

The covering number and the uniformity of the ideal $\mathcal{I}_f$

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1 Introduction

For the ideal $\mathcal{SN}$ of strongly measure zero subsets of the real line, the cardinal coefficients have been studied[1]. But its cofinality had not been studied. In general, it may be larger than the continuum. Yorioka studied its cofinality(see [2]). One of his results is that the value of $\text{cof}(\mathcal{SN})$ is equal to the dominating number for $\omega_1^{\omega_1}$ under the continuum hypothesis. In the process, he introduced ideals $\mathcal{I}_f$ for $f \in \omega^\omega$. These ideals were used in the proof. We are interested in the ideals $\mathcal{I}_f$ themselves. These ideals are subideals of the null ideal $\mathcal{N}$ and include $\mathcal{SN}$. The properties of these ideals depend on f .

In this paper, we discuss the following contents. In section 3, we show a characterization of $\text{cov}(\mathcal{I}_f) \geq \mathfrak{b}$. In section 4, we define a forcing notion which has the countable chain condition. And with the results of section 3 we show that its ω_2 -stage finite support iteration by bookkeeping method lifts up $\text{cov}(\mathcal{I}_f)$ from a ground model with the continuum hypothesis. In section 5, we introduce a sufficient condition not to lift up $\text{cov}(\mathcal{I}_f)$ for forcing notions which satisfy axiom A.

2 Definitions and notation

Throughout this paper, we use the standard terminology for forcing of set theory and cardinal coefficients (see[1]). We regard the set of all reals as the Cantor set 2^ω . We denote by $\mathcal{M}$ and $\mathcal{N}$ the set of all meager subsets of 2^ω and the set of all null subsets of 2^ω respectively.

For functions f, g in ω^ω we write " $f \leq g$ " to mean that g dominates f everywhere, that is, $f(n) \leq g(n)$ for all $n < \omega$. And we let " $f \leq^* g$ " mean that g eventually dominates f , that is, there exists an $n < \omega$ such that $f(m) \leq g(m)$ holds for all $m < \omega$ larger than n . We denote by $\mathcal{S}$ the set of all non-decreasing functions d in ω^ω which diverges to infinity and $d(0) = 0$. We denote by $\mathbb{C}$ and $\mathbb{D}$ the Cohen forcing notion and the dominating forcing notion respectively[1]. For each ideal (or family if there is not a problem in particular) $\mathcal{I}$ on 2^ω which contains all singletons, we denote by $\text{add}(\mathcal{I})$, $\text{cov}(\mathcal{I})$, $\text{non}(\mathcal{I})$ and $\text{cof}(\mathcal{I})$ the additivity, covering number,

uniformity and cofinality of $\mathcal{I}$ respectively which means that:

1. $\text{add}(\mathcal{I}) = \min \{ |\mathcal{A}| \mid \mathcal{A} \subset \mathcal{I} \cup \mathcal{A} \notin \mathcal{I} \},$
2. $\text{cov}(\mathcal{I}) = \min \{ |\mathcal{A}| \mid \mathcal{A} \subset \mathcal{I} \cup \mathcal{A} = 2^\omega \},$
3. $\text{non}(\mathcal{I}) = \min \{ |Y| \mid Y \subset 2^\omega, Y \notin \mathcal{I} \},$
4. $\text{cof}(\mathcal{I}) = \min \{ |\mathcal{A}| \mid \mathcal{A} \subset \mathcal{I} \forall B \in \mathcal{I} \exists A \in \mathcal{A} (B \subset A) \}.$

We have that $\text{add}(\mathcal{I}) \leq \text{cov}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ and $\text{add}(\mathcal{I}) \leq \text{non}(\mathcal{I}) \leq \text{cof}(\mathcal{I})$ for each ideal or family $\mathcal{I}$ on 2^ω which contains all singletons.

We define some notation before we define the ideals $\mathcal{I}_f$ and $\mathcal{K}_f$.

Definition 2.1 Let f, g be functions in ω^ω .

1. We define the order " $\ll$ " on ω^ω by
 $f \ll g$ iff $\forall k < \omega \exists N < \omega \forall n \geq N (f(n^k) \leq g(n))$,
2. We define the order " $\lll$ " on ω^ω by
 $g_\sigma(n) = |\sigma(n)|$ for all $n < \omega$,
3. For $\sigma \in (2^{<\omega})^\omega$, define the subset $Y(\sigma) \subset 2^\omega$ by
 $Y(\sigma) = \bigcap_{n < \omega} \bigcup_{m \geq n} [\sigma(m)]$, where $[s] = \{ x \in 2^\omega \mid s \subset x \}$ for each $s \in 2^{<\omega}$.

