

The Onsager-Machlup functions for diffusion processes

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For the n -dimensional Wiener measure P^W , we often write formally as

$$P^W(\cdot) = \int \text{const.} \exp\left[-\frac{1}{2} \int_0^T \|\dot{\phi}_t\|^2 dt\right] \mathcal{Q}(\phi)$$

the functional $\exp\left[-\frac{1}{2} \int_0^T \|\dot{\phi}_t\|^2 dt\right]$ representing thus an ideal density with respect to a fictitious uniform measure on the space of all continuous paths $\phi = (\phi_t) : [0, T] \rightarrow \mathbb{R}^n$. Our problem is concerned with defining such densities for general diffusion processes. If this density is of the form

$$\text{const.} \exp\left[-\int_0^T L(\phi_t, \dot{\phi}_t) dt\right]$$

with a function L on the tangent bundle, L is naturally regarded as Lagrangian associated with the diffusion process. In physical literatures, such a function is called the Onsager-Machlup function of the diffusion process, cf. [2],[4],[6],[7].

As a mathematical problem, it has the following two aspects. First, we would like to find a function L such that the transition probability density $p(t, x, y)$ of the diffusion

process admits a representation

$$p(t, x, y) = \int_{\{\phi: \phi_0=x, \phi_t=y\}} \text{const.} \exp\left[-\int_0^t L(\phi_s, \dot{\phi}_s) ds\right] \mathcal{Q}(\phi)$$

by a so-called path integral or continuum integral. This problem is closely related to the asymptotic evaluation of $p(t, x, y)$ as $t \downarrow 0$ and was studied in , e.g., [1], [4]. It seems, however, that a rigorous justification is still open.

The second aspect is that, since a diffusion process is a well-defined mathematical object as a probability measure on the space of paths, L might be obtained by evaluating asymptotically the measure of those paths which lie in a small tube around a given smooth curve. Such an evaluation was attempted by Stratonovich [8] where the functional thus obtained is called the probability functional of the diffusion process. Let us rigorously formulate the problem. Let (X_t, P_x) be a locally conservative, non-singular diffusion process on $\underbrace{M}_{\text{a manifold}}$ ("locally conservative" means that the infinitesimal generator of the diffusion does not contain the 0-th order term and "non-singular" means that it is strictly elliptic: the case of the non-zero 0-th order term can be treated by the Feynman-Kac formula). We assume that the coefficients of the infinitesimal generator are smooth. Then a Riemannian structure is naturally induced by the diffusion coefficients so that the generator is $\frac{1}{2} \Delta_M + b$ where Δ_M is the Laplace-Beltrami operator and b is a vector field and an intrinsic metric defining the tube should be the Riemannian distance $\rho(x, y)$. We may thus formulate the problem as follows: let M be a Riemannian manifold of the dimension n

, (X_t, P_x) be the diffusion process on M with the infinitesimal generator $\frac{1}{2} \Delta_M + b$ and $\phi = (\phi_t) : [0, T] \rightarrow M$ be a smooth curve. Find an asymptotic formula for the probability

$$P_{\phi_0}(\rho(X_t, \phi_t) < \varepsilon \text{ for all } t \in [0, T])$$

as $\varepsilon \downarrow 0$.

An answer is given in the following theorem.

THEOREM

$$P_{\phi_0}(\rho(X_t, \phi_t) < \varepsilon \text{ for all } t \in [0, T])$$

$$\sim C \exp(-\lambda_1 T / \varepsilon^2) \exp[-\int_0^T L(\phi_t, \dot{\phi}_t) dt] \quad \text{as } \varepsilon \downarrow 0,$$

where L is a function on the tangent bundle TM defined by

$$L(x, \dot{x}) = \frac{1}{2} \|\dot{x}\|^2 + \frac{1}{2} \operatorname{div} b(x) - \frac{1}{12} R(x).$$

Here $\|\cdot\|$ is the Riemannian norm in the tangent space $T_x(M)$,

$R(x)$ is the scalar curvature (i.e. the trace of the Ricci

curvature), $C = \psi_1(0) \int_D \psi_1(x) dx$ and $(\psi_m(x), \lambda_m)_{m=1,2,\dots}$

is the eigen-system for $-\frac{1}{2} \Delta_{R^n}$ (Δ_{R^n} : the Laplacian in R^n)

in the unit ball D of R^n with Dirichlet's boundary condition.

The above theorem was conjectured and verified to hold in Einstein spaces by Y. Takahashi [9] through probabilistic method using e.g. Girsanov's theorem, Levy's stochastic area integrals and stochastic Stokes's theorem. The final proof was given in [10] by accomplishing Takahashi's probabilistic approach and

also in [3] by using purely analytical method.

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