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# Weak and Strong Convergence Theorems for Generalized Nonlinear Mappings in Hilbert Spaces
# (ヒルベルト空間における非線形写像の弱収束・強収束定理)

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**Abstract.** In this article, using strongly asymptotically invariant sequences, we first prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. Next, we prove a weak convergence theorem of Mann's type [24] for the mappings. Furthermore, using the idea of mean convergence by Shimizu and Takahashi [25, 26], we prove a strong convergence theorem of Halpern's type [8] for the mappings. The nonlinear ergodic theorem and the strong convergence theorem in this article generalize the Kawasaki and Takahashi nonlinear ergodic theorem [19] and the Hojo and Takahashi strong convergence theorem [11], respectively.

## 1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a non-empty subset of $H$. For a mapping $T : C \to H$, we denote by $F(T)$ the set of fixed points of $T$. Kocourek, Takahashi and Yao [20] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings [7], nonspreading mappings [21, 22] and hybrid mappings [31]. A mapping $T : C \to H$ is said to be *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers; see also [1]. We call such a mapping an $(\alpha, \beta)$*-generalized hybrid* mapping. Kocourek, Takahashi and Yao [20] and Hojo and Takahashi [11] proved the following nonlinear ergodic and strong convergence theorems for generalized hybrid mappings, respectively.

**Theorem 1.1** ([20]). *Let $H$ be a real Hilbert space, let $C$ be a non-empty, closed and convex subset of $H$, let $T$ be a generalized hybrid mapping from $C$ into itself with $F(T) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(T)$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

*converges weakly to $p \in F(T)$, where $p = \lim_{n\to\infty} PT^n x$.*

**Theorem 1.2** ([11]). *Let $C$ be a non-empty, closed and convex subset of a real Hilbert space $H$. Let $T$ be a generalized hybrid mapping of $C$ into itself. Let $u \in C$ and define two sequences*

*$\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

*for all $n = 1, 2, ...,$ where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where $P$ is the metric projection of $H$ onto $F(T)$.*

Very recently, Kawasaki and Takahashi [19] introduced a broader class of nonlinear mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping $T$ from $C$ into $H$ is said to be *widely more generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0 \end{aligned} \tag{1.1}$$

for all $x, y \in C$. Such a mapping $T$ is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-*widely more generalized hybrid* mapping; see also [18]. An $(\alpha, \beta, \gamma, \delta, 0, 0, 0)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [20] if $\alpha + \beta = -\gamma - \delta = 1$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. In [19], Kawasaki and Takahashi proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [3] for such new mappings in a Hilbert space. In particular, by using their fixed point theorems, they proved directly Browder and Petryshyn's fixed point theorem [5] for strict pseudo-contractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [20] for super generalized hybrid mappings.

In this article, using strongly asymptotically invariant sequences, we first prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. Next, we prove a weak convergence theorem of Mann's type [24] for the mappings. Furthermore, using the idea of mean convergence by Shimizu and Takahashi [25, 26], we prove a strong convergence theorem of Halpern's type [8] for the mappings. The nonlinear ergodic theorem and the strong convergence theorem in this article generalize the Kawasaki and Takahashi nonlinear ergodic theorem [] and the Hojo and Takahashi strong convergence theorem [11].

## 2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let $H$ be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [30], we have that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|y\|^2 - \|x\|^2 \le 2\langle y - x, y\rangle, \tag{2.1}$$

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we know that for $x, y, u, v \in H$

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let $C$ be a non-empty subset of $H$. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. Let $C$ be a non-empty, closed and convex subset of $H$ and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping $P_C$ is called the *metric projection* of $H$ onto $C$. It is known that $P_C$ is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \tag{2.4}$$

for all $x, y \in H$; see [30] for more details. For proving main results in this article, we also need the following lemmas proved in Takahashi and Toyoda [32] and Aoyama, Kimura, Takahashi and Toyoda [2].

