

Butterflies from information metric

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ABSTRACT: We study time evolution of distance between thermal states excited by local operators, with different external couplings. We find that growth of the distance implies growth of commutators of operators, signifying the local excitations are scrambled. We confirm this growth of distance by holographic computation, by evaluating volume of codimension 1 extremal volume surface. We find that the distance increases exponentially as $e^{\frac{2\pi t}{\beta}}$. Our result implies that, in chaotic system, trajectories of excited thermal states exhibit high sensitivity to perturbation to the Hamiltonian, and the distance between them will be significant at the scrambling time. We also confirm the decay of two point function of holographic Wilson loops on thermofield double state.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Conformal Field Theory, 1/N Expansion

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1 Introduction

The study of chaos in terms of AdS/CFT correspondence [1] has revealed new perspectives on quantum field theory and gravity. Its relation to black hole information paradox [2, 3] and Fast scrambling conjecture [4–6] is of particular interest.

In this article, we study scrambling of local excitation in chaotic system. Consider a thermal system at equilibrium excited by a local operator W . Sometime after the excitation, the excitation will lose its identity, so that it can not be distinguished from other excitations by local measurement. We can distinguish them only by measuring total system, so that the excitation is scrambled. One way to define indistinguishability is to measure the growth of complexity of the excitation $W(t)$. When the perturbation $W(-t_w)$ is scrambled, then it should be a complicated sum of variety of local operators. Complexity of operators can be measured by the growth of expectation value of square of commutator

$$\langle [W(t), V][W(t), V]^\dagger \rangle_\beta \quad (1.1)$$

for all local V with $[W, V] = 0$. The expectation value is small at early time, and grows as $e^{\lambda_L t}$ at late time, where λ_L is called Lyapunov exponent. Such growth of commutators was already considered in [7], and was interpreted as diagnosis of chaos, because semiclassically the commutator $\langle [q(t), p(0)][q(t), p(0)]^\dagger \rangle$ is equal to the square of Poisson bracket

$$\{q(t), p(0)\}_P^2 = \left(\frac{\delta q(t)}{\delta q(0)} \right)^2, \quad (1.2)$$

whose exponential growth implies high sensitivity of classical trajectories to initial configurations. λ_L is known to obey a bound $\lambda_L \leq \frac{2\pi}{\beta}$ in large N theory [6]. The bound is saturated by the holographic CFT dual to classical Einstein gravity, and the dual geometry can be approximated by shock wave geometry [5, 8–12]. Lyapunov exponents of various theories are computed in [13–20]. Related studies include [21–28].

In this paper, we will characterize scrambling of operator by a quantum information theoretic quantity called Fisher information metric and quantum fidelity. Quantum fidelity is a measure of similarity between two quantum states. It is defined by square of absolute value of inner product of two states

$$|\langle \psi_{\lambda+\delta\lambda} | \psi_{\lambda} \rangle|^2. \quad (1.3)$$

Decay of quantum fidelity implies the distance between states becomes larger. We call the infinitesimal part of the fidelity $G_{\lambda\lambda}$ as Fisher information metric,

$$|\langle \psi_{\lambda+\delta\lambda} | \psi_{\lambda} \rangle|^2 = 1 - 2G_{\lambda\lambda}(\delta\lambda)^2 + \mathcal{O}((\delta\lambda)^3). \quad (1.4)$$

Fisher information metric is a metric on the manifold of quantum states, parametrized by parameter λ . Fisher information represents distance between states when they are close to each other. By definition, Fisher information metric is positive and is covariant quantity. In the context of AdS/CFT, the holographic dual of inner product of states with different marginal couplings was proposed in [29], and it was argued that volume of codimension 1 surface with extremal volume in dual geometry is proportional to the Fisher information metric.

