

# Asymptotic Properties of Solutions to the Homogeneous Navier-Stokes Equations in $\mathbf{R}^3$

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*Key-Words:* Navier-Stokes equations, global solution, asymptotic properties, fast decays

**Abstract.** We show as the main result of the paper that if  $w$  is a weak global solution of homogeneous Navier-Stokes equations satisfying the strong energy inequality and  $\beta \in (3/4, 1)$ , then there exist  $t_0 \geq 0$ ,  $C_0 \geq 0$  and  $\delta_0 > 0$  such that

$$\frac{\|A^\beta w(t)\| + \|w(t)\|}{\|A^\beta w(t+\delta)\| + \|w(t+\delta)\|} \leq C_0$$

for all  $t \geq t_0$  and  $\delta \in [0, \delta_0]$ . So, measuring  $w$  in the graph norm  $\|A^\beta w\| + \|w\|$  and starting at time  $t_0$ , we exclude fast decays of  $w$  on short time intervals.

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## 1. Introduction

In this paper we study some asymptotic properties of weak global solutions of the Cauchy problem for the Navier-Stokes equations in the space domain  $\Omega = \mathbf{R}^3$ :

$$\frac{\partial w}{\partial t} - \Delta w + w \cdot \nabla w + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times (0, \infty), \quad (1)$$

$$\nabla \cdot w = 0, \quad w(x, 0) = w_0(x), \quad (2)$$

with  $w_0 \in L^2(\mathbf{R}^3)^3$ ,  $\nabla \cdot w_0 = 0$ . By a weak global solution  $w$  we mean a function

$$w \in C_w([0, \infty); L^2(\mathbf{R}^3)^3) \cap L^2_{loc}((0, \infty); W^{1,2}(\mathbf{R}^3)^3) \quad (3)$$

with  $\nabla \cdot w = 0$ , which satisfies the integral relation

$$(w(t), \phi(t)) + \int_0^t \left[ - \left( w(s), \frac{\partial \phi}{\partial s}(s) \right) + (\nabla w(s), \nabla \phi(s)) + (w(s) \cdot \nabla w(s), \phi(s)) \right] ds = (w_0, \phi(0)), \quad t > 0,$$

for all smooth vector fields  $\phi$  with compact support and  $\nabla \cdot \phi = 0$ .  $(\cdot, \cdot)$  denotes the scalar product and  $\|\cdot\|$  denotes the norm in  $L^2(\mathbf{R}^3)^3$ .  $C_w$  denotes the space of weakly continuous functions. The existence of weak global solutions is well known (see [1] or [7]).

From now on we suppose that the solutions satisfy the strong energy inequality

$$\|w(t)\|^2 + 2 \int_s^t \|\nabla w(\sigma)\|^2 d\sigma \leq \|w(s)\|^2$$

for  $s = 0$  and almost all  $s > 0$ , and all  $t \geq s$ .

It is known (see [4]) that the global weak solutions with the strong energy inequality become strong after a finite time:

$$\text{there is some } T_0 = T_0(\|w_0\|) \geq 0, \text{ such that } w \in C([T_0, \infty); L^p) \text{ for every } p \in [2, \infty). \quad (4)$$

The following theorem is the main result of the paper.

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**Theorem 1** Let  $\beta \in (3/4, 1)$ ,  $w_0 \in L^2(\mathbf{R}^3)^3$ ,  $\nabla \cdot w_0 = 0$ ,  $w_0 \neq 0$ . Let  $w$  be a weak global solution of (1) and (2) satisfying the strong energy inequality and let  $T_0$  be from (4). Then there exist  $C_0 > 1$  and  $\delta_0 \in (0, 1)$  such that

$$\frac{\|A^\beta w(t)\| + \|w(t)\|}{\|A^\beta w(t + \delta)\| + \|w(t + \delta)\|} \leq C_0, \quad \forall t \geq T_0 + 2, \quad \forall \delta \in (0, \delta_0]. \quad (5)$$