**Lemma 2.1** ([32]). *Let $D$ be a non-empty, closed and convex subset of $H$. Let $P$ be the metric projection from $H$ onto $D$. Let $\{u_n\}$ be a sequence in $H$. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

**Lemma 2.2** ([2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \gamma_n + \beta_n$$

*for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.*

Let $\ell^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(\ell^\infty)^*$ (the dual space of $\ell^\infty$). Then we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $\ell^\infty$ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean $\mu$ is called a *Banach limit* on $\ell^\infty$ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on $\ell^\infty$. If $\mu$ is a Banach limit on $\ell^\infty$, then for $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [28] for the proof of existence of a Banach limit and its other elementary properties. For $f \in \ell^\infty$, define $\ell_1 : \ell^\infty \to \ell^\infty$ as follows:

$$\ell_1 f(k) = f(1 + k), \quad \forall k \in \mathbb{N}.$$

A sequence $\{\mu_n\}$ of means on $\ell^\infty$ is said to be *strongly asymptotically invariant* if

$$\|\ell_1^* \mu_n - \mu_n\| \to 0,$$

where $\ell_1^*$ is the adjoint operator of $\ell_1$. See [6] for more details. The following definition which was introduced by Takahashi [27] is crucial in the fixed point theory. Let $h$ be a bounded function of $\mathbb{N}$ into $H$. Then, for any mean $\mu$ on $\ell^\infty$, there exists a unique element $h_\mu \in H$ such that

$$\langle h_\mu, z \rangle = (\mu)_k \langle h(k), z \rangle, \quad \forall z \in H.$$

Such $h_\mu$ is contained in $\overline{\text{co}}\{h(k) : k \in \mathbb{N}\}$, where $\overline{\text{co}}A$ is the closure of convex hull of $A$. In particular, let $T$ be a mapping of a subset $C$ of a Hilbert space $H$ into itself such that $\{T^k x : k \in \mathbb{N}\}$ is bounded for some $x \in C$. Putting $h(k) = T^k x$ for all $k \in \mathbb{N}$, we have that there exists $z_0 \in H$ such tat

$$\mu_k \langle T^k x, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

We denote such $z_0$ by $T_\mu x$. From Kawasaki and Takahashi [19], we also know the following fixed point theorem for widely more generalized hybrid mappings in a Hilbert space.

**Theorem 2.3** ([19]). *Let $H$ be a Hilbert space, let $C$ be a non-empty, closed and convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from $C$ into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

*for all $x, y \in C$. Suppose that it satisfies the following condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

*Then $T$ has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of $T$ is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ under the conditions (1) and (2).*

## 3 Nonlinear ergodic theorems

In this section, using the technique developed by Takahashi [27], we prove a mean convergence theorem for widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following three lemmas. The following lemma was proved by Kawasaki and Takahashi [19].

**Lemma 3.1** ([19]). *Let $H$ be a real Hilbert space, let $C$ be a closed and convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from $C$ into itself such that $F(T) \neq \emptyset$ and it satisfies the condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\zeta + \eta \geq 0$ *and* $\alpha + \beta > 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\varepsilon + \eta \geq 0$ *and* $\alpha + \gamma > 0$.

*Then $T$ is quasi-nonexpansive.*

The following two lemmas by Hojo and Takahashi [12] are crucial in the proof of our main theorem in this section.

**Lemma 3.2** ([12]). *Let $C$ be a non-empty, closed and convex subset of a real Hilbert space $H$. Let $T$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from $C$ into itself such that $F(T) \neq \emptyset$. Suppose that it satisfies the following condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

*Let $\{\mu_\nu\}$ be a srongly asymptotically invariant net of means on $\ell^\infty$. For any $x \in C$, define $S_{\mu_\nu}x$ as follows:*

$$\langle S_{\mu_\nu}x, y\rangle = (\mu_\nu)_k\langle T^k x, y\rangle, \quad \forall y \in H.$$

*Then $\lim_\nu \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0$. In addition, if $C$ is bounded, then*

$$\lim_\nu \sup_{x\in C} \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0.$$

**Lemma 3.3** ([12]). *Let $H$ be a Hilbert space and let $C$ be a non-empty, closed and convex subset of $H$. Let $T: C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$.

*If $x_\nu \rightharpoonup z$ and $x_\nu - Tx_\nu \to 0$, then $z \in F(T)$.*

Now we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space which was proved by Hojo and Takahashi [12].

