^0} f_{\varepsilon_m^{\rho}, \rho}(X_u^0) du + \sum_{r < s_*^n(t)} \int_0^{\xi(\mathbf{e}_r^\uparrow)} f_{\varepsilon_m^{\rho}, \rho}(\mathbf{e}_r^\uparrow(u)) du \right. \right. \\ &\quad \left. \left. + \int_0^{s - J^\uparrow(s_*^n(t) -)} f_{\varepsilon_m^{\rho}, \rho}(\mathbf{e}_{s_*^n(t)}^\uparrow(u)) du \right) > t \right\}, \quad \text{a.s.}, \end{aligned} \quad (4.162)$$

$$\widetilde{L}_{\widetilde{\tau}(t)}^n(x) = \lim_{m \rightarrow \infty} \sum_{r \leq s_*^n(t)} \int_0^{\xi(\mathbf{e}_r^\downarrow)} f_{\varepsilon_m^x, x}(\mathbf{e}_r^\downarrow(u)) du, \quad \text{a.s., for all } x \in \widetilde{T}^a \setminus \{a\}, \quad (4.163)$$

$$\begin{aligned} \widetilde{L}_{\widetilde{\tau}(t)}^n(y) &= \lim_{m \rightarrow \infty} \left(\int_0^{H_a} f_{\varepsilon_m^y, y}(\widetilde{X}_u^0) du + \sum_{r < s_*^n(t)} \int_0^{\xi(\mathbf{e}_r^\uparrow)} f_{\varepsilon_m^y, y}(\mathbf{e}_r^\uparrow(u)) du \right. \\ &\quad \left. + \int_0^{\widetilde{\tau}(t) - J^\uparrow(s_*^n(t) -)} f_{\varepsilon_m^y, y}(\mathbf{e}_{s_*^n(t)}^\uparrow(u)) du \right), \quad \text{a.s., for all } y \in \widetilde{T}_{\leq n} \setminus \widetilde{T}^a, \end{aligned} \quad (4.164)$$

where “a.s.” in (4.161)-(4.164) means “ $\tilde{P}_\rho$ -almost surely”. Let $Y^\downarrow$ be the process defined by replacing $J(\cdot)$ and $\mathbf{e}$ in (4.158) with $J^\downarrow(\cdot)$ and $\mathbf{e}^\downarrow$, respectively. Let $\tilde{X}^\downarrow$ be the process defined by replacing Y with $Y^\downarrow$. Then, by [51, Theorem 4.1 and Corollary 5.1], the process $\tilde{X}^\downarrow = (\tilde{X}_t^\downarrow, t \geq 0, \tilde{P}_x, x \in \tilde{T}^a)$ is a Brownian motion on $\tilde{T}^a$. Thus, by (4.161)-(4.164) and independence of $\mathbf{e}^\uparrow$ and $\mathbf{e}^\downarrow$, together with [80, Theorem 3.6.3], the law of $\{\tilde{L}_{\tilde{\tau}(t)}^n(x) : x \in \tilde{T}^a\}$ under $\tilde{P}_\rho(\cdot|\mathcal{F}^\uparrow)$ is the same as that of $\left\{\tilde{L}_{\tilde{\tau}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(a))}^\downarrow(x) : x \in \tilde{T}^a\right\}$ under $\tilde{P}_a$. $\square$

4.7.2 Proof of Lemma 4.2.2

Proof of Lemma 4.2.2. Let $\{z_1, z_2\}, \{w_1, w_2\}$ be edges of T with $x \in I_{\{z_1, z_2\}}$ and $y \in I_{\{w_1, w_2\}}$. We have

$$\begin{aligned} \mathbb{E}(\tilde{h}_x \tilde{h}_y) &= \tilde{E}_x(\tilde{L}_{H_\rho}^n(y)) \\ &= \tilde{E}_x\left(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y) + \tilde{L}_{H_\rho}^n(y) \circ \theta_{H_{z_1} \wedge H_{z_2}}\right) \\ &= \tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) + P_x(H_{z_1} < H_{z_2})E_{z_1}(\tilde{L}_{H_\rho}^n(y)) + P_x(H_{z_2} < H_{z_1})E_{z_2}(\tilde{L}_{H_\rho}^n(y)) \\ &= \tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) + \frac{d(x, z_2)}{d(z_1, z_2)}E_y(\tilde{L}_{H_\rho}^n(z_1)) + \frac{d(x, z_1)}{d(z_1, z_2)}E_y(\tilde{L}_{H_\rho}^n(z_2)). \end{aligned} \quad (4.165)$$

Similar arguments implies that for each $i \in \{1, 2\}$,

$$\tilde{E}_y(\tilde{L}_{H_\rho}^n(z_i)) = \frac{d(y, w_2)}{d(w_1, w_2)}\tilde{E}_{w_1}(\tilde{L}_{H_\rho}^n(z_i)) + \frac{d(y, w_1)}{d(w_1, w_2)}\tilde{E}_{w_2}(\tilde{L}_{H_\rho}^n(z_i)). \quad (4.166)$$

(In (4.166), we have used the equality $\tilde{E}_y(\tilde{L}_{H_{w_1} \wedge H_{w_2}}^n(z_i)) = 0$ for each $i \in \{1, 2\}$.) Note that by (4.12) and [40, Lemma 2.1], for all $i, j \in \{1, 2\}$,

$$\tilde{E}_{w_i}(\tilde{L}_{H_\rho}^n(z_j)) = E_{w_i}(L_{H_\rho}^n(z_j)) = d(\rho, w_i) + d(\rho, z_j) - d(w_i, z_j). \quad (4.167)$$

We have

$$\tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) = 0, \quad \text{if } I_{\{z_1, z_2\}} \neq I_{\{w_1, w_2\}}. \quad (4.168)$$

Assume that $I_{\{z_1, z_2\}} = I_{\{w_1, w_2\}}$. Note that a Brownian motion on $\tilde{T}_{\leq n}$ starting at x killed upon $H_{z_1} \wedge H_{z_2}$ has the same law as a standard Brownian motion on $(0, \frac{1}{2})$ starting at $d(x, z_1)$ killed upon $H_0 \wedge H_{1/2}$. Let $\{L_t^B(x) : t \geq 0\}$ be a

local time at x of a standard Brownian motion $\{B_t, t \geq 0, P_y^B, y \in \mathbb{R}\}$ on $\mathbb{R}$. By this and [86, Chapter VI, Exercise 2.8], we have

$$\begin{aligned}\tilde{E}_x(\tilde{L}_{H_{z_1} \wedge H_{z_2}}^n(y)) &= E_{d(x, z_1)}^B(L_{H_0 \wedge H_{1/2}}^B(d(y, z_1))) \\ &= \frac{d(x, z_2)}{d(z_1, z_2)} d(y, z_1) + \frac{d(x, z_1)}{d(z_1, z_2)} d(y, z_2) - d(x, y), \\ &\quad \text{if } I_{\{z_1, z_2\}} = I_{\{w_1, w_2\}}.\end{aligned}\tag{4.169}$$