Let us present in this connection a theorem proved in [5]:

**Theorem 2** Let  $w_0 \in D(A)$ ,  $w_0 \neq 0$ . Let  $w$  be a strong global solution of the Navier-Stokes equations (1) and (2) in a smooth and bounded domain  $\Omega \subset \mathbf{R}^3$  endowed with the homogeneous Dirichlet boundary conditions. If  $k, l, m \in N \cup \{0\}$ , then there exist  $C = C(k, l, m) > 1$ ,  $t_0 = t_0(k, l, m) \geq 0$  and  $\delta_0 \in (0, 1)$  such that

$$\left\| \frac{d^k w}{dt^k}(t) \right\|_{m,2} \leq C \left\| \frac{d^l w}{dt^l}(t + \delta) \right\|, \quad \forall t \geq t_0, \quad \forall \delta \in [0, \delta_0].$$

It is clear that the result from Theorem 2 for the case of a bounded domain is stronger than the result presented in Theorem 1. In this paper we do not have the ambition to prove an analogical version of Theorem 2 for the whole space  $\mathbf{R}^3$  and Theorem 1 is only the first step in this direction. Let us also remark that unlike the case of a bounded domain, we do not have the inequality  $\|B(w, w)\| \leq \|A^{1/2}w\| \|A^\beta w\|$ , which must be replaced by  $\|B(w, w)\| \leq \|A^{1/2}w\| (\|A^\beta w\| + \|w\|)$  (see the second section for the notation). It leads to the form of the left hand side in (5). Therefore, Theorem 1 says that if we measure the solution  $w$  in the graph norm  $\|A^\beta \cdot\| + \|\cdot\|$ , then, starting at time  $T_0 + 2$ , fast decays of  $w$  on short time intervals are excluded. Let us remark, that the question of fast decays of solutions on short time intervals was raised and studied in [3].

## 2. Notations

$L^q = L^q(\mathbf{R}^3)$ ,  $q \geq 1$ : the Lebesgue spaces with the norm  $\|\cdot\|_q$ . If  $q = 2$ , we denote  $\|\cdot\| = \|\cdot\|_2$ .

$W^{s,q} = W^{s,q}(\mathbf{R}^3)$ ,  $s \geq 0$ ,  $q \geq 2$ : the Sobolev spaces endowed with the norm  $\|\cdot\|_{s,q}$ .

$L_\sigma^2$ : the closure of  $\{\varphi \in C_0^\infty(\mathbf{R}^3)^3; \nabla \cdot \varphi = 0\}$  in  $L^2(\mathbf{R}^3)^3$ .

$P_\sigma$ : orthogonal projection of  $L^2(\mathbf{R}^3)^3$  onto  $L_\sigma^2$ .

$A$ : the Stokes operator on  $L_\sigma^2$ ,  $\mathcal{D}(A) = \{u \in W^{2,2}; \nabla \cdot u = 0\}$ ,  $Au = -\Delta u$ ,  $\forall u \in \mathcal{D}(A)$ .

$A^\alpha$ ,  $\alpha \geq 0$ : the fractional powers of the Stokes operator.

$e^{-At}$ ,  $t \geq 0$ : the Stokes semigroup generated by the Stokes operator  $-A$ .

$B(w, w) = P_\sigma(w \cdot \nabla w)$ .

the graph norm  $\|w\|_\beta = \|A^\beta w\| + \|w\|$ .

## 3. Auxiliary results

At first, let us present several known properties of weak global solutions which will be used in this paper. According to [8], if  $w$  is a weak global solutions of (1) and (2) satisfying the strong energy inequality and if  $w_0 \in L^2(\mathbf{R}^3)^3 \cap L^p(\mathbf{R}^3)^3$  with  $p \in [1, 2)$  then

$$\|w(t)\| \leq C(1+t)^{-\frac{6-3p}{4p}}, \quad t \geq 0.$$

Using the results from [2] and [8] we can disregard the assumption  $p \in [1, 2)$  and derive that