Define the subsets $\mathcal{S}(f)$, $\mathcal{T}(f)$ and $\mathcal{U}(f)$ of $(2^{<\omega})^\omega$ by

$$\begin{aligned} \mathcal{S}(f) &= \{ \sigma \in (2^{<\omega})^\omega \mid g_\sigma \gg f \}, \\ \mathcal{T}(f) &= \{ \sigma \in (2^{<\omega})^\omega \mid g_\sigma = f \}. \end{aligned}$$

Definition 2.2 Let $f \in \omega^\omega$. Define the families $\mathcal{I}_f, \mathcal{J}_f$ and $\mathcal{K}_f$ on 2^ω by

$$\begin{aligned} \mathcal{I}_f &= \{ X \subset 2^\omega \mid \exists \sigma \in \mathcal{S}(f) X \subset Y(\sigma) \}, \\ \mathcal{J}_f &= \{ X \subset 2^\omega \mid \exists \sigma \in \mathcal{T}(f) X \subset Y(\sigma) \}. \end{aligned}$$

The following definition is not necessary for the definition of ideal $\mathcal{I}_f$. But it is the very useful.

Definition 2.3 Let $f \in \omega^\omega$. For each $d \in S$, we define the functions $g_d^{(f)}$ and $h_d^{(f)} \in \omega^\omega$ by

$$g_d^{(f)}(n) = f(n^{k+2}) \text{ if } n \in [d(k), d(k+1))$$

for all $n < \omega$, respectively. If $g \in \omega^\omega$ is $g = g_d^{(f)}$ for some $d \in \omega^{\uparrow\omega}$, then we say " g is generated by d (and f) for $\ll$ ".

3 $\text{cov}(\mathcal{I}_f)$, $\text{cov}(\mathcal{J}_f)$ and bounding number $\mathfrak{b}$

In this section, we show that the ideal $\mathcal{I}_f$ and the family $\mathcal{J}_f$ are related to bounding number $\mathfrak{b}$ intimately. For each $d \in S$, $g_d^{(f)} \gg f$ holds where $g_d^{(f)}$ was introduced in chapter 2. In addition, for each $g \gg f$ there exists a $d \in S$ by the definitions of $g_d^{(f)}$ and $\ll$ such that $g_d^{(f)} \leq^* g$. Therefore, the following hold.

Lemma 3.1 *For each family $\mathcal{F} \subset \omega^\omega$ such that $|\mathcal{F}| < \mathfrak{b}$ and $\forall g \in \mathcal{F} (g \gg f)$, there exists $d \in S$ such that $\forall g \in \mathcal{F} (g_d^{(f)} \leq^* g)$.*

Proof of Lemma 3.1 Let $\mathcal{F} \subset \omega^\omega$ satisfy $\forall g \in \mathcal{F} (g \gg f)$ and $|\mathcal{F}| < \mathfrak{b}$. For each $g \in \mathcal{F}$, there exists $d_g \in S$ such that $g_{d_g}^{(f)} \leq g$. Since $|\mathcal{F}| < \mathfrak{b}$, the family $\{d_g \mid g \in \mathcal{F}\}$ is bounded family in ω^ω . So there exists $d \in S$ which dominates for all functions in $\{d_g \mid g \in \mathcal{F}\}$. $\square$ (Lemma 3.1)

Lemma 3.2 *There exists a family $\mathcal{F} \subset \omega^\omega$ such that $|\mathcal{F}| = \mathfrak{b}$ and $\forall g \in \mathcal{F} (g \gg f)$ and $\forall h \gg f \exists g \in \mathcal{F} (h \not\leq^* g)$.*

Proof of Lemma 3.2 Take a unbounded family $\mathcal{B} \subset S$. Then a family $\{g_d^{(f)} \mid d \in \mathcal{B}\}$ is as desired. $\square$ (Lemma 3.2)

For all $d \in S$, $\text{cov}(\mathcal{I}_f) \leq \text{cov}(\mathcal{J}_{g_d^{(f)}})$ holds by $\mathcal{I}_f = \bigcup_{g \gg f} \mathcal{J}_g = \bigcup_{d \in S} \mathcal{J}_{g_d^{(f)}}$. By this, if $\text{cov}(\mathcal{I}_f)$ is larger than $\mathfrak{b}$, then $\text{cov}(\mathcal{J}_{g_d^{(f)}})$ is larger than $\mathfrak{b}$ for all $d \in S$. The inverse holds.