Thus, by (4.165)-(4.169), we have

$$\mathbb{E}(\tilde{h}_x \tilde{h}_y) = d(\rho, x) + d(\rho, y) - d(x, y). \quad \square$$

4.7.3 Tail of maximum of local time revisited

To show that the random measure Z_∞ in Theorem 4.1.1 is non-atomic, we use a version of Proposition 4.3.1(i):

Proposition 4.7.1 *There exists $c_1 > 0$ such that for all $n \in \mathbb{N}$, $t > 0$, $y > 0$, and $A \subset T_n$,*

$$P_\rho \left(\max_{v \in A} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \leq c_1 \left(\frac{|A|}{b^n} \right)^{5/8} (1 + y) e^{-2\sqrt{\log b} \, y}.\tag{4.170}$$

Proof. Recall (4.26) and (4.28). We have

$$\begin{aligned}& P_\rho \left(\max_{v \in A} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \\ & \leq \tilde{P}_\rho \left(\text{there exists } v \in A, \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} \leq g_{y+(b^n|A|^{-1})^{1/8}, t, n}(s), \text{ for all } s \leq n, \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in \left[\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + (b^n|A|^{-1})^{1/8} + 1 \right] \right) \\ & \quad + \tilde{P}_\rho \left(G_{y+(b^n|A|^{-1})^{1/8}}^n(t) \right) \\ & = J_1 + J_2.\end{aligned}\tag{4.171}$$

By an argument similar to (4.44), we have

$$J_1 \leq c_1 (|A|b^{-n})^{5/8} (1 + y) e^{-2\sqrt{\log b} \, y}.\tag{4.172}$$

By Lemma 4.3.2, we have

$$J_2 \leq c_2 (b^n/|A|)^{1/8} (1+y) e^{-2\sqrt{\log b} y} e^{-2\sqrt{\log b} (b^n/|A|)^{1/8}}. \quad (4.173)$$

By (4.171)-(4.173), we have (4.170). $\square$

In the proof of Proposition 4.4.1, we need a version of Proposition 4.3.1(ii):

Proposition 4.7.2 *There exist $c_1 > 0$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$, $t \geq t_0$, and $y \in [0, 2\sqrt{n}]$,*

$$P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq c_1 e^{-2\sqrt{\log b} y}. \quad (4.174)$$

Remark 4.7.3 *The assumption of t in Proposition 4.7.2 is weaker than that of Proposition 4.3.1(ii). This is the main requirement in the proof of Proposition 4.4.1.*

Fix $\varepsilon \in (0, 1/4)$. For $n \in \mathbb{N}$, $t > 0$, $y > 0$, $0 < r < n$, and $v \in T_n$, set

$$\begin{aligned} A_{v,r}^n(t) := & \left\{ \sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \in \sqrt{t} + \ell_{y,t,n}(s) + [-g_n(s), -f_n(s)], \right. \\ & \text{for all } r \leq s \leq n-r, \sqrt{\tilde{L}_{\tau(t)}^n(v)} \in \sqrt{t} + a_n(t) + [y, y+1), \\ & \left. \sqrt{\tilde{L}_{\tau(t)}^n(v_{s'})} \geq \sqrt{t} + \ell_{y,t,n}(s') - r^{\frac{1}{2}+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n-r, n] \right\}, \end{aligned}$$

where

$$\ell_{y,t,n}(s) := \left(\frac{a_n(t)}{n} + \frac{y}{n} \right) s, \quad 0 \leq s \leq n,$$

$$f_n(s) := \min \{ s^{1/2-\varepsilon}, (n-s)^{1/2-\varepsilon} \}, \quad g_n(s) := \min \{ s^{1/2+\varepsilon}, (n-s)^{1/2+\varepsilon} \}.$$

To prove Proposition 4.7.2, we will apply the second moment method to $\sum_{v \in T_n} 1_{A_{v,r}^n(t)}$. We first need the following:

Lemma 4.7.4 *There exist $r_0 \in \mathbb{N}$, $c_1 = c_1(r_0) > 0$ such that for all $r_0 \leq r \leq n/4$, $t > 4r^{1+4\varepsilon}$, $y \in [0, 2\sqrt{n}]$, and $v \in T_n$,*

$$\tilde{P}_\rho(A_{v,r}^n(t)) \geq c_1 b^{-n} e^{-2\sqrt{\log b} y}. \quad (4.175)$$

Proof. Fix $r_0 \leq r \leq n/4$, $t > 4r^{1+4\varepsilon}$, and $y \in [0, 2\sqrt{n}]$, where we take $r_0 \in \mathbb{N}$ large enough. By Lemma 4.2.5 and (4.16), we have

$$\begin{aligned}
& \sqrt{\frac{\sqrt{t} + a_n(t) + y + 1}{\sqrt{t}}} \tilde{P}_\rho(A_{v,r}^n(t)) \\
& \geq c_1(r_0) P_0^B \left(\ell_{y,t,n}(s) - g_n(s) \leq \frac{B_s}{\sqrt{2}} < \ell_{y,t,n}(s) - f_n(s), \text{ for all } r \leq s \leq n-r, \right. \\
& \quad \left. B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1), \right. \\
& \quad \left. B_{s'}/\sqrt{2} \geq \ell_{y,t,n}(s') - r^{1/2+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n-r, n] \right), \tag{4.176}
\end{aligned}$$

where we have used the following: under the event that $\sqrt{X_s/2} \geq \sqrt{t} + \ell_{y,t,n}(s) - g_n(s)$ for all $s \in [r, n-r]$ and $\sqrt{X_{s'}/2} \geq \sqrt{t} + \ell_{y,t,n}(s') - r^{1/2+2\varepsilon}$ for all $s' \in [0, r] \cup [n-r, n]$, we have $\exp\left(-\frac{3}{8} \int_0^n \frac{ds}{X_s}\right) \geq c_1(r_0)$. Since $\{B_s - B_n \frac{s}{n} : 0 \leq s \leq n\}$ is independent of B_n and has the same law as that of a standard Brownian bridge from 0 to 0 on $[0, n]$, the right-hand side of (4.176) is bounded from below by

$$\begin{aligned}
& c_1(r_0) P_{0 \rightarrow 0}^n \left(-g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n-r, \right. \\
& \quad \left. X_{s'}/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n-r, n] \right) \\
& \cdot P_0^B \left(B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1] \right), \tag{4.177}
\end{aligned}$$

where for $T > 0$ and $p, q \in \mathbb{R}$, $P_{p \rightarrow q}^T$ is a probability law on $(C[0, T], \mathcal{B}(C[0, T]))$ ($C[0, T]$ is the space of all continuous functions on $[0, T]$ and $\mathcal{B}(C[0, T])$ is the σ -field generated by cylinder sets in $C[0, T]$) under which the coordinate process $\{X_s : 0 \leq s \leq T\}$ is a standard Brownian bridge from p to q on $[0, T]$.