$$\|w(t)\| \leq C(1+t)^{-\mu}, \quad t \geq 0$$

for any  $\mu \in (0, 1/2)$  where  $C$  possibly depends on  $\mu$ . Applying now a result from [4], we get that for  $m, k \in N$  and  $\mu \in (0, 1/2)$  there is  $C_{m,k} = C_{m,k}(\mu, C)$ , independent of  $T_0$ , such that

$$\left\| D^m \frac{d^k w}{dt^k}(t) \right\| \leq C_m(t - T_0 - 2)^{-\mu - m/2 - k}, \quad t \geq T_0 + 1. \quad (6)$$

The following inequality can be derived as a consequence of Hölder inequality and Lemma 2.4.3 from [6]: if  $\gamma \in [3/4, 1)$  then there exists  $c > 0$  such that

$$\|B(u, u)\| \leq c\|A^{1/2}u\| \|u\|_\gamma, \quad \forall u \in \mathcal{D}(A^\gamma). \quad (7)$$

Finally, if  $\gamma \in [3/4, 1)$  then there exists  $c > 0$  such that

$$\|A^{1/2}u\| \leq c\|u\|_\gamma, \quad \forall u \in \mathcal{D}(A^\gamma). \quad (8)$$

#### 4. Proofs of the main results

We prove at first the following lemma. Its corollary is substantial for the proof of Theorem 1.

**Lemma 3** *If  $w \in \mathcal{D}(A^\alpha)$ ,  $w \neq 0$ ,  $t \geq 0$  and  $0 \leq \beta \leq \alpha$  then*

$$\frac{\|A^\alpha w\|}{\|A^\beta e^{-At}w\|} \geq \frac{\|A^\alpha e^{-At}w\|}{\|A^\beta e^{-2At}w\|}.$$

**Proof:** Let  $E_\lambda$ ,  $\lambda \geq 0$  be the resolution of identity for the Stokes operator  $A$ . Then

$$\|A^\beta e^{-At}w\|^2 = \int_0^\infty \lambda^{2\beta} e^{-2\lambda t} d\|E_\lambda w\|^2, \quad t \geq 0. \quad (9)$$

By the Hölder inequality we get easily that

$$\begin{aligned} \|A^\beta e^{-At}w\|^2 &= \int_0^\infty \lambda^{2\beta} e^{-2\lambda t} d\|E_\lambda w\|^2 \leq \\ &\left( \int_0^\infty \lambda^{2\beta} d\|E_\lambda w\|^2 \right)^{1/2} \left( \int_0^\infty \lambda^{2\beta} e^{-4\lambda t} d\|E_\lambda w\|^2 \right)^{1/2} = \|A^\beta w\| \|A^\beta e^{-2At}w\| \end{aligned}$$

and immediately

$$\frac{\|A^\beta w\|}{\|A^\beta e^{-At}w\|} \geq \frac{\|A^\beta e^{-At}w\|}{\|A^\beta e^{-2At}w\|}. \quad (10)$$

We will show further that the function  $t \mapsto \|A^\alpha e^{-At}w\|^2 / \|A^\beta e^{-At}w\|^2$  is non-increasing. Firstly, for every  $\gamma \geq 0$

$$\frac{d}{dt} \|A^\gamma e^{-At}w\|^2 = -2\|A^{\gamma+1/2}e^{-At}w\|^2, \quad t > 0$$

and therefore

$$\frac{d}{dt} \frac{\|A^\alpha e^{-At}w\|^2}{\|A^\beta e^{-At}w\|^2} = \frac{2\|A^\alpha e^{-At}w\|^2 \|A^{\beta+1/2}e^{-At}w\|^2 - 2\|A^{\alpha+1/2}e^{-At}w\|^2 \|A^\beta e^{-At}w\|^2}{\|A^\beta e^{-At}w\|^4}, \quad t > 0.$$