Theorem 3.1 $\text{cov}(\mathcal{I}_f) \geq \mathfrak{b}$ iff $\text{cov}(\mathcal{J}_{g_d^{(f)}}) \geq \mathfrak{b}$ for all $d \in S$.

Proof of Theorem 3.1 $\implies$: As above.

$\impliedby$: Assume $\text{cov}(\mathcal{I}_f) < \mathfrak{b}$. There exists a family $\mathcal{F}$ such that $|\mathcal{F}| = \text{cov}(\mathcal{I}_f) < \mathfrak{b}$ and $\bigcup \mathcal{F} = 2^\omega$. For each $X \in \mathcal{F}$, there exists σ_X such that $X \subset Y(\sigma_X)$. By Lemma 3.1, there exists $d \in S$ such that $\forall X \in \mathcal{F} g_d^{(f)} \leq^* g_{\sigma_X}$. For each $X \in \mathcal{F}$, define $\tau_X \in T(g_d^{(f)})$ by $\tau_X(n) = \sigma_X(n) \upharpoonright g_d^{(f)}(n)$. Then a family $\{Y(\tau_X) \mid X \in \mathcal{F}\} \subset \mathcal{J}_{g_d^{(f)}}$ covers 2^ω . $\square$ (Theorem 3.1)

However, it is easily proved that $\text{cov}(\mathcal{I}_f) \geq \mathfrak{b}$ is independent from ZFC. $\text{cov}(\mathcal{I}_f) = \omega_1$ and $\mathfrak{b} = \mathfrak{c}$ hold in a generic model which is obtained by a forcing notion satisfying Laver property from a ground model with the continuum hypothesis. Also $\text{cov}(\mathcal{I}_f) = \mathfrak{b} = \omega_1$ holds in a generic model which is obtained by the Cohen forcing notion of any weight from a ground model with continuum hypothesis.

4 The forcing notion $\mathbb{P}(d)$ for $d \in S$ and $\text{cov}(\mathcal{I}_f)$ and $\text{non}(\mathcal{I}_f)$

In this section, we discuss the covering number and the uniformity of ideal $\mathcal{I}_f$ in the model obtained by a certain iteration of the forcing notion $\mathbb{P}(d)$. We define the forcing notion $\mathbb{P}(d)$ for $d \in S$.

Definition 4.1 Let $d \in S$. Define the forcing notion $\mathbb{P}(d)$ by

$$\begin{aligned} \mathbb{P}(d) &= \left\{ (s, F) \in 2^{<\omega} \times \left[T(g_d^{(f)}) \right]^{<\omega} \mid |s| = f(|F|) \right\}, \\ (s, F) &\leq (s', F') \\ &\iff 1. s \supset s' F \supset F' \\ &\quad 2. \forall \sigma \in F' \forall n \in |F| \setminus |F'| \ [s \upharpoonright [f(n), f(n+1)) \neq \sigma(n+1) \upharpoonright [f(n), f(n+1))]. \end{aligned}$$

Lemma 4.1 For all $d \in S$, the forcing notion $\mathbb{P}(d)$ is σ -linked. So it has the countable chain condition.

Proof of Lemma 4.1 Since $g_d^{(f)}(n+1) - g_d^{(f)}(n) > n$ for all $n < \omega$, holds that $\forall (s, F) \in \mathbb{P}(d) \forall F' \in [T(g_d^{(f)})]^{<\omega} \exists (t, H) \leq (s, F) (H = F \cup F')$.

Let $N < \omega$ and $g = g_d^{(f)}$. For each $t \in 2^{g(N)}$, $\psi \in \prod_{n \in [N, 2N)} [2^{g(n+1)-g(n)}]^{<N}$, define a subset $B_{t,\psi}$ of $\mathbb{P}(d)$ by
 $B_{t,\psi} = \{ (s, F) \in \mathbb{P}(d) \mid s = t\psi = \langle \{ \sigma(n+1) \upharpoonright [g(n), g(n+1)) \mid \sigma \in F \} \mid n \in [|F|, 2|F|) \rangle \}$.
 Clearly $\mathbb{P}(d) = \bigcup_{N < \omega} \bigcup \{ B_{t,\psi} \mid t \in 2^{g(N)} \psi \in \prod_{n \in [N, 2N)} [2^{g(n+1)-g(n)}]^{<N} \}$. We show that for all $N < \omega$, $t \in 2^{g(N)}$ and $\psi \in \prod_{n \in [N, 2N)} [2^{g(n+1)-g(n)}]^{<N}$, any two distinct conditions in $B_{t,\psi}$ are compatible. Let $(s, F), (s', F')$ be in $B_{t,\psi}$ and $(s, F) \neq (s', F')$. By the definition of $B_{t,\psi}$,