We estimate two probabilities in (4.177). By using the Gaussian density of B_n , we have

$$P_0^B \left(B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1] \right) \geq c_2 b^{-n} n \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}} e^{-2\sqrt{\log b} y}. \quad (4.178)$$

For the rest of the proof, we estimate the other probability. We have

$$\begin{aligned} & P_{0 \rightarrow 0}^n \left(-g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n - r, \right. \\ & \quad \left. X_{s'}/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } s' \in [0, r] \cup [n - r, n] \right) \\ &= E_{0 \rightarrow 0}^n \left[1_{\{-g_n(r) \leq X_r/\sqrt{2} < -f_n(r) - 1, -g_n(n-r) \leq X_{n-r}/\sqrt{2} < -f_n(n-r) - 1\}} \right. \\ & \quad \cdot P_{X_r \rightarrow X_{n-r}}^{n-2r} \left(-g_n(s+r) \leq \frac{X_s}{\sqrt{2}} < -f_n(s+r) - 1, \text{ for all } s \leq n - 2r \right) \\ & \quad \cdot P_{0 \rightarrow X_r}^r \left(X_s/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } 0 \leq s \leq r \right) \\ & \quad \left. \cdot P_{X_{n-r} \rightarrow 0}^r \left(X_s/\sqrt{2} \geq -r^{1/2+2\varepsilon}, \text{ for all } 0 \leq s \leq r \right) \right] \\ &\geq c_3(r_0) P_{0 \rightarrow 0}^n \left(-g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n - r \right), \end{aligned} \quad (4.179)$$

where we have used the following in the second inequality: for $x_1, x_2 > 0$ and $T > 0$,

$$P_{0 \rightarrow 0}^T \left(X_s > -\frac{s}{T}x_1 - \frac{T-s}{T}x_2, \quad 0 \leq s \leq T \right) = 1 - \exp \left(-\frac{2x_1x_2}{T} \right), \quad (4.180)$$

(see, for example, [25, Lemma 2.2]) which implies that the last two probabilities in the expectation in the second display of (4.179) are bounded below by constants depending on r_0 .

To estimate the probability on the right-hand side of (4.179), we use the following: for any $C > 0$ and $\delta > 1/2$, as $r \rightarrow \infty$

$$P_{0 \rightarrow 0}^T (X_s \leq C(s \wedge (T - s))^\delta, \text{ for all } r \leq s \leq T - r | X_s \geq 0, \text{ for all } r \leq s \leq T - r)$$

$$\rightarrow 1 \text{ uniformly in } T, \quad (4.181)$$

and for any $C' > 0$ and $0 < \delta' < 1/2$, as $r \rightarrow \infty$

$$\frac{P_{0 \rightarrow 0}^T (X_s > C'(s \wedge (T - s))^{\delta'}, \text{ for all } r \leq s \leq T - r)}{P_{0 \rightarrow 0}^T (X_s > 0, \text{ for all } r \leq s \leq T - r)} \rightarrow 1, \quad \text{uniformly in } T. \quad (4.182)$$

See [25, Lemma 2.7 and Proposition 6.1]. We have for $r_0 \leq r < n/4$ with r_0 large enough

$$\begin{aligned} & P_{0 \rightarrow 0}^n \left(-g_n(s) \leq X_s/\sqrt{2} < -f_n(s) - 1, \text{ for all } r \leq s \leq n - r \right) \\ &= P_{0 \rightarrow 0}^n \left(\sqrt{2}(f_n(s) + 1) \leq X_s \leq \sqrt{2}g_n(s), \text{ for all } r \leq s \leq n - r \right) \\ &\geq P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n - r) \\ &- P_{0 \rightarrow 0}^n \left(X_s \geq 0, \text{ for all } r \leq s \leq n - r, X_{s'} > \sqrt{2}g_n(s') \text{ for some } s' \in [r, n - r] \right) \\ &- P_{0 \rightarrow 0}^n \left(X_s \geq 0, \text{ for all } r \leq s \leq n - r, X_{s'} < 2f_n(s') \text{ for some } s' \in [r, n - r] \right) \\ &\geq P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n - r) \\ &- \frac{1}{4}P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n - r) - \frac{1}{4}P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n - r) \\ &= \frac{1}{2}P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n - r) \\ &\geq \frac{1}{2}P_{0 \rightarrow 0}^n (X_s \geq 0, \text{ for all } r \leq s \leq n - r, X_r, X_{n-r} \in [\sqrt{r}/2, \sqrt{r}]) \\ &= \frac{1}{2}E_{0 \rightarrow 0}^n \left[1_{\{X_r, X_{n-r} \in [\sqrt{r}/2, \sqrt{r}]\}} P_{X_r \rightarrow X_{n-r}}^{n-2r} (X_s \geq 0, \text{ for all } 0 \leq s \leq n - 2r) \right] \\ &\geq \frac{c_4(r_0)}{n} P_{0 \rightarrow 0}^n (X_r, X_{n-r} \in [\sqrt{r}/2, \sqrt{r}]) \geq \frac{c_5(r_0)}{n}, \end{aligned} \quad (4.183)$$

where we have used the symmetry of a Brownian bridge in the first equality, (4.181)-(4.182) in the third inequality, and (4.180) in the 7th inequality.

Thus, by (4.176)-(4.179) and (4.183), we have (4.175). $\square$

Next, we need the following:

Lemma 4.7.5 *There exist $r_0 \in \mathbb{N}$ and $c_1 > 0$ such that for all $t > 0$, $r_0 \leq r \leq n/4$, $y \in [0, 2\sqrt{n}]$, $k \in \{r, r+1, \dots, n-r-1\}$, $v \in T_n$, and $(\sqrt{t} + \ell_{y,t,n}(k) - g_n(k))^2 \leq q \leq (\sqrt{t} + \ell_{y,t,n}(k) - f_n(k))^2$,*

$$\tilde{P}_{v_k} \left[\sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(q)}^\downarrow}(v_s) < \sqrt{t} + \ell_{y,t,n}(s) - f_n(s), \text{ for all } k \leq s \leq n - r, \right.$$

$$\begin{aligned}
& \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow(q)}^\downarrow}(v) \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \\
& \leq c_1 r^{1/2+\varepsilon} \sqrt{\frac{\sqrt{t} + \ell_{y,t,n}(k) - f_n(k)}{\sqrt{t} + a_n(t) + y} \frac{g_n(k)}{(n-k-r)\sqrt{n-k}}} b^{-(n-k)} \\
& \quad \cdot e^{\frac{3 \log n}{2n}(n-k)} e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}(n-k)} e^{-\frac{2a_n(t)}{n}y \frac{n-k}{n}} e^{-\frac{2a_n(t)}{n}f_n(k)}, \quad (4.184)
\end{aligned}$$

where $\{\tilde{L}_s^\downarrow(x) : (s, x) \in [0, \infty) \times \tilde{T}^{v_k}\}$ is a local time of a Brownian motion on $\tilde{T}^{v_k} \subset \tilde{T}_{\leq n}$ and $\tilde{\tau}^\downarrow(p) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(v_k) > p\}$.