Further,

$$\|A^\alpha e^{-At}w\|^2 \|A^{\beta+1/2}e^{-At}w\|^2 \leq \|A^{\alpha+1/2}e^{-At}w\|^2 \|A^\beta e^{-At}w\|^2,$$

as follows from the moment inequality

$$\|A^y u\| \leq \|A^z u\|^{\frac{x-y}{x-z}} \|A^x u\|^{\frac{y-z}{x-z}},$$

which holds for every  $0 \leq z < y < x$  and  $u \in D(A^x)$ . So,

$$\frac{d}{dt} \frac{\|A^\alpha e^{-At}w\|^2}{\|A^\beta e^{-At}w\|^2} \leq 0, \quad t > 0$$

and due to the continuity from the right at 0 we get that the above mentioned function is non-increasing. It means especially, that

$$\frac{\|A^\alpha w\|^2}{\|A^\beta w\|^2} \geq \frac{\|A^\alpha e^{-At} w\|^2}{\|A^\beta e^{-At} w\|^2}, \quad t \geq 0. \quad (11)$$

Using now (10) and (11), we get

$$\frac{\|A^\alpha w\|}{\|A^\beta e^{-At} w\|} = \frac{\|A^\alpha w\|}{\|A^\beta w\|} \frac{\|A^\beta w\|}{\|A^\beta e^{-At} w\|} \geq \frac{\|A^\alpha e^{-At} w\|}{\|A^\beta e^{-At} w\|} \frac{\|A^\beta e^{-At} w\|}{\|A^\beta e^{-2At} w\|} = \frac{\|A^\alpha e^{-At} w\|}{\|A^\beta e^{-2At} w\|},$$

which completes the proof of the lemma.  $\circ$

**Corollary 4** *If  $w \in \mathcal{D}(A^\alpha)$ ,  $w \neq 0$ ,  $t \geq 0$  and  $0 \leq \beta \leq \alpha$  then*

$$\frac{|||w|||_\alpha}{|||e^{-At} w|||_\beta} \geq \frac{|||e^{-At} w|||_\alpha}{|||e^{-2At} w|||_\beta}.$$

**Proof:** The proof of the corollary follows immediately from Lemma 3 and from the elementary fact that if  $\frac{\alpha_1}{\beta_1} \geq \frac{\beta_1}{\gamma_1}$  and  $\frac{\alpha_2}{\beta_2} \geq \frac{\beta_2}{\gamma_2}$  for some positive  $\alpha_i, \beta_i, \gamma_i, i = 1, 2$ , then  $\frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} \geq \frac{\beta_1 + \beta_2}{\gamma_1 + \gamma_2}$ .  $\circ$

Throughout the proof of Theorem 1  $c$  denotes the generic constant which can change from line to line.

**Proof of Theorem 1:** Let the assumptions of Theorem 1 be fulfilled. We will use the method from [5]. We denote

$$H = \max_{t \in [T_0 + 2, \infty)} |||w(t)|||_\beta.$$

It follows from (6) that  $H < \infty$ . Since  $\|A^\beta w(t)\| \neq 0$  for all  $t \in [T_0 + 2, \infty)$ , there exist  $C'_0 > 1$  and  $\delta'_0 \in (0, 1)$  such that

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} \leq C'_0, \quad \forall t \in [T_0 + 2, T_0 + 4], \quad \forall \delta \in (0, \delta'_0]. \quad (12)$$

We set now  $D_0 = 6C'_0$  and let  $\delta_0 \in (0, \delta'_0]$  be such a number that

$$4Hc \left( D_0 e^{\frac{5D_0}{2(D_0-1)}} \right)^3 \left( \frac{\delta_0^{1-\beta}}{1-\beta} + \delta_0 \right) \leq 1. \quad (13)$$

We will prove now the following proposition:

**Proposition P:** Let  $t > T_0 + 4$ ,  $\delta \in (0, \delta_0]$ . Let further

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} = C \in \left( D_0, D_0 e^{\frac{5D_0}{2(D_0-1)}} \right) \quad (14)$$

and

$$|||w(t)|||_\beta \geq |||w(s)|||_\beta, \quad \forall s \in [t, t + \delta]. \quad (15)$$

