

ON SOME INTEGRAL MONOID RINGS

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1. INTRODUCTION

Let $\mathbf{Rng}$ denote the category of rings with identity and let $\mathbf{Mon}$ denote the category of monoids. Let $\mathbf{Z}$ denote the ring of rational integers and define a functor $F : \mathbf{Mon} \rightarrow \mathbf{Rng}$ by $F(M) = \mathbf{Z}[M]$, the integral monoid ring of M . Then F is a left adjoint functor of the forgetful functor $U : \mathbf{Rng} \rightarrow \mathbf{Mon}$. We consider the functor $FU : \mathbf{Rng} \rightarrow \mathbf{Rng}$ and we study which properties of rings are preserved by the functor FU .

2. EXAMPLES

Let $\mathbf{Z}$ denote the ring of rational integers and let $\mathbf{Q}$ denote the field of rational numbers. Let $(R, \cdot, +)$ be a ring and consider the monoid rings $\mathbf{Z}[(R, \cdot)]$ and $\mathbf{Q}[(R, \cdot)]$. We briefly denote these rings by $\mathbf{Z}[R]$ and $\mathbf{Q}[R]$. In this section we consider some examples.

Example 1 Consider the polynomial ring $GF(3)[x]$ over the Galois field $GF(3)$ of three elements. We can easily see the monoid $(GF(3)[x], \cdot)$ is isomorphic to the monoid $(\mathbf{Z}, \cdot)$. Hence $\mathbf{Z}[(GF(3)[x])] is isomorphic to $\mathbf{Z}[\mathbf{Z}]$.$

The monoid ring $\mathbf{Z}[R]$ of a ring R is determined by the monoid structure of the ring R . So we consider some properties of rings which depend only on the monoid structure of rings and we ask whether those properties are preserved by the the functor $FU : \mathbf{Rng} \rightarrow \mathbf{Rng}$ or not. More generally we can consider the following problem.

Problem. Let P be some property on (monoids of) rings. If a ring R has property P , then what can be said about the structure of $\mathbf{Z}[R]$ and $\mathbf{Q}[R]$?

A ring R is said to be *prime* if $aRb \neq 0$ for all nonzero $a, b \in R$. The following example shows that primeness does not preserved by the functor FU .

Example 2 Let $\mathbf{Q}$ denote the field of rational numbers. Then we can easily see that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}[(GF(3))] \cong \mathbf{Q}[(GF(3))]$ is isomorphic to $\mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q}$.

A ring R is called a *von Neumann regular* ring if for each $a \in R$ there exists $x \in R$ such that $a = axa$. The following example shows that the von Neumann regularity does not preserved by the functor $\mathbf{Rng} \rightarrow \mathbf{Rng}; R \rightarrow \mathbf{Q}[R]$.

Example 3 Let D be a division ring. Then $\mathbf{Q}[D]$ is isomorphic to $\mathbf{Q} \oplus \mathbf{Q}[D^*]$. It is well-known that, for a group G , the group ring $\mathbf{Q}[G]$ is von Neumann regular if and only if G is locally finite. Hence $\mathbf{Q}[D^*]$ is von Neumann regular if and only if D is an algebraic extension of a finite field.

When R is a noncommutative ring, it is not easy to see the structure of the ring $\mathbf{Z}[R]$.

Example 4 Let $M_2(GF(2))$ denote the ring of 2×2 matrices over the field $GF(2)$. Then we can prove that $\mathbf{Z}[M_2(GF(2))]$ is a semiprime ring. In fact we can see that $\mathbf{Q}[M_2(GF(2))]$ is isomorphic to the semisimple Artinian ring $\mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q} \oplus M_2(\mathbf{Q}) \oplus M_3(\mathbf{Q})$.

Example 5 Let $T_2(GF(2))$ denote the ring of 2×2 upper triangular matrices over the field $GF(2)$. Then we can see that $\mathbf{Q}[T_2(GF(2))]$ is isomorphic to the ring $\mathbf{Q} \oplus \mathbf{Q} \oplus T_3(\mathbf{Q})$.

Conjecture 1. Let K be a finite field and consider the ring $M_n(K)$ of $n \times n$ matrices over K . Then $\mathbf{Q}[M_n(K)]$ is a semisimple Artinian ring.

3. STRUCTURE OF $\mathbf{Z}[R]$

Let R be a ring. Then every element of the monoid ring $\mathbf{Z}[R]$ can be written as a finite sum of the form $\sum_{r \in R} a_r \hat{r}$ where $a_r \in \mathbf{Z}$. In this section we consider general structure of $\mathbf{Z}[R]$.

Proposition 1. *Let R be a ring and let I be an ideal of R .*

(1) *Let A denote the ideal of $Z[R]$ generated by $\{\widehat{r+s} - \hat{r} - \hat{s} \mid r, s \in R\}$. Then $Z[R]/A \cong R$.*

(2) *Let B denote the ideal of $Z[R]$ generated by $\{\hat{r} - \hat{s} \mid r - s \in I\}$. Then $Z[R]/B \cong Z[R/I]$.*

Consider the following condition on a ring R :

(*) For any $x, y \in R$, $xy = 1$ implies $yx = 1$.

If R is a left (or right) Noetherian ring, R satisfies condition (*).

Proposition 2. *Let R be a ring with condition (*) and let R^* denote the group of units in R . Let C denote the ideal of $Z[R]$ generated by $\{\hat{r} \mid r \in R - R^*\}$. Then $Z[R]/C \cong Z[R^*]$.*

4. SEMIPRIMENESS

In this section we consider the semiprimeness of $Z[R]$.

Assume that a ring R has a nonzero ideal I with $I^2 = 0$. Let $\bar{I}$ denote the ideal of $Z[R]$ generated by $\{\hat{r} - \hat{0} \mid r \in I\}$. Then we can easily see that $\bar{I}^2 = 0$. Therefore we have the following.

Proposition 3. *Let R be a ring. If $Z[R]$ is semiprime then R is semiprime.*

A commutative ring R is semiprime if and only if it has no nonzero nilpotent elements.

Theorem 1. *Let R be a commutative ring. Then $Z[R]$ is semiprime if and only if R is semiprime.*

Corollary 1. *Let S be a subsemigroup of a commutative semiprime ring R . Then the integral monoid ring $Z[S]$ is semiprime.*

Proposition 4. *Let R be a PI domain. Then $Z[R]$ is a semiprime ring.*

Conjecture 2. *Let R be a ring. Then $Z[R]$ is semiprime if and only if R is semiprime.*