Proof. Let P_k be the left-hand side of (4.184). Recall the probability measure $P_{0 \rightarrow 0}^T$ defined in the proof of Lemma 4.7.4. By Lemma 4.2.5 and (4.16), we have

$$\begin{aligned}
& \sqrt{\frac{\sqrt{t} + a_n(t) + y}{\sqrt{q}}} P_k \\
& \leq P_0^B \left[B_s/\sqrt{2} < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - f_n(k+s), \text{ for all } s \leq n-k-r, \right. \\
& \quad \left. B_{n-k}/\sqrt{2} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \sqrt{t} - \sqrt{q} + a_n(t) + y + 1] \right] \\
& \leq P_{0 \rightarrow 0}^{n-k} \left(X_s/\sqrt{2} < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - \frac{s}{n-k}(\sqrt{t} - \sqrt{q} + a_n(t) + y), \right. \\
& \quad \left. \text{for all } 0 \leq s \leq n-k-r \right) \\
& \quad \cdot P_0^B \left(B_{n-k}/\sqrt{2} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \sqrt{t} - \sqrt{q} + a_n(t) + y + 1] \right), \quad (4.185)
\end{aligned}$$

where we have used the fact that the process $\{B_s - \frac{s}{n-k} \cdot B_{n-k} : s \leq n-k\}$ is independent of B_{n-k} and has the same law as that of a standard Brownian bridge from 0 to 0 on $[0, n-k]$.

We estimate two probabilities on the right-hand side of (4.185). By using the Gaussian density of B_{n-k} ,

$$P_0^B \left(B_{n-k}/\sqrt{2} \in [\sqrt{t} - \sqrt{q} + a_n(t) + y, \sqrt{t} - \sqrt{q} + a_n(t) + y + 1] \right)$$

$$\leq \frac{c_1}{\sqrt{n-k}} b^{-(n-k)} e^{\frac{3 \log n}{2n}(n-k)} e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}(n-k)} e^{-\frac{2a_n(t)}{n}y \frac{n-k}{n}} e^{-\frac{2a_n(t)}{n}f_n(k)}. \quad (4.186)$$

To estimate the other probability, we use the following: for any $x_1, x_2 \in [0, \infty)$, $r_1, r_2 \in [0, \infty)$, and $T > r_1 + r_2$,

$$\begin{aligned} & P_{0 \rightarrow 0}^T \left(X_s \leq \left(1 - \frac{s}{T}\right) x_1 + \frac{s}{T} x_2, \quad r_1 \leq s \leq T - r_2 \right) \\ & \leq \frac{2}{T - r_1 - r_2} \left\{ \left(1 - \frac{r_1}{T}\right) x_1 + \frac{r_1}{T} x_2 + \sqrt{r_1} \right\} \left\{ \frac{r_2}{T} x_1 + \left(1 - \frac{r_2}{T}\right) x_2 + \sqrt{r_2} \right\}. \end{aligned} \quad (4.187)$$

See [6, Lemma 3.4]. By (4.187), we have

$$\begin{aligned} & P_{0 \rightarrow 0}^{n-k} \left(\frac{X_s}{\sqrt{2}} < \sqrt{t} - \sqrt{q} + \ell_{y,t,n}(k+s) - \frac{s}{n-k}(\sqrt{t} - \sqrt{q} + a_n(t) + y) \right. \\ & \quad \left. \text{for all } 0 \leq s \leq n-k-r \right) \\ & \leq c_2 \frac{r^{1/2+\varepsilon}}{n-k-r} g_n(k). \end{aligned} \quad (4.188)$$

Thus, by (4.185), (4.186), and (4.188), we have (4.184). $\square$

Lemma 4.7.6 (i) *There exist $c_1 > 0$ and $r_0 \in \mathbb{N}$ such that for all $t > 0$, $r_0 \leq r \leq n/4$, $y \in [0, 2\sqrt{n}]$, $k \in \{r, r+1, \dots, n-r-1\}$, and $v, u \in T_n$ with $|v \wedge u| = k$ such that*

$$\begin{aligned} & \tilde{P}_\rho(A_{v,r}^n(t) \cap A_{u,r}^n(t)) \\ & \leq c_1 r^{3/2+\varepsilon} b^{-2n+k} \sqrt{\frac{\sqrt{t} + \ell_{y,t,n}(k) - f_n(k)}{\sqrt{t} + a_n(t) + y}} \frac{g_n(k)}{(n-k-r)\sqrt{n-k}} \\ & \quad \cdot e^{\frac{3 \log n}{2n}(n-k)} e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}(n-k)} e^{-\frac{4a_n(t)}{n}y} e^{\frac{2a_n(t)}{n}y \frac{k}{n}} e^{-\frac{2a_n(t)}{n}f_n(k)}. \end{aligned} \quad (4.189)$$

(ii) *There exist $r_0 \in \mathbb{N}$, $c_2 = c_2(r_0) > 0$, $n_0 = n_0(r_0) \in \mathbb{N}$ such that for all $t > 0$, $n \geq n_0$, $r_0 \leq r \leq 2r_0$, $y \in [0, 2\sqrt{n}]$, $k \in \{0, \dots, r\}$, and $v, u \in T_n$ with $|v \wedge u| = k$,*

$$\tilde{P}_\rho(A_{v,r}^n(t) \cap A_{u,r}^n(t)) \leq c_2 b^{-2n+r} e^{-4\sqrt{\log b} \, y}. \quad (4.190)$$

Proof. For $w \in \{v, u\}$, set

$$\begin{aligned} \tilde{C}_w^\downarrow := & \left\{ \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(v_k))}(w_s) < \sqrt{t} + \ell_{y,t,n}(s) - f_n(s), \text{ for all } k \leq s \leq n-r, \right. \\ & \left. \sqrt{\tilde{L}_{\tilde{\tau}^\downarrow}^\downarrow(\tilde{L}_{\tilde{\tau}(t)}^n(v_k))}(w) \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right\}. \end{aligned} \quad (4.191)$$

where $\{\tilde{L}_s^\downarrow(x) : (s, x) \in [0, \infty) \times \tilde{T}^{v_k}\}$ is a local time of a Brownian motion on $\tilde{T}^{v_k} \subset \tilde{T}_{\leq n}$ and $\tilde{\tau}^\downarrow(q) := \inf\{s \geq 0 : \tilde{L}_s^\downarrow(v_k) > q\}$. By using Lemma 4.2.1 in a way similar to (4.58), we have

$$\begin{aligned} & \tilde{P}_\rho(A_{v,r}^n(t) \cap A_{u,r}^n(t)) \\ & \leq \tilde{E}_\rho \left[1_{\left\{ \sqrt{t} + \ell_{y,t,n}(s) - g_n(s) \leq \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \ell_{y,t,n}(s) - f_n(s), \text{ for all } r \leq s \leq k \right\}} \right. \\ & \quad \left. \tilde{P}_{v_k}(\tilde{C}_v^\downarrow) \cdot \tilde{P}_{v_k}(\tilde{C}_u^\downarrow) \right] \\ & \leq c_1 r^{1/2+\varepsilon} \sqrt{\frac{\sqrt{t} + \ell_{y,t,n} - f_n(k)}{\sqrt{t} + a_n(t) + y}} \frac{g_n(k)}{(n-k-r)\sqrt{n-k}} b^{-(n-k)} \\ & \quad \cdot e^{\frac{3 \log n}{2n}(n-k)} e^{\frac{\log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right)}{2n}(n-k)} e^{-\frac{2a_n(t)}{n}y \frac{n-k}{n}} e^{-\frac{2a_n(t)}{n}f_n(k)} \\ & \quad \cdot \tilde{P}_\rho \left[\sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \ell_{y,t,n}(s) - f_n(s), \text{ for all } r \leq s \leq n-r, \right. \\ & \quad \left. \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right], \end{aligned} \quad (4.192)$$

where in the last inequality, we have used Lemma 4.7.5 for $\tilde{P}_{v_k}(\tilde{C}_u^\downarrow)$ and Lemma 4.2.1.