$$\begin{aligned} s &= s' = t \upharpoonright |F| = |F'| = N \\ &\langle \{ \sigma(n+1) \upharpoonright [g(n), g(n+1)) \mid \sigma \in F \} \mid n \in [|F|, 2|F|) \rangle \\ &= \langle \{ \sigma(n+1) \upharpoonright [g(n), g(n+1)) \mid \sigma \in F' \} \mid n \in [|F'|, 2|F'|) \rangle = \psi. \end{aligned}$$

There exists $(u, H) \leq (s, F)$ such that $H = F \cup F'$. Clearly $|F'| < |H| \leq 2N$. To prove $(u, H) \leq (s', F')$, let $\sigma \in F'$ and $n \in |H| \setminus |F'|$. Since $(u, H) \leq (s, F)$, $u \upharpoonright [g(n), g(n+1)) \neq \tau(n+1) \upharpoonright [g(n), g(n+1))$ for all $\tau \in F$, that is, $u \upharpoonright [g(n), g(n+1)) \notin \psi(n)$.

But $\sigma(n+1) \upharpoonright [g(n), g(n+1)) \in \psi(n)$.

Therefore $u \upharpoonright [g(n), g(n+1)) \neq \sigma(n+1) \upharpoonright [g(n), g(n+1))$.

□(Lemma 4.1)

For each $d \in S$, $\sigma \in T(g_d^{(f)})$ and $n < \omega$, define the subsets $D_\sigma, E_n \subset \mathbb{P}(d)$ as follows:

$$\begin{aligned} D_\sigma &= \{ (s, F) \in \mathbb{P}(d) \mid \sigma \in F \}, \\ E_n &= \{ (s, F) \in \mathbb{P}(d) \mid |F| \geq n \}. \end{aligned}$$

Lemma 4.2 For all $s \in S$, $\sigma \in T(g_d^{(f)})$ and $n < \omega$, the subsets D_σ and E_n are dense open sets in $\mathbb{P}(d)$.

Proof of Lemma 4.2 Let $\sigma \in T(g_d^{(f)})$, $n < \omega$ and $(s, F) \in \mathbb{P}(d)$. Take $F' \subset T(g_d^{(f)})$ such that $\sigma \in F'$ and $|F| \geq n$. There exists $(t, H) \leq (s, F)$ such that $H = F \cup F'$. Since $\sigma \in H$ and $|H| \geq n$, $(t, H) \in D_\sigma$ and $(t, H) \in E_n$. □(Lemma 4.2)

We are interested in the generic model of $\mathbb{P}(d)$. Let $d \in S$ and $\dot{G}$ be the canonical generic $\mathbb{P}(d)$ -name. Define $\mathbb{P}(d)$ -name $\dot{a}_{\dot{G}}$ by

$$\Vdash_{\mathbb{P}(d)} \dot{a}_{\dot{G}} = \bigcup \left\{ s \mid \exists F (s, F) \in \dot{G} \right\} \in 2^\omega.$$

Lemma 4.3 For all $d \in S$, $\Vdash_{\mathbb{P}(d)} \forall \sigma \in T(g_d^{(f)}) \cap \mathbf{V} (\dot{a}_{\dot{G}} \notin Y(\sigma))$.

Proof of Lemma 4.3 Let $d \in S$, $\sigma \in T(g_d^{(f)})$ and $(s, F) \in \mathbb{P}(d)$. By Lemma 4.2, there exists $(s', F') \leq (s, F)$ such that $\sigma \in F'$. To prove that $(s', F') \Vdash_{\mathbb{P}(d)} \sigma(n) \notin \dot{a}_{\dot{G}}$ for all $n > |F'|$, let $n > |F'|$. By Lemma 4.2, there exists $(s'', F'') \leq (s', F')$ such that $|F''| \geq n$. Then $(s'', F'') \Vdash_{\mathbb{P}(d)} "s'' \subset \dot{a}_{\dot{G}} s'' \upharpoonright [g_d^{(f)}(n-1), g_d^{(f)}(n)) \neq \sigma(n) \upharpoonright [g_d^{(f)}(n-1), g_d^{(f)}(n))"$. Therefore $(s'', F'') \Vdash_{\mathbb{P}(d)} \sigma(n) \notin \dot{a}_{\dot{G}}$. $\square$ (Lemma 4.3)

Lemma 4.4 For all $d \in S$, $\Vdash_{\mathbb{P}(d)} 2^\omega \cap \mathbf{V} \in \mathcal{J}_{g_d^{(f)}}$.