Then there exists  $t^* \in [t - \delta, t]$  such that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} \frac{\left( 1 - \frac{|||w(t)|||_\beta}{2H} \right)^2}{\left( 1 + \frac{|||w(t)|||_\beta}{2H} \right)}. \quad (16)$$

**Proof of Proposition P:** Let (14) and (15) be fulfilled. We can suppose that

$$\max_{s \in [t-\delta, t]} |||w(s)|||_\beta < C |||w(t)|||_\beta, \quad (17)$$

because otherwise (16) would be satisfied immediately. We begin with the integral representation of  $w$ :

$$w(t + \delta) = e^{-A\delta} w(t) + \int_0^\delta e^{-A(\delta-s)} B(w(t+s), w(t+s)) ds, \quad (18)$$

$$w(t) = e^{-A\delta}w(t-\delta) + \int_0^\delta e^{-A(\delta-s)}B(w(t-\delta+s), w(t-\delta+s)) ds. \quad (19)$$

Applying gradually (7), (8) and (17) we obtain that

$$\begin{aligned} & |||w(t) - e^{-A\delta}w(t-\delta)|||_\beta \leq \\ & \int_0^\delta c((\delta-s)^{-\beta} + 1) \|B(w(t-\delta+s), w(t-\delta+s))\| ds \leq \\ & \int_0^\delta c((\delta-s)^{-\beta} + 1) \|A^{1/2}w(t-\delta+s)\| |||w(t-\delta+s)|||_\beta ds = \\ & |||w(t)|||_\beta \int_0^\delta c((\delta-s)^{-\beta} + 1) \frac{\|A^{1/2}w(t-\delta+s)\|}{|||w(t-\delta+s)|||_\beta} \times \\ & \frac{|||w(t-\delta+s)|||_\beta}{|||w(t)|||_\beta} |||w(t-\delta+s)|||_\beta ds \leq |||w(t)|||_\beta^2 cC^2 \int_0^\delta ((\delta-s)^{-\beta} + 1) ds. \end{aligned}$$

So we can get from (13) and (14) that

$$\begin{aligned} & |||w(t) - e^{-A\delta}w(t-\delta)|||_\beta \leq |||w(t)|||_\beta \left[ 2HcC^2 \left( \frac{\delta_0^{1-\beta}}{1-\beta} + \delta_0 \right) \right] \frac{|||w(t)|||_\beta}{2H} \leq \\ & |||w(t)|||_\beta \frac{|||w(t)|||_\beta}{2H} \end{aligned} \quad (20)$$

and also

$$\begin{aligned} & |||w(t) - e^{-A\delta}w(t-\delta)|||_\beta \leq |||w(t+\delta)|||_\beta \left[ 4HcC^3 \left( \frac{\delta_0^{1-\beta}}{1-\beta} + \delta_0 \right) \right] \times \\ & \frac{|||w(t)|||_\beta}{4H} \leq |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{4H}. \end{aligned} \quad (21)$$

(21) now gives immediately that

$$|||e^{-A\delta}w(t) - e^{-2A\delta}w(t-\delta)|||_\beta \leq |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{4H}. \quad (22)$$

It follows from (18), (7), (8), (14), (15) and (13) that

$$\begin{aligned} & |||w(t+\delta) - e^{-A\delta}w(t)|||_\beta \leq \\ & \int_0^\delta (c(\delta-s)^{-\beta} + 1) \|A^{1/2}w(t+s)\| |||w(t+s)|||_\beta ds = \\ & |||w(t+\delta)|||_\beta \int_0^\delta (c(\delta-s)^{-\beta} + 1) \frac{\|A^{1/2}w(t+s)\|}{|||w(t+s)|||_\beta} \frac{|||w(t+s)|||_\beta}{|||w(t+\delta)|||_\beta} \times \\ & |||w(t+s)|||_\beta ds \leq |||w(t+\delta)|||_\beta |||w(t)|||_\beta cC \int_0^\delta ((\delta-s)^{-\beta} + 1) ds = \\ & |||w(t+\delta)|||_\beta \left[ 4HcC \left( \frac{\delta_0^{1-\beta}}{1-\beta} + \delta_0 \right) \right] \frac{|||w(t)|||_\beta}{4H} \leq |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{4H}. \end{aligned} \quad (23)$$