We estimate the probability on the right-hand side of (4.192). By using Lemma 4.2.5 and (4.16) in a way similar to (4.185), we have

$$\sqrt{\frac{\sqrt{t} + a_n(t) + y}{\sqrt{t}}} \tilde{P}_\rho \left[\sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v_s)} < \sqrt{t} + \ell_{y,t,n}(s) - f_n(s), \right.$$

$$\begin{aligned}
& \left[\text{for all } r \leq s \leq n-r, \sqrt{\tilde{L}_{\tilde{\tau}(t)}^n(v)} \in [\sqrt{t} + a_n(t) + y, \sqrt{t} + a_n(t) + y + 1] \right] \\
& \leq P_{0 \rightarrow 0}^n (X_s < 1, \text{ for all } r \leq s \leq n-r) \cdot P_0^B \left(\frac{B_n}{\sqrt{2}} \in a_n(t) + y + [0, 1) \right) \\
& \leq c_2 r b^{-n} e^{-\frac{2a_n(t)}{n}y} \sqrt{\frac{\sqrt{t} + n}{\sqrt{t}}}, \tag{4.193}
\end{aligned}$$

where in the last inequality, we have used the Gaussian density of B_n and (4.187). Thus, by (4.192) and (4.193), we have (4.189).

The proof of (4.190) is similar to that of (4.189). We omit the detail. $\square$

Proof of Proposition 4.7.2. Fix any $r_0 \leq r \leq 2r_0$ and $t > 4(2r_0)^{1+4\epsilon}$, where we take $r_0 \in \mathbb{N}$ large enough. We first assume that $n \geq n_0$, where we take $n_0 = n_0(r_0) > 8r_0$ large enough. Fix any $y \in [0, 2\sqrt{n}]$. Set

$$Z := \sum_{v \in T_n} 1_{A_{v,r}^n(t)}.$$

We have

$$P_\rho \left(\max_{v \in T_n} \sqrt{L_{\tau(t)}^n(v)} \geq \sqrt{t} + a_n(t) + y \right) \geq \tilde{P}_\rho(Z \geq 1) \geq \frac{(\tilde{E}_\rho[Z])^2}{\tilde{E}_\rho[Z^2]}. \tag{4.194}$$

By Lemma 4.7.4, we have

$$\tilde{E}_\rho[Z] \geq c_1 e^{-2\sqrt{\log b} y}. \tag{4.195}$$

Next, we obtain an upper bound of $\tilde{E}_\rho[Z^2]$. We have

$$\tilde{E}_\rho[Z^2] = \tilde{E}_\rho[Z] + \sum_{k=0}^{n-1} \sum_{\substack{v, u \in T_n, \\ |v \wedge u| = k}} \tilde{P}_\rho(A_{v,r}^n(t) \cap A_{u,r}^n(t)). \tag{4.196}$$

By (4.193), we have

$$\tilde{E}_\rho[Z] \leq c_2 e^{-2\sqrt{\log b} y}. \tag{4.197}$$

By Lemma 4.7.6, we have

$$\sum_{k=0}^{n-r-1} \sum_{\substack{v, u \in T_n, \\ |v \wedge u| = k}} \tilde{P}_\rho(A_{v,r}^n(t) \cap A_{u,r}^n(t)) \leq c_3 e^{-2\sqrt{\log b} y}. \tag{4.198}$$

By (4.193), we have

$$\sum_{k=n-r}^{n-1} \sum_{\substack{v,u \in T_n, \\ |v \wedge u|=k}} \tilde{P}_\rho(A_{v,r}^n(t) \cap A_{u,r}^n(t)) \leq \sum_{k=n-r}^{n-1} \sum_{\substack{v,u \in T_n, \\ |v \wedge u|=k}} \tilde{P}_\rho(A_{v,r}^n(t)) \leq c_4 e^{-2\sqrt{\log b} y}. \quad (4.199)$$

Thus, by (4.194)-(4.199), we have (4.174) for $n \geq n_0$.

Next, we consider the case $n \leq n_0$. Fix any $y \in [0, 2\sqrt{n}]$, $v_0 \in T_n$, and $\delta \in (0, 1)$. By Lemma 4.2.5 and (4.16), we have

$$\begin{aligned} & P_\rho \left(\max_{u \in T_n} \sqrt{L_{\tau(t)}^n(u)} \geq \sqrt{t} + a_n(t) + y \right) \\ & \geq \tilde{P}_\rho \left(\sqrt{\tilde{L}_{\tau(t)}^n(v_s)} \geq \delta \sqrt{t}, \ 0 \leq s \leq n, \ \sqrt{\tilde{L}_{\tau(t)}^n(v)} \in \sqrt{t} + a_n(t) + y + [0, 1) \right) \\ & \geq c_5 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} \\ & \cdot P_0^B \left(\frac{B_s}{\sqrt{2}} \geq -(1 - \delta)\sqrt{t}, \text{ for all } s \leq n, \ B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1] \right) \\ & \geq c_5 \sqrt{\frac{\sqrt{t}}{\sqrt{t} + a_n(t) + y + 1}} P_0^B \left(B_n/\sqrt{2} \in [a_n(t) + y, a_n(t) + y + 1] \right) \\ & \cdot P_{0 \rightarrow 0}^n \left(X_s/\sqrt{2} \geq -(1 - \delta)\sqrt{t} - (a_n(t) + y)\frac{s}{n}, \text{ for all } 0 \leq s \leq n \right) \\ & \geq c_6 \geq c_6 e^{-2\sqrt{\log b} y}, \end{aligned} \quad (4.200)$$

where we have used the fact that under the event that $\sqrt{X_s/2} \geq \delta\sqrt{t}$ for all $0 \leq s \leq n$, we have $\exp\left(-\frac{3}{8} \int_0^n \frac{ds}{X_s}\right) \geq e^{-\frac{3n_0}{16\delta^2 t_0}} =: c_5$ in the second inequality, the independence of $B_s - \frac{s}{n}B_n$ and B_n in the third inequality, and the Gaussian density of B_n and (4.180) in the 4th inequality. $\square$

4.7.4 Tail of maximum of BRW

In the proof of Proposition 4.5.1, we use tail estimates of the maximum of the BRW on T at time n . Let $(h_v)_{v \in T}$ be a BRW on T defined in Section 4.1.