Quantum fidelity has been a useful tool to study chaos in quantum systems [30–39]. When two slightly different states evolve with identical Hamiltonians, the fidelity between two states is conserved because of linearity and unitarity of quantum mechanics, so “butterfly effect” does not appear. Nonetheless, butterfly effect plays role when we evolve two states with slightly different Hamiltonians $H_{\lambda} = H_0 + \lambda V$ and $H_{\lambda+\delta\lambda} = H_0 + (\lambda + \delta\lambda)V$. λ represents a coupling of external environment or internal imperfections. In this case inner product

$$|\langle \psi_{\lambda+\delta\lambda} | e^{iH_{\lambda+\delta\lambda}t} e^{-iH_{\lambda}t} | \psi_{\lambda} \rangle| \quad (1.5)$$

is not generically conserved, and it decays rapidly if either H_{λ} or $H_{\lambda+\delta\lambda}$ is chaotic as in figure 1. The decay of fidelity signifies difficulty of revival of quantum state by imperfect time reversal procedure, and is also a measure of strength of decoherence.

In this article, we consider time evolution of distance between thermal states excited by local operator with slightly different Hamiltonians. Assuming the difference between Hamiltonians is infinitesimally small, we find that the distance for short time is proportional to expectation values of commutators of local operators. This implies that, the rapid growth of distance signifies the excitation is scrambled. In particular, in large N chaotic theory, we show that growth of Fisher information metric is proportional to $e^{\frac{2\pi}{\beta}t}$, from holographic computation [29]. Intuitive interpretation of our result is that the trajectories of thermal states excited by local operators are highly sensitive to external environment or internal imperfections, so it diagnoses butterfly effect. In addition, we study two point functions between Wilson loop operators on different boundaries. We confirm that it decays rapidly.

Note. The volume of maximal volume surface in shock wave geometry was computed up to $\mathcal{O}(1)$ constant in [40], in order to confirm local perturbation does not change complexity significantly. Our study is focused on that $\mathcal{O}(1)$ constant. Also, similarity between timescales in scrambling and the decay of Loschmidt echo was pointed out in [26].

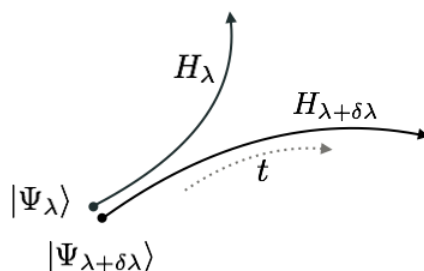


Figure 1. Orbits of states with slightly different Hamiltonians. For chaotic theory or chaotic perturbation to the original Hamiltonian, the distance between states grows exponentially even if original states are the same or close to each other.

Here we provide a direct connection between them and give a concrete example by holographic computation.

2 Fisher information metric

In this section, we introduce fidelity and Fisher information metric. We point out that growth of Fisher information metric of thermofield double states, perturbed by local operators, corresponds to growth of commutator between excitation and external perturbation. This growth of commutators implies the initial excitation is scrambled.

We consider time evolution of Fisher information metric of excited states. The inner product of $e^{-iH_\lambda t} A e^{iH_\lambda t} |\psi_\lambda\rangle$ and $e^{-iH_{\lambda+\delta\lambda} t} A e^{iH_{\lambda+\delta\lambda} t} |\psi_{\lambda+\delta\lambda}\rangle$ is

$$\langle \psi_{\lambda+\delta\lambda} | e^{-iH_{\lambda+\delta\lambda} t} A e^{iH_{\lambda+\delta\lambda} t} e^{-iH_\lambda t} A e^{iH_\lambda t} | \psi_\lambda \rangle. \quad (2.1)$$

The decay of this inner product can be understood as a butterfly effect. When $|\psi_\lambda\rangle = |\psi_{\lambda+\delta\lambda}\rangle$, this inner product can be interpreted differently. In this case, the quantity is called Polarization echo, and is given by

$$\langle \psi_\lambda | e^{iH_{\lambda+\delta\lambda} t} e^{-iH_\lambda t} A e^{iH_\lambda t} e^{-iH_{\lambda+\delta\lambda} t} A | \psi_\lambda \rangle. \quad (2.2)$$

When $|\psi_\lambda\rangle$ is an eigenstate of A , Polarization echo can be used to measure the distance between a state $e^{iH_\lambda t} e^{-iH_{\lambda+\delta\lambda} t} |\psi_\lambda\rangle$ and eigenspace of A which $|\psi_\lambda\rangle$ belongs to.