(22) and (23) provide the estimate

$$\begin{aligned} & |||e^{-2A\delta}w(t-\delta) - w(t+\delta)|||_\beta \leq |||e^{-2A\delta}w(t-\delta) - e^{-A\delta}w(t)|||_\beta + \\ & |||e^{-A\delta}w(t) - w(t+\delta)|||_\beta \leq |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{2H}. \end{aligned} \quad (24)$$

It follows now from Corollary 4 and (20) and (24) that

$$|||w(t - \delta)|||_\beta \geq \frac{|||e^{-A\delta}w(t - \delta)|||_\beta^2}{|||e^{-2A\delta}w(t - \delta)|||_\beta} \geq \frac{|||w(t)|||_\beta^2 \left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{|||w(t + \delta)|||_\beta \left(1 + \frac{|||w(t)|||_\beta}{2H}\right)}.$$

If we put  $t^* = t - \delta$ , (16) is proved. The proof of Proposition P is finished and we can continue in the proof of Theorem 1.

Let us fix  $t \in [T_0 + 2, \infty)$ ,  $\delta \in (0, \delta_0]$  and suppose that

$$|||w(t)|||_\beta > H/D_0 \text{ and} \quad (25)$$

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} \geq D_0 \frac{1 + 1/2}{(1 - 1/2)^2} = 6D_0. \quad (26)$$

Since  $D_0 > C'_0$  and  $\delta_0 \leq \delta'_0$ , it follows from (12) and (26) that  $t > T_0 + 4$ . We can also suppose without loss of generality that

$$|||w(t)|||_\beta = \max_{s \in [t, t + \delta]} |||w(s)|||_\beta$$

and (by possible decreasing of  $\delta$ )

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} = 6D_0.$$

Let us notice that  $6D_0 < D_0 e^{\frac{5D_0}{2(D_0-1)}}$  ( $D_0 > 1$ ) and the conditions (14) and (15) are satisfied. By Proposition P there exists  $t^* \in [t - \delta, t)$  so that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} \frac{\left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1 + \frac{|||w(t)|||_\beta}{2H}\right)} \geq 6D_0 \frac{(1 - 1/2)^2}{1 + 1/2} = D_0.$$

Thus, by (25),  $|||w(t^*)|||_\beta \geq D_0 |||w(t)|||_\beta > D_0 H/D_0 = H$  and it is the contradiction with the definition of  $H$ . Let  $D_1 = 6D_0$ . We proved

**Proposition  $P_1$ :** Let  $t \in [T_0 + 2, \infty)$ ,  $\delta \in (0, \delta_0]$  and  $|||w(t)|||_\beta > H/D_0$ . Then

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} < D_1.$$

We define now

$$D_n = D_{n-1} \frac{1 + \frac{1}{2D_0 D_1 \dots D_{n-2}}}{\left(1 - \frac{1}{2D_0 D_1 \dots D_{n-2}}\right)^2}, \quad \forall n \in N, n \geq 2. \quad (27)$$

We have

$$6 < D_0 < D_1 < \dots < D_{n-1} < D_n, \quad \forall n \in N, \quad (28)$$

$$D_n = 6D_0 \prod_{j=0}^{n-2} \frac{1 + \frac{1}{2D_0 D_1 \dots D_j}}{\left(1 - \frac{1}{2D_0 D_1 \dots D_j}\right)^2} \leq D_0 \prod_{j=0}^{n-1} \frac{1 + \frac{1}{2D_0^j}}{\left(1 - \frac{1}{2D_0^j}\right)^2}, \quad \forall n \geq 2$$

and

$$\ln D_n \leq \ln D_0 + \sum_{j=0}^{n-1} \ln \left(1 + \frac{1}{2D_0^j}\right) - 2 \ln \left(1 - \frac{1}{2D_0^j}\right), \quad \forall n \geq 1.$$