Lemma 4.7.7 (i) *There exist $c_1, c_2 \in (0, \infty)$ such that for all $y > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{P} \left(\max_{v \in T_n} h_v / \sqrt{2} > \sqrt{\log b} \, n - \frac{3}{4\sqrt{\log b}} \log n + y \right) \leq c_1(1+y)e^{-2\sqrt{\log b} \, y} e^{-c_2 \frac{y^2}{n}}.$$

(ii) *There exists $c_3 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $y \in [1, \sqrt{n}]$,*

$$\mathbb{P} \left(\max_{v \in T_n} h_v / \sqrt{2} > \sqrt{\log b} \, n - \frac{3}{4\sqrt{\log b}} \log n + y \right) \geq c_3 y e^{-2\sqrt{\log b} \, y}.$$

Lemma 4.7.7(ii) is a special version of Lemma 2.7 of [27]. One can easily modify the proof of Lemma 3.8 in [26] to prove Lemma 4.7.7(i). We omit the details.

4.7.5 Proof of Lemma 4.2.6

Following an argument in the proof of [26, Lemma 3.6], we give a proof of Lemma 4.2.6 for the sake of completeness.

Proof of Lemma 4.2.6. (i) Fix any interval $I = (\underline{I}, \overline{I}] \subset (-\infty, z + z^{1/20}]$.

We consider the cases $s/2 > z + z^{1/20} + 1$ and $s/2 \leq z + z^{1/20} + 1$ separately.

Case 1: $s/2 > z + z^{1/20} + 1$. For $0 \leq j \leq \lfloor s \rfloor$, we set

$$\psi_{s,z}^*(j) := \max_{j \leq r \leq (j+1) \wedge s} \left(z + z^{1/20} + C(r \wedge (s-r))^{1/20} \right). \quad (4.201)$$

We have

$$\begin{aligned} & P_0^B \left(B_s / \sqrt{2} \in I, B_r / \sqrt{2} \leq z + z^{1/20} + C(r \wedge (s-r))^{1/20}, \text{ for all } 0 \leq r \leq s \right) \\ & - P_0^B \left(B_s / \sqrt{2} \in I, B_r / \sqrt{2} \leq z + z^{1/20}, \text{ for all } 0 \leq r \leq s \right) \\ & = P_0^B \left(B_s / \sqrt{2} \in I, B_{r'} / \sqrt{2} > z + z^{1/20}, \text{ for some } 0 \leq r' \leq s, \right. \\ & \quad \left. B_r / \sqrt{2} \leq z + z^{1/20} + C(r \wedge (s-r))^{1/20}, \text{ for all } 0 \leq r \leq s \right) \\ & \leq \sum_{j=0}^{\lfloor s \rfloor} P_0^B \left(B_s / \sqrt{2} \in I, B_{r'} / \sqrt{2} > z + z^{1/20}, \text{ for some } 0 \leq r' \leq s, \right. \\ & \quad \left. \frac{B_r}{\sqrt{2}} \leq z + z^{\frac{1}{20}} + C(r \wedge (s-r))^{\frac{1}{20}}, \text{ for all } r \leq s, \max_{j \leq r \leq (j+1) \wedge s} B_r = \max_{0 \leq r \leq s} B_r \right) \end{aligned}$$

$$\leq \sum_{j=0}^{\lfloor s \rfloor} P_0^B \left(B_s/\sqrt{2} \in I, B_r/\sqrt{2} \leq \psi_{s,z}^*(j), \text{ for all } 0 \leq r \leq s, \right. \\ \left. z + z^{1/20} < \max_{j \leq r \leq (j+1) \wedge s} B_r/\sqrt{2} \leq \psi_{s,z}^*(j) \right). \quad (4.202)$$

Fix $1 \leq j \leq \lfloor s \rfloor - 2$. We estimate the j th term on the right-hand side of (4.202). Recall the density (4.21). For $x \leq \psi_{s,z}^*(j)$, we set

$$\tau_x := \inf \left\{ r \in [0, 1] : \frac{B_r}{\sqrt{2}} > z + z^{1/20} - x \right\}.$$

By the Markov property at time j , we have

$$\begin{aligned} & P_0^B \left(\frac{B_s}{\sqrt{2}} \in I, \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j), \text{ for all } r \leq s, z + z^{1/20} < \max_{j \leq r \leq j+1} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) \right) \\ &= E_0^B \left[1_{\{B_r/\sqrt{2} \leq \psi_{s,z}^*(j), \text{ for all } 0 \leq r \leq j\}} \right. \\ &\quad \left. P_{B_j}^B \left(B_{s-j}/\sqrt{2} \in I, B_r/\sqrt{2} \leq \psi_{s,z}^*(j), \text{ for all } r \leq s-j, \right. \right. \\ &\quad \left. \left. z + z^{1/20} < \max_{0 \leq r \leq 1} B_r/\sqrt{2} \leq \psi_{s,z}^*(j) \right) \right] \\ &= \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\frac{B_{s-j}}{\sqrt{2}} \in I - x, \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) - x, \text{ for all } r \leq s-j, \right. \\ &\quad \left. z + z^{1/20} - x < \max_{0 \leq r \leq 1} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) - x \right) dx \\ &\leq \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) \\ &\quad E_0^B \left[1_{\{\tau_x \leq 1, B_{\tau_x}/\sqrt{2} \leq \psi_{s,z}^*(j) - x\}} \right. \\ &\quad \left. P_{B_{\tau_x}}^B \left(\frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) - x, \text{ for all } r \leq s-j-\tau_x, \frac{B_{s-j-\tau_x}}{\sqrt{2}} \in I - x \right) \right] dx \\ &\leq \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r/\sqrt{2} > z + z^{1/20} - x \right) \\ &\quad \sup_{j \leq s' \leq j+1} \sup_{z+z^{1/20}-x \leq y \leq \psi_{s,z}^*(j)-x} P_0^B \left(B_r/\sqrt{2} \leq \psi_{s,z}^*(j) - x - y, \text{ for all } r \leq s-s', \right. \\ &\quad \left. x + y + B_{s-s'}/\sqrt{2} \in I \right) dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r / \sqrt{2} > z + z^{1/20} - x \right) dx \\
&\quad \cdot \sup_{j \leq s' \leq j+1} \int_I \mu_{s-s',\psi_{s,z}^*(j)-z-z^{1/20}}(w - z - z^{1/20}) dw, \tag{4.203}
\end{aligned}$$

where we have used the strong Markov property at τ_x in the third inequality, and in the 4th inequality, we have used the following: for all $x \leq \psi_{s,z}^*(j)$, $j \leq s' \leq j+1$, $z + z^{1/20} - x \leq y \leq \psi_{s,z}^*(j) - x$, and $w \in I$,

$$\mu_{s-s',\psi_{s,z}^*(j)-x-y}(w - x - y) \leq \mu_{s-s',\psi_{s,z}^*(j)-z-z^{1/20}}(w - z - z^{1/20}).$$