Polarization echo and related quantities are studied and measured in various experiments, including Nuclear Magnetic Resonance, Microwave billiards, Elastic waves in metals, and so on. Fisher information metric is a useful tool in quantum estimation theory, in particular, its reciprocal gives lower bound of variance of an estimate of deterministic parameter in Cramer-Rao bound. Also, it is known that they can be used as diagnosis of non-Markovianity [41, 42]. Fisher information metric and fidelity are also used as order parameter for quantum phase transitions [43, 44].

Thermofield double state of a QFT is defined as a purification of thermal density matrix. Explicitly, it is defined by

$$|\Psi_{\text{TFD}}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-\frac{\beta E_i}{2}} |E_i\rangle_L \otimes |E_i\rangle_R, \quad (2.3)$$

where β is inverse temperature and $|E_i\rangle$ is eigenstate of QFT Hamiltonian. The TFD state lives in tensor product of two identical QFT Hilbert spaces, and tracing out either side of Hilbert space gives the thermal density matrix of the system. Time evolution of TFD state is given by

$$\begin{aligned} |\Psi_{\text{TFD}}(t)\rangle &= e^{-i(H_L+H_R)t}|\Psi_{\text{TFD}}\rangle \\ &= \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-(\frac{\beta}{2}+2it)E_i} |E_i\rangle_L \otimes |E_i\rangle_R, \end{aligned} \quad (2.4)$$

where $Z(\beta)$ is thermal partition function. We consider scalar excitation W at $t = -t_w$ on TFD state, which lives in left QFT Hilbert space. Then the perturbed TFD state at $t = 0$ is given by

$$|\Psi_{\text{TFD}}(\lambda, W)\rangle = \frac{1}{\sqrt{Z_W(\beta)}} e^{-iH_{\text{tot}}^\lambda t_w} W e^{iH_{\text{tot}}^\lambda t_w} |\Psi_{\text{TFD}}(\lambda)\rangle, \quad (2.5)$$

with $H_{\text{tot}} = H_L + H_R$. Tracing out right Hilbert space of this state gives correct excited thermal density matrix

$$e^{-iH^\lambda t_w} W e^{iH^\lambda t_w} \rho(\beta) e^{-iH^\lambda t_w} W e^{iH^\lambda t_w}. \quad (2.6)$$

Therefore, Fisher information metric of excited TFD state gives upper bound on Bures metric of excited thermal density matrices.

Let's consider deformation parametrized by coupling λ , of the QFT Hamiltonian. Then the inner product and the Fisher information metric of TFD state $|\text{TFD}(t, \lambda)\rangle$ corresponding to (2.1) with $|\psi\rangle \neq |\psi'\rangle$ is given by

$$|\langle \Psi_{\text{TFD}}(\lambda + \delta\lambda, W) | \Psi_{\text{TFD}}(\lambda, W) \rangle| = 1 - G_{\lambda\lambda}(\delta\lambda)^2 + O((\delta\lambda)^3). \quad (2.7)$$