Proof of Lemma 4.4 This is directly followed from the fact that $\mathbb{P}(d)$ adds Cohen reals in $\prod_{n < \omega} 2^{g_d^{(f)}(n)}$. $\square$ (Lemma 4.4)

To define a finite support iteration of $\mathbb{P}(d)$, let κ be an uncountable regular cardinal and π be a bijection from κ onto $\kappa \times \kappa$ such that if $\pi(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$ for all $\alpha < \kappa$. Let π_0 and π_1 be the first and second coordinate of the value of π respectively.

Assume the continuum hypothesis. We define $\mathbb{P}_\kappa$ by κ -stage finite support iteration $\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa \rangle$ as follows:
Assume that P_β and the P_β -names $\dot{d}_\xi^\beta$ for $\xi < \kappa$ with $\Vdash_\beta \langle \dot{d}_\xi^\beta \mid \xi < \kappa \rangle$ be an enumeration of S'' are defined for all $\beta \leq \alpha$ in α -stage. Define $\Vdash_\alpha \dot{Q}_\alpha \simeq \mathbb{P}(\dot{d}_{\pi_1(\alpha)}^{\pi_0(\alpha)}) \star \mathbb{D}$.

Theorem 4.1 (CH) $\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = \mathfrak{b} = \kappa \wedge \forall d \in S \text{ cov}(\mathcal{J}_{g_d^{(f)}}) = \mathfrak{c}$.

Therefore, it holds that $\Vdash_{\mathbb{P}_\kappa} \text{cov}(\mathcal{I}_f) = \mathfrak{c}$ by theorem 3.1.

Proof of Theorem 4.1 Clearly $\mathfrak{c} = \mathfrak{b} = \kappa$ in $\mathbf{V}[G_\kappa]$. Let $d \in S$, $\lambda < \mathfrak{c}$ and a family $\{X_\delta \mid \delta < \lambda\} \subset \mathcal{J}_{g_d^{(f)}}$ in $\mathbf{V}[G_\kappa]$. There exists $\alpha < \kappa$ such that X_δ is coded by $\sigma_\delta \in T(g_d^{(f)})$ for each $\delta < \lambda$ in $\mathbf{V}[G_\alpha]$. By Lemma 4.3, $\{Y(\sigma_\delta) \mid \delta < \lambda\}$ does not cover 2^ω in $\mathbf{V}[G_{\alpha+1}]$. Hence $\{X_\delta \mid \delta < \lambda\}$ does not cover 2^ω in $\mathbf{V}[G_\kappa]$. $\square$ (Theorem 4.1)

Theorem 4.2 (CH) $\Vdash_{\mathbb{P}_{\omega_2}} \text{non}(\mathcal{I}_f) = \mathfrak{c}$

Proof of Theorem 4.2 Clearly by Lemma 4.4.

$\square$ (Theorem 4.2)

5 Property E and $\text{cov}(\mathcal{I}_f) = \omega_1$

In this section, we introduce a certain property for forcing notions which satisfy axiom A. A forcing notion with this property does not add a real which is not covered by all elements of $\mathcal{S}(f)$ in ground model. This property is preserved in an iterated forcing. So the countable support iteration of forcing notions with this property does not lift up $\text{cov}(\mathcal{I}_f)$. For example, the infinitely equal forcing notion $\mathbb{EE}$ satisfies this property.

Definition 5.1 Let forcing notion P satisfy axiom A by the fusion orders $\langle \leq_n \mid n < \omega \rangle$. P has property E if there exists $\varphi \in \omega^{P \times \omega}$ such that

- (1) for all $p \in P$ and $n < \omega$, if $p \Vdash_P \dot{a} \in \mathbf{V}$ then
there exist $q \leq_n p$ and a finite set B such that $|B| \leq \varphi(p, n)$ and $q \Vdash_P \dot{a} \in B$,
- (2) for all $p, q \in P$ and $n < \omega$, if $q \leq_n p$ then $\varphi(q, n) = \varphi(p, n)$.

Lemma 5.1 Suppose that the axiom A forcing notion P has property E.

Then $\Vdash_P "2^\omega \subset \bigcup \{Y(\tau) \mid \tau \in T(g) \cap \mathbf{V}\}"$ for all strictly increasing function $g \in \omega^\omega$. Therefore, $\Vdash_P "2^\omega \subset \bigcup \{Y(\tau) \mid \tau \in \mathcal{S}(f) \cap \mathbf{V}\}"$.