We will estimate the integrals on the right-hand side of (4.203). By the density (4.21) and (4.40), we have

$$\begin{aligned}
&\int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r / \sqrt{2} > z + z^{1/20} - x \right) dx \\
&\leq \int_{z+z^{1/20}}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) dx \\
&\quad + \int_{-\infty}^{z+z^{1/20}} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r / \sqrt{2} > z + z^{1/20} - x \right) dx \\
&\leq c_1 A j^{-3/2} e^{-\frac{(z+z^{1/20})^2}{s}}, \tag{4.204}
\end{aligned}$$

where $A := (j \wedge (s-j))^{\frac{1}{10}} \psi_{s,z}^*(j) + j$ if $1 \leq j \leq \lfloor z + z^{\frac{1}{20}} \rfloor$ and $A := (j \wedge (s-j))^{\frac{1}{10}} \psi_{s,z}^*(j)$ if $\lfloor z + z^{\frac{1}{20}} \rfloor < j \leq \lfloor s \rfloor - 2$. Similarly, for all $j \leq s' \leq j+1$, we have

$$\begin{aligned}
&\mu_{s-s',\psi_{s,z}^*(j)-z-z^{1/20}}(w - z - z^{1/20}) \\
&\leq c_2 (s-j-1)^{-3/2} (\psi_{s,z}^*(j) - w) (\psi_{s,z}^*(j) - z - z^{1/20}) e^{-\frac{(z+z^{1/20}-w)^2}{s}}. \tag{4.205}
\end{aligned}$$

Thus, by (4.203)-(4.205), we have

$$\sum_{j=1}^{\lfloor s \rfloor - 2} P_0^B \left(\frac{B_s}{\sqrt{2}} \in I, \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j), \text{ for all } 0 \leq r \leq s, \right.$$

$$\begin{aligned}
& z + z^{1/20} < \max_{j \leq r \leq j+1} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) \\
& \leq c_3 s^{-3/2} \int_I z e^{-\frac{w^2}{s}} (z + z^{1/20} - w + 1_{\{w \geq z + z^{1/20} - 1\}}) dw. \tag{4.206}
\end{aligned}$$

By the density (4.21), we have for all $j \in \{0, \lfloor s \rfloor - 1, \lfloor s \rfloor\}$,

$$\begin{aligned}
& P_0^B \left(\frac{B_s}{\sqrt{2}} \in I, \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j), \text{ for all } r \leq s, z + z^{\frac{1}{20}} < \max_{j \leq r \leq (j+1) \wedge s} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) \right) \\
& \leq P_0^B \left(B_r/\sqrt{2} \leq z + z^{1/20} + 3^{1/20}C, \text{ for all } 0 \leq r \leq s, B_s/\sqrt{2} \in I \right) \\
& \leq c_4 s^{-3/2} \int_I z e^{-\frac{x^2}{s}} (z + z^{1/20} - x + 1_{\{x \geq z + z^{1/20} - 1\}}) dx. \tag{4.207}
\end{aligned}$$

Similarly, we have

$$\int_I \mu_{s,z+z^{1/20}}(x) dx \leq c_5 s^{-3/2} \int_I z(z + z^{1/20} - x) e^{-\frac{x^2}{s}} dx. \tag{4.208}$$

Thus, by (4.202) and (4.206)-(4.208), we have (4.22) with $f = 1_I$.

Case 2: $0 < s/2 \leq z + z^{1/20} + 1$. By the density (4.21), we have

$$\begin{aligned}
& P_0^B \left(B_s/\sqrt{2} \in I, B_r/\sqrt{2} \leq z + z^{1/20} + C(r \wedge (s-r))^{1/20}, \text{ for all } r \leq s \right) \\
& \leq P_0^B \left(B_s/\sqrt{2} \in I, B_r/\sqrt{2} \leq z + z^{1/20} + c_6 z^{1/20}, \text{ for all } 0 \leq r \leq s \right) \\
& \leq c_7 s^{-3/2} \int_I z e^{-\frac{x^2}{s}} \left(z + z^{1/20} - x + z^{1/20} 1_{\{x \geq z + \frac{1}{2} z^{1/20}\}} \right) dx. \tag{4.209}
\end{aligned}$$

Thus, (4.22) holds for $f = 1_I$.

By the monotone class theorem and the monotone convergence theorem, (4.22) holds for general nonnegative measurable function f on $(-\infty, z + z^{1/20}]$.

(ii) Fix any $z \geq z_0$, where we take $z_0 > 0$ large enough. Fix any interval $I = (\underline{I}, \overline{I}] \subset (z - \frac{s}{4z}, 0]$. Recall (4.201). By an argument similar to (4.202), we have

$$P_0^B \left(B_s/\sqrt{2} \in I, B_r/\sqrt{2} \leq z + z^{1/20} + C(r \wedge (s-r))^{1/20}, \text{ for all } 0 \leq r \leq s \right)$$

$$\begin{aligned}
& - P_0^B \left(B_s/\sqrt{2} \in I, B_r/\sqrt{2} \leq z, \text{ for all } 0 \leq r \leq s \right) \\
& \leq \sum_{j=0}^{\lfloor s \rfloor} P_0^B \left(\frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j), \text{ for all } r \leq s, \frac{B_s}{\sqrt{2}} \in I, z < \max_{j \leq r \leq (j+1) \wedge s} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) \right).
\end{aligned} \tag{4.210}$$

We estimate the probabilities on the right-hand side of (4.210). Fix $1 \leq j \leq \lfloor s \rfloor - 2$. By an argument similar to (4.203), we have

$$\begin{aligned}
& P_0^B \left(\frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j), \text{ for all } r \leq s, \frac{B_s}{\sqrt{2}} \in I, z < \max_{j \leq r \leq (j+1) \wedge s} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) \right) \\
& \leq \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r/\sqrt{2} > z - x \right) dx \\
& \quad \cdot \sup_{j \leq s' \leq j+1} \int_I \mu_{s-s',\psi_{s,z}^*(j)-z}(w - z) dw.
\end{aligned} \tag{4.211}$$

By an argument similar to (4.204) and (4.205), we have

$$\begin{aligned}
& \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r/\sqrt{2} > z - x \right) dx \\
& \leq c_8 j^{-3/2} \psi_{s,z}^*(j) (\psi_{s,z}^*(j) - z)^2 e^{-\frac{z^2}{j}} \\
& \quad + c_8 j^{-3/2} \psi_{s,z}^*(j) \left(\frac{z}{j+1} + z^{1/20} + \max_{j \leq r \leq j+1} (r \wedge (s-r))^{1/20} \right) e^{-\frac{z^2}{j+1}},
\end{aligned} \tag{4.212}$$

$$\begin{aligned}
& \sup_{j \leq s' \leq j+1} \int_I \mu_{s-s',\psi_{s,z}^*(j)-z}(w - z) dw \\
& \leq c_9 (s - j - 1)^{-3/2} (\psi_{s,z}^*(j) - z) \int_I (\psi_{s,z}^*(j) - w) e^{-\frac{(z-w)^2}{s-j}} dw.
\end{aligned} \tag{4.213}$$

Note that since $s > 4z(z - \underline{I})$ by assumption, we have $\lfloor \frac{s}{2} \rfloor > \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor$. By (4.212) and (4.213), we have

$$\sum_{j=\left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor}^{\lfloor s \rfloor - \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor} \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j,\psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r/\sqrt{2} > z - x \right) dx$$

$$\begin{aligned}
& \cdot \sup_{j \leq s' \leq j+1} \int_I \mu_{s-s', \psi_{s,z}^*(j)-z}(w-z) dw \\
& \leq c_{10} s^{-3/2} z^{3/10} \log z \int_I (z + z^{1/20} - w) e^{-\frac{w^2}{s}} dw.
\end{aligned} \tag{4.214}$$