We assume that the theory has time reversal symmetry. Then

$$\begin{aligned} &|\langle \Psi_{\text{TFD}}(\lambda + \delta\lambda, W) | \Psi_{\text{TFD}}(\lambda, W) \rangle| \\ &= \frac{1}{\sqrt{Z_W(\beta, \lambda) Z_W(\beta, \lambda + \delta\lambda)}} \left| \text{Tr} \left[e^{-\frac{\beta}{2} H_L^{\lambda+\delta\lambda}} W^{\lambda+\delta\lambda}(-t_w) W^\lambda(-t_w) e^{-\frac{\beta}{2} H_L^\lambda} \right] \right|, \end{aligned} \quad (2.8)$$

where we defined $W^\lambda(t) = e^{iH^\lambda t} W e^{-iH^\lambda t}$. The Fisher information metric can be decomposed into two parts,

$$G_{\lambda\lambda} = G_{\lambda\lambda}^{(W:0)} + G_{\lambda\lambda}^{(W:c)}, \quad (2.9)$$

where

$$G_{\lambda\lambda}^{(W:0)} = \frac{1}{2} \int_0^\beta dt_1 dt_2 \text{Tr} \left[\frac{e^{-\beta H}}{Z_W(\beta, \lambda)} e^{Ht_1} V e^{-Ht_1} e^{Ht_2} V e^{-Ht_2} W(-t_w)^2 \right] - \frac{3}{8} \left(\int_0^\beta dt \text{Tr} \left[\frac{e^{-\beta H}}{Z_W(\beta, \lambda)} e^{Ht} V e^{-Ht} W(-t_w)^2 \right] \right)^2, \quad (2.10)$$

$$G_{\lambda\lambda}^{(W:c)} = \text{Re} \left[\frac{1}{2} \int_0^{t_w} dt_1 dt_2 \text{Tr} \left[\frac{e^{-\beta H}}{Z_W(\beta, \lambda)} [W(-t_w), V(-t_1)] \cdot [W(-t_w), V(-t_2)]^\dagger \right] + \frac{i}{2} \int_0^{\frac{\beta}{2}} dt_E \int_0^{t_w} dt \text{Tr} \left[\frac{e^{-\beta H}}{Z_W(\beta, \lambda)} e^{Ht_E} V e^{-Ht_E} [[W(-t_w), V(-t)], W(-t_w)] \right] \right] - \frac{1}{2} \left(\int_0^{t_w} dt \text{Tr} \left[\frac{e^{-\beta H}}{Z_W(\beta, \lambda)} [W(-t_w), V(-t)] W(-t_w) \right] \right)^2. \quad (2.11)$$

$G_{\lambda\lambda}^{(W:c)}$ is negligible when the theory is not chaotic, so that $[W(-t_w), V(-t)] \approx 0$ holds. Therefore, the growth or decay of $G_{\lambda\lambda}^{(W:c)}$ implies scrambling is taking place.

Let us consider large N limit. The relaxation time t_r of a thermal QFT can be determined from the behavior of two point function

$$\langle W(t) W(0) \rangle \sim \mathcal{O} \left(e^{-\frac{t}{t_r}} \right). \quad (2.12)$$

In holographic CFT, t_r is given by β , which is significantly small compare to scrambling time $t_s = \frac{\beta}{2\pi} \log N^2$. We consider particular class of large N QFT with such hierarchy.

In [6], it was shown that CFT dual to Einstein gravity has largest Lyapunov exponent, among large N theories with such hierarchy.

Under these assumptions,

$$G_{\lambda\lambda}^{(W:0)} \approx \frac{1}{2} \int_0^\beta dt_1 dt_2 \text{Tr} \left[\frac{e^{-\beta H}}{Z(\beta, \lambda)} e^{Ht_1} V e^{-Ht_1} e^{Ht_2} V e^{-Ht_2} \right]. \quad (2.13)$$

Therefore, $G_{\lambda\lambda}^{(W:0)}$ coincides with unexcited Fisher information metric. In particular, $G_{\lambda\lambda}^{(W:0)}$ is independent of excitation W and is time independent. This fact implies that growth of Fisher information metric comes purely from scrambling in large N theory.