Proof of Lemma 5.1 Let $p \in P$ satisfy $p \Vdash_P \dot{x} \in 2^\omega$ and $g \in \omega^\omega$ be strictly increasing. By induction on $j < \omega$, define three sequences $\langle p_j \in P \mid j < \omega \rangle$, $\langle m_j < \omega \mid j < \omega \rangle$ and $\langle A_j \mid j < \omega \rangle$ as follows:

- (i) $p_0 = p$,
- (ii) $p_{j+1} \leq_j p_j$,
- (iii) $m_j = \sum_{i < j} \varphi(p_i, i)$,
- (iv) $A_j \subset 2^{g(m_j + \varphi(p_j, j))}$,
- (v) $|A_j| \leq \varphi(p_j, j)$,
- (vi) $p_{j+1} \Vdash_P \dot{x} \restriction (m_j + \varphi(p_j, j)) \in A_j$,

for all $j < \omega$. For each $j < \omega$, let $\{s_l^j \mid l < \varphi(p_j, j)\}$ be an enumeration of A_j . There exists $q \in P$ such that $\forall j < \omega$ $q \leq_j p_j$.

We define $\sigma \in (2^{<\omega})^\omega$ by for each $n < \omega$, $\sigma(n) = s_l^j \restriction g(n)$ where $n = m_j + l$. To prove that $q \Vdash_P \dot{x} \in Y(\sigma)$, let $n < \omega$. There exists $j < \omega$ such that $m_j \geq n$. Since $q \Vdash_P \dot{x} \restriction (m_j + \varphi(p_j, j)) \in A_j$, there exist $q' \leq q$ and $l < \varphi(p_j, j)$ such that $q' \Vdash_P \dot{x} \restriction (m_j + \varphi(p_j, j)) = s_l^j \restriction \sigma(m_j + l)$. $\square$ (Lemma 5.1)

Let $\delta \leq \omega_2$. Let $P_\delta = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \delta \rangle$ be a δ -stage countable support iteration such that $\dot{Q}_\alpha$ is defined by the forcing notion with property E for all $\alpha < \delta$. For $n < \omega$ and $F \in [\delta]^{<\omega}$, $p \in P_\delta$ is (n, F) -good if there exists $h \in \omega^F$ such that $p \restriction \gamma \Vdash_\gamma \dot{\varphi}_\gamma(p(\gamma), n) \leq h(\gamma)$ for all $\gamma \in F$ where $\dot{\varphi}_\gamma$ is P_γ -name for the function φ appeared in the definition of property E for $\dot{Q}_\gamma$.

Lemma 5.2 Let $\delta \leq \omega_2$. For all $n < \omega$ and $F \in [\delta]^{<\omega}$, the set $\{p \in P_\delta \mid p \text{ is } (n, F)\text{-good}\}$ is (n, F) -dense open in P_δ .

Proof of Lemma 5.2 Since the property E implies the strongly ω^ω -bounding, we can prove easily by induction on $\delta \leq \omega_2$. $\square$ (Lemma 5.2)

By the lemma above, we may suppose only the condition that is (n, F) -good. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and p with (n, F) -good, define $h_{p,n,F} \in \omega^F$ by

- (a) $p \restriction \gamma \Vdash_\gamma \dot{\varphi}_\gamma(p(\gamma), n) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$,
- (b) if $q \leq_{n,F} p$ then $h_{q,n,F}(\gamma) \leq h_{p,n,F}(\gamma)$ for all $\gamma \in F$.

Lemma 5.3 Let $\delta \leq \omega_2$. There exists $\tilde{\varphi}_\delta \in \omega^{P_\delta \times \omega \times [\delta]^{<\omega}}$ such that

- (1) for all $n < \omega$, $F \in [\delta]^{<\omega}$ and p with (n, F) -good, if $p \Vdash_\delta \dot{a} \in \mathbf{V}$ then there exist $q \leq_{n,F} p$ and a finite set B such that $|B| \leq \tilde{\varphi}_\delta(p, n, F)$ and $q \Vdash_\delta \dot{a} \in B$,
- (2) for all $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$, if $q \leq_{n,F} p$ then $\tilde{\varphi}_\delta(q, n, F) \leq \tilde{\varphi}_\delta(p, n, F)$.

Proof of Lemma 5.3 We prove by induction on $\delta \leq \omega_2$. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and p with (n, F) -good, we define $\tilde{\varphi}_\delta(p, n, F)$ as follows:

Case 1 : δ is limit ordinal.