By (4.212)-(4.213) and $s > 4z^2$, in the case $1 \leq j \leq \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor$ or $\lfloor s \rfloor - \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor \leq j \leq \lfloor s \rfloor - 2$, we have

$$\begin{aligned}
& \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j, \psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r / \sqrt{2} > z - x \right) dx \\
& \leq \begin{cases} c_{11} j^{-3/2} z^2 e^{-\frac{(\log z)^2}{2}}, & \text{if } 1 \leq j \leq \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor, \\ c_{11} s^{-3/2} z^{6/5}, & \text{if } \lfloor s \rfloor - \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor \leq j \leq \lfloor s \rfloor - 2, \end{cases}
\end{aligned} \tag{4.215}$$

$$\begin{aligned}
& \sup_{j \leq s' \leq j+1} \int_I \mu_{s-s', \psi_{s,z}^*(j)-z}(w-z) dw \\
& \leq c_{12} A' z^{1/10} \int_I (z + z^{1/20} - w + (j \wedge (s-j))^{1/20}) e^{-\frac{w^2}{s}} dw,
\end{aligned} \tag{4.216}$$

where $A' := s^{-3/2}$ if $1 \leq j \leq \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor$ and $A' := (s-j)^{-3/2} e^{-\frac{(\log z)^2}{2}}$ if $\lfloor s \rfloor - \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor \leq j \leq \lfloor s \rfloor - 2$. Thus, by (4.215) and (4.216), we have

$$\begin{aligned}
& \sum_{j=1}^{\left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor} \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j, \psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r / \sqrt{2} > z - x \right) dx \\
& \cdot \sup_{j \leq s' \leq j+1} \int_I \mu_{s-s', \psi_{s,z}^*(j)-z}(w-z) dw \\
& \leq c_{13} s^{-3/2} z^{21/10} e^{-\frac{(\log z)^2}{2}} \int_I (z + z^{1/20} - w) e^{-\frac{w^2}{s}} dw,
\end{aligned} \tag{4.217}$$

$$\sum_{j=\lfloor s \rfloor - \left\lfloor \frac{z^2}{(\log z)^2} \right\rfloor}^{\lfloor s \rfloor - 2} \int_{-\infty}^{\psi_{s,z}^*(j)} \mu_{j, \psi_{s,z}^*(j)}(x) P_0^B \left(\max_{0 \leq r \leq 1} B_r / \sqrt{2} > z - x \right) dx$$

$$\begin{aligned}
& \cdot \sup_{j \leq s' \leq j+1} \int_I \mu_{s-s', \psi_{s,z}^*(j)-z}(w-z) dw \\
& \leq c_{14} s^{-3/2} z^{13/10} e^{-\frac{(\log z)^2}{2}} \int_I (z + z^{1/20} - w) e^{-\frac{w^2}{s}} dw.
\end{aligned} \tag{4.218}$$

Next, we estimate the probabilities on the right-hand side of (4.210) for $j \in \{0, \lfloor s \rfloor - 1, \lfloor s \rfloor\}$. In the case $j = 0$, by an argument similar to (4.211), we have

$$\begin{aligned}
& P_0^B \left(\frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(0), \text{ for all } 0 \leq r \leq s, \frac{B_s}{\sqrt{2}} \in I, z < \max_{0 \leq r \leq 1} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(0) \right) \\
& \leq P_0^B \left(\max_{0 \leq r \leq 1} B_r / \sqrt{2} > z \right) \cdot \sup_{0 \leq s' \leq 1} \int_I \mu_{s-s', \psi_{s,z}^*(0)-z}(w-z) dw \\
& \leq c_{15} s^{-3/2} z^{1/20} e^{-z^2} \int_I (z + z^{1/20} - w) e^{-\frac{w^2}{s}} dw,
\end{aligned} \tag{4.219}$$

where we have used (4.40) in the last inequality. In the case $j \in \{\lfloor s \rfloor - 1, \lfloor s \rfloor\}$, by an argument similar to (4.211), we have

$$\begin{aligned}
& P_0^B \left(\frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j), \text{ for all } r \leq s, \frac{B_s}{\sqrt{2}} \in I, z < \max_{j \leq r \leq (j+1) \wedge s} \frac{B_r}{\sqrt{2}} \leq \psi_{s,z}^*(j) \right) \\
& \leq P_0^B \left(B_r / \sqrt{2} \leq z + z^{1/20} + 2^{1/20} C, \text{ for all } 0 \leq r \leq s, \right. \\
& \quad \left. z < \max_{s-2 \leq r \leq s} B_r / \sqrt{2} \leq z + z^{1/20} + 2^{1/20} C, B_s / \sqrt{2} \in I \right) \\
& \leq \int_z^{z+z^{1/20}+2^{1/20}C} \mu_{s-2, z+z^{1/20}+2^{1/20}C}(x) P_0^B \left(B_2 / \sqrt{2} + x \in I \right) dx \\
& + \int_{-\infty}^z \mu_{s-2, z+z^{1/20}+2^{1/20}C}(x) P_0^B \left(\max_{0 \leq r \leq 2} B_r / \sqrt{2} > z - x, B_2 / \sqrt{2} + x \in I \right) dx \\
& \leq c_{16} s^{-3/2} z e^{-\frac{z^2}{2}} \int_I (z + z^{1/20} - w) e^{-\frac{w^2}{s}} dw,
\end{aligned} \tag{4.220}$$

where we have used the Gaussian density of B_2 and (4.48) in the last inequality.

Next, we estimate $\int_I \mu_{s,z}(x) dx$. For all $s > 4z(z - \underline{I})$, we have

$$\int_I \mu_{s,z}(x) dx = \frac{1}{\sqrt{\pi s}} \int_I e^{-\frac{x^2}{s}} \left(1 - e^{-\frac{4z(z-x)}{s}} \right) dx$$

$$\begin{aligned}
&\geq \frac{1}{\sqrt{\pi s}} \int_I e^{-\frac{x^2}{s}} \left(\frac{4z(z-x)}{s} - \frac{8z^2(z-x)^2}{s^2} \right) dx \\
&\geq \frac{2}{\sqrt{\pi}} s^{-3/2} \int_I e^{-\frac{x^2}{s}} z(z-x) dx, \tag{4.221}
\end{aligned}$$

where we have used the inequality $1 - e^{-u} \geq u - u^2/2$ for $u \geq 0$ in the second inequality, and $s > 4z(z-x)$ for all $x \in I$ in the last inequality.

Thus, by (4.210)-(4.211), (4.214), and (4.217)-(4.221), we have (4.23) with $f = 1_I$ and $\delta_z = c_{18} z^{-7/10} \log z$. By the monotone class theorem and the monotone convergence theorem, we have (4.23) for general nonnegative measurable function f on $(z - \frac{s}{4z}, 0]$. $\square$

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