**Theorem 3.4** ([12]). *Let $H$ be a real Hilbert space, let $C$ be a non-empty, closed and convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from $C$ into itself such that $F(T) \neq \emptyset$. Suppose that $T$ satisfies the condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

*Let $\{\mu_\nu\}$ be a srongly asymptotically invariant net of means on $\ell^\infty$ and let $P$ be the metric projection of $H$ onto $F(T)$. Then for any $x \in C$, the net $\{S_{\mu_\nu}x\}$ converges weakly to a fixed point $p$ of $T$, where $p = \lim_{n\to\infty} PT^n x$.*

Using Theorem 3.4, we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space which was proved by Kawasaki and Takahashi [19].

**Theorem 3.5** ([19]). *Let $H$ be a real Hilbert space, let $C$ be a non-empty, closed and convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from $C$ into itself such that $F(T) \neq \emptyset$ and it satisfies the condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, $\zeta + \eta \geq 0$ *and* $\alpha + \beta > 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\varepsilon + \eta \geq 0$ *and* $\alpha + \gamma > 0$.

*Then for any $x \in C$ the Cesàro means*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

*converge weakly to a fixed point $p$ of $T$ and $p = \lim_{n\to\infty} PT^n x$, where $P$ is the metric projection of $H$ onto $F(T)$.*

*Proof.* For any $f = (x_0, x_1, x_2, \dots) \in \ell^\infty$, define

$$\mu_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} x_k, \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is an asymptotically invariant sequence of means on $\ell^\infty$; see [28, p.78]. Furthermore, we have that for any $x \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x.$$

Therefore, we have the desired result from Theorem 3.4. □

## 4 Weak convergence theorems of Mann's type

In this section, we prove a weak convergence theorem of Mann's type [24] for widely more generalized hybrid mappings in a Hilbert space. Let $C$ be a non-empty, closed and convex subset of a Hilbert space $H$. Then we know from Lemma 3.1 that an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping $T$ from $C$ into itself with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$,

is quasi-nonexpansive. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [17]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since $C$ is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

$z$ is a fixed point of $T$ and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.2) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
&\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
&= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
&= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 = 0
\end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex. Using Lemma 3.1 and the technique developed by Ibaraki and Takahashi [14, 15], we can prove the following weak convergence theorem.

**Theorem 4.1** ([9]). *Let $H$ be a Hilbert space and let $C$ be a non-empty, closed and convex subset of $H$. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$- widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

*Let $P$ be the mertic projection of $H$ onto $F(T)$. Let $\{\mu_n\}$ be a srongly asymptotically invariant sequence of means on $\ell^\infty$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n} x_n, \quad n \in \mathbb{N}.$$

*Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n\to\infty} Px_n$.*

Using Theorem 4.1, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

**Theorem 4.2** ([9]). *Let $H$ be a Hilbert space and let $C$ be a non-empty, closed and convex subset of $H$. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\mu_n\}$ be a srongly asymptotically invariant sequence of means on $\ell^\infty$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \quad n \in \mathbb{N}.$$

*Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.*

*Proof.* Since $T : C \to C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - Ty\|^2 + (1-\beta)\|x - Ty\|^2$$

for all $x, y \in C$. We have that an $(\alpha, \beta)$-generalized hybrid mapping is an $(\alpha, 1-\alpha, -\beta, -(1-\beta), 0, 0, 0)$-widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.1. Therefore, we have the desired result from Theorem 4.1. $\square$

## 5 Strong Convergence Theorems

In this section, using the idea of mean convergence by Shimizu and Takahashi [25] and [26], we prove the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

**Theorem 5.1** ([9]). *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $T$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping of $C$ into itself which satisfies the following condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \gamma > 0, \quad \varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0, \quad \zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

*Let $\{\mu_n\}$ be a srongly asymptotically invariant sequence of means on $\ell^\infty$. Let $u \in C$ and define sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \\ z_n = T_{\mu_n}x_n \end{cases}$$

*for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu$, where $P$ is the metric projection of $H$ onto $F(T)$.*

Using Theorem 5.1, as in the proof of Theorem 4.2, we can show the result (Theorem 1.2) in Introduction which was obtained by Hojo and Takahashi [11].

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