Let $\alpha = \max(F) + 1$. Then $F \subset \alpha$. By induction hypothesis, there exists $\tilde{\varphi}_\alpha \in \omega^{P_\alpha \times \omega \times [\alpha]^{<\omega}}$ such that (1) and (2). So we define $\tilde{\varphi}_\delta(p, n, F)$ by $\tilde{\varphi}_\alpha(p \restriction \alpha, n, F)$.

We show that (1) and (2). (1): Let $p \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $p \Vdash_\delta \dot{a} \in \mathbf{V}$. Suppose $\alpha = \max(F) + 1$. Since $p \restriction \alpha \Vdash_\alpha \dot{b} \in \mathbf{V}$ “ $\dot{b} \in P_{\alpha\delta} \dot{f} \Vdash_{\alpha\delta} \dot{a} = \dot{b}$ ” for some P_α -name $\dot{b}$ and $\dot{f}$, there exist $r \leq_{n,F} p \restriction \alpha$, finite set B and $g \in P_{\alpha\delta}$ such that $|B| \leq \tilde{\varphi}_\alpha(p \restriction \alpha, n, F) = \tilde{\varphi}_\delta(p, n, F)$ and $r \Vdash_\alpha \dot{b} \in B \wedge \dot{f} = g$. Let $q = r \cup g$. Then $q \leq_{n,F} p$ and $q \Vdash_\delta \dot{a} \in B$.

(2): Let $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $q \leq_{n,F} p$. Suppose $\alpha = \max(F) + 1$. Then since $q \restriction \alpha \leq_{n,F} p \restriction \alpha$,

$$\begin{aligned} \tilde{\varphi}_\delta(q, n, F) &= \tilde{\varphi}_\alpha(q \restriction \alpha, n, F) \\ &\leq \tilde{\varphi}_\alpha(p \restriction \alpha, n, F) \\ &= \tilde{\varphi}_\delta(p, n, F) \end{aligned}$$

Case 2 : $\delta = \gamma + 1$.

In the case of $F \subset \gamma$, we define in the same way as the case of that δ is limit ordinal.

Suppose $\gamma \in F$.

By induction hypothesis, there exists $\tilde{\varphi}_\gamma$ such that for all $p' \in P_\gamma$, $n' < \omega$ and $F' \in [\gamma]^{<\omega}$, if $p \restriction \gamma \Vdash_\gamma \dot{a} \in \mathbf{V}$, there exist $r \leq_{n,F \cap \gamma} p'$ and B such that $|B| \leq \tilde{\varphi}_\gamma(p \restriction \gamma, n, F \cap \gamma)$ and $r \Vdash_\gamma \dot{a} \in B$. So we define $\tilde{\varphi}_\delta(p, n, F)$ by $\tilde{\varphi}_\gamma(p \restriction \gamma, n, F \cap \gamma) \cdot h_{p,n,F}(\gamma)$.

We show that (1) and (2). (1): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and p with (n, F) -good. In the case of $F \subset \gamma$, we can show in the same way as the case of that δ is limit ordinal.

Assume $\gamma \in F$. Also there exist P_γ -names $\dot{q}$ and $\dot{B}$ such that $p \restriction \gamma \Vdash_\gamma " \dot{q} \leq_n p(\gamma) \wedge \dot{B} \subset \mathbf{V} \wedge |\dot{B}| \leq \dot{\varphi}_\gamma(p(\gamma), n) \wedge \dot{q} \Vdash_{\dot{Q}_\gamma} \dot{a} \in \dot{B} "$. By p is (n, F) -good, $p \restriction \gamma$ is $(n, F \cap \gamma)$ -good and $p \restriction \gamma \Vdash_\gamma |\dot{B}| \leq \dot{\varphi}_\gamma(p(\gamma), n) \leq h_{p,n,F}(\gamma)$. Let $\langle \dot{b}_j \mid j < h_{p,n,F}(\gamma) \rangle$ be a sequence of P_γ -names for an enumeration of $\dot{B}$. That is $p \restriction \gamma \Vdash_\gamma \{ \dot{b}_j \mid j < h_{p,n,F}(\gamma) \} = \dot{B} \subset \mathbf{V}$. By induction on $j < h_{p,n,F}$, we construct two sequences $\langle r_j \mid j < h_{p,n,F} \rangle$ and $\langle B_j \mid j < h_{p,n,F} \rangle$ such that (let $r_{-1} = p \restriction \gamma$)