It follows from the elementary properties of the function  $x \rightarrow \ln(1+x)$  that

$$\ln D_n < \ln D_0 + \sum_{j=0}^{n-1} \left( \frac{1}{2D_0^j} + 4\frac{1}{2D_0^j} \right) < \ln D_0 + \frac{5D_0}{2(D_0-1)}$$

and

$$D_n < D_0 e^{\frac{5D_0}{2(D_0-1)}}, \quad \forall n \in N. \quad (29)$$

We will prove now that for every  $n \in N$  the following proposition is valid:

**Proposition  $P_n$ :** Let  $t \in [T_0 + 2, \infty)$ ,  $\delta \in (0, \delta_0]$  and

$$|||w(t)|||_\beta > \frac{H}{D_0 D_1 \dots D_{n-1}}.$$

Then

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_n.$$

We will use the mathematical induction. Proposition  $P_1$  has already been proved. Let us suppose that  $P_n$  holds for some  $n \in N$  and we will prove the validity of  $P_{n+1}$ . Thus, let  $t \in [T_0 + 2, \infty)$ ,  $\delta \in (0, \delta_0]$  and  $|||w(t)|||_\beta > H/D_0 D_1 \dots D_n$ . We can suppose that

$$|||w(t)|||_\beta \leq H/D_0 D_1 \dots D_{n-1}, \quad (30)$$

since otherwise we would apply Proposition  $P_n$ , get  $|||w(t)|||_\beta / |||w(t+\delta)|||_\beta < D_n < D_{n+1}$  and Proposition  $P_{n+1}$  would be proved. We suppose by contradiction that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \geq D_{n+1}. \quad (31)$$

It follows then from (12) and (28) that  $t > T_0 + 4$ . We can suppose without loss of generality that

$$|||w(t)|||_\beta \geq |||w(s)|||_\beta, \quad \forall s \in [t, t+\delta] \quad (32)$$

and also

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} = D_{n+1}. \quad (33)$$

Due to (28), (29), (32) and (33) we see that (14) and (15) are satisfied. Therefore, Proposition P, (33), (30) and (27) yield that there exists  $t^* \in [t-\delta, t)$  so that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \frac{\left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1 + \frac{|||w(t)|||_\beta}{2H}\right)} \geq D_{n+1} \frac{\left(1 - \frac{1}{2D_0 D_1 \dots D_{n-1}}\right)^2}{\left(1 + \frac{1}{2D_0 D_1 \dots D_{n-1}}\right)} = D_n. \quad (34)$$

If we use the assumptions of Proposition  $P_{n+1}$  we obtain that

$$|||w(t^*)|||_\beta \geq D_n |||w(t)|||_\beta > D_n \frac{H}{D_0 D_1 \dots D_n} = \frac{H}{D_0 D_1 \dots D_{n-1}}$$

and according to Proposition  $P_n$  we get that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} < D_n,$$

which is the contradiction to (34). Therefore, (31) does not hold, in fact

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_{n+1}$$

and Proposition  $P_{n+1}$  is proved. We proved that Proposition  $P_n$  holds for every  $n \in N$ .

We now finish the proof of Theorem 1. Let us fix  $t \in [T_0 + 2, \infty)$  and  $\delta \in (0, \delta_0]$ . Then there exists  $n \in N$  so that  $|||w(t)|||_\beta > \frac{H}{D_0 D_1 \dots D_{n-1}}$ . By Proposition  $P_n$  and by (29) we get that

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} < D_n < D_0 e^{\frac{5D_0}{2(D_0-1)}}.$$

Setting  $C_0 = D_0 e^{\frac{5D_0}{2(D_0-1)}}$  the proof of Theorem 1 is complete.  $\bigcirc$

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