- (a) $r_j \leq_{n, F \cap \gamma} r_{j-1}$ for all $j < h_{p,n,F}(\gamma)$,
- (b) $|B_j| \leq \tilde{\varphi}_\gamma(r_{j-1}, n, F \cap \gamma) \leq \tilde{\varphi}_\gamma(p \restriction \gamma, n, F \cap \gamma)$ for $j < h_{p,n,F}$,
- (c) $r_j \Vdash_\gamma \dot{b}_j \in B_j$ for all $j < h_{p,n,F}$

Let $q = r_{h_{p,n,F}(\gamma)-1} \cup \{(\gamma, \dot{q})\}$ and $B = \bigcup \{ B_j \mid j < h_{p,n,F}(\gamma) \}$. Clearly $q \Vdash_\delta \dot{a} \in B$ and

$$\begin{aligned} |B| &\leq \sum_{j < h_{p,n,F}} |B_j| \\ &\leq \sum_{j < h_{p,n,F}} \tilde{\varphi}_\gamma(p \restriction \gamma, n, F \cap \gamma) \\ &= \tilde{\varphi}_\gamma(p \restriction \gamma, n, F \cap \gamma) \cdot h_{p,n,F}(\gamma) \\ &= \tilde{\varphi}_\delta(p, n, F). \end{aligned}$$

(2): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and p, q satisfy (n, F) -good and $q \leq_{n,F} p$. In the case of $F \subset \gamma$, we can show in the same way as the case of that δ is limit ordinal. Suppose $\gamma \in F$. Then

$$\begin{aligned} \tilde{\varphi}_\delta(q, n, F) &= \tilde{\varphi}_\gamma(q \restriction \gamma, n, F \cap \gamma) \cdot h_{q,n,F}(\gamma) \\ &\leq \tilde{\varphi}_\gamma(p \restriction \gamma, n, F \cap \gamma) \cdot h_{p,n,F}(\gamma) \\ &= \tilde{\varphi}_\delta(p, n, F). \end{aligned}$$

□(Lemma5.3)

Theorem 5.1 $\Vdash_{P_{\omega_2}} 2^\omega \subset \bigcup \{ Y(\tau) \mid \tau \in \mathcal{T}(g) \cap \mathbf{V} \}$, for all strictly increasing function $g \in \omega^\omega$. therefore, $\Vdash_{P_{\omega_2}} 2^\omega \subset \bigcup \{ Y(\tau) \mid \tau \in \mathcal{S}(f) \cap \mathbf{V} \}$.

Proof of Theorem5.1 By Lemma5.3, we can show in the same way as Lemma5.1.

□(Theorem5.1)

Corollary 5.4 (CH) $\Vdash_{P_{\omega_2}} " \text{cov}(\mathcal{I}_f) = \text{cov}(\mathcal{J}_g) = \omega_1 "$ for all strictly increasing function $g \in \omega^\omega$.

6 The diagram of cardinal coefficients of $\mathcal{I}_f$

In this section, we give the results for the cardinal coefficients of ideal $\mathcal{I}_f$ of the forcing notions that we studied. Let κ be an uncountable regular cardinal. We express the parts which we do not yet understand in ‘?’.

forcing notions	add	cov	non	cof	$\mathfrak{b}$	$\mathfrak{d}$	$\mathfrak{c}$
$\mathbb{O}(f)_\kappa$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	κ
$\mathbb{P}_\kappa$	?	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	?	?	κ
$\mathbb{C}_\kappa$	ω_1	ω_1	$\mathfrak{c}$	$\mathfrak{c}$	ω_1	$\mathfrak{c}$	κ
$\mathbb{EE}_{\omega_2}$	ω_1	ω_1	$\mathfrak{c}$	$\mathfrak{c}$	ω_1	ω_1	ω_2
$\mathbb{S}_{\omega_2}$	ω_1	ω_1	ω_1	ω_1	ω_1	ω_1	ω_2

$\mathbb{O}(f)_\kappa$: the κ -stage finite support iteration of the forcing notion $\mathbb{O}(f)$ introduced by T.Yorioka,
 $\mathbb{P}_\kappa$: the κ -stage finite support iteration of the forcing notion $\mathbb{P}(d)$ by bookkeeping method,
 $\mathbb{C}_\kappa$: the Cohen forcing notion which adds κ many Cohen reals,
 $\mathbb{EE}_{\omega_2}$: the ω_2 -stage countable support iteration of the infinitely equal forcing notion,
 (the infinitely equal forcing notion has property E),
 $\mathbb{S}_{\omega_2}$: the ω_2 -stage countable support iteration of the Sacks forcing notion.

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