We can also consider similar quantity, corresponding to (2.1) with $|\psi\rangle = |\psi'\rangle$, which purely contains chaotic terms. We consider the inner product of two states

$$|\Psi_{\text{TFD}}(\lambda, W)\rangle = \frac{1}{\sqrt{Z_W(\beta)}} e^{-iH_{\text{tot}}^\lambda t_w} W e^{iH_{\text{tot}}^\lambda t_w} |\Psi_{\text{TFD}}(\lambda)\rangle, \quad (2.14)$$

$$|\Psi_{\text{TFD}}(\lambda, \delta\lambda, W)\rangle = \frac{1}{\sqrt{Z_W(\beta, \delta\lambda)}} e^{-iH_{\text{tot}}^{\lambda+\delta\lambda} t_w} W e^{iH_{\text{tot}}^{\lambda+\delta\lambda} t_w} |\Psi_{\text{TFD}}(\lambda)\rangle, \quad (2.15)$$

where $Z_W(\beta, \delta\lambda)$ is the normalization constant. Infinitesimal part of the inner product is defined by

$$\begin{aligned} & |\langle \Psi_{\text{TFD}}(\lambda, W) | \Psi_{\text{TFD}}(\lambda, \delta\lambda, W) \rangle| \\ &= 1 - G_{\lambda\lambda}^S(\delta\lambda)^2 + \mathcal{O}((\delta\lambda)^3) \\ &= \frac{1}{\sqrt{Z_W(\beta, \lambda) Z_W(\beta, \delta\lambda)}} |\text{Tr} [e^{-\beta H_L^\lambda} W^{\lambda+\delta\lambda}(-t_w) W^\lambda(-t_w)]|. \end{aligned} \quad (2.16)$$

$G_{\lambda\lambda}^S$ can be explicitly written as

$$G_{\lambda\lambda}^S = \text{Re} \left[\frac{1}{2} \int_0^{t_w} dt_1 dt_2 \text{Tr} \left[\frac{e^{-\beta H}}{Z_W(\beta, \lambda)} [W(-t_w), V(-t_1)] \cdot [W(-t_w), V(-t_2)]^\dagger \right] \right] \\ - \frac{1}{2} \left(\int_0^{t_w} dt \text{Tr} \left[\frac{e^{-\beta H}}{Z_W(\beta, \lambda)} [W(-t_w), V(-t)] W(-t_w) \right] \right)^2. \quad (2.17)$$

Compared to (2.9), this quantity does not include constant divergent terms, but only contains terms proportional to commutators of operators. Therefore, we can use this quantity to measure the growth of commutators, therefore scrambling.

3 Holographic caluculation

In this section, we will show Fisher information metric grows rapidly in holographic systems, using proposed holographic dual of Fisher information metric.

The metric of $d+1$ eternal black hole is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{S^{d-1}}^2, \quad (3.1)$$

where $f(r)$ is given by $r^2 - R^2$ for $d=2$ with $R^2 = 8G_N M$, and $r^2 + 1 - a^2/r^{d-2}$ for higher dimensions with $a^2 = \frac{G_N M}{\text{Vol}(S^{d-1})}$. R is radius of black hole horizon, and AdS radius is set to 1. The inverse Hawking temperature β of this black hole is given by $\beta = \frac{4\pi}{f'(R)}$. In the Kruskal coordinate, the metric is

$$ds^2 = -\frac{f(r)}{f'(R)^2} e^{-f'(R)r_*(r)} du dv + r^2 d\Omega_{S^{d-1}}^2, \quad (3.2)$$

where u and v are given by

$$u^2 = e^{f'(R)(r_*(r)-t)} \\ v^2 = e^{f'(R)(r_*(r)+t)}. \quad (3.3)$$

and $r_*(r)$ is defined to satisfy $dr_* = f(r)^{-1}dr$ and $r_*(0) = r_*(\infty) = 0$. The state dual to eternal black hole at $t=0$ is thermofield double state [45, 46]. We consider a few particles with total energy E are thrown homogeneously into eternal black hole at $t = -t_w$ from left boundary. The proper energy of the particles at $t=0$ slice is $E_p \approx \frac{E}{R} e^{\frac{2\pi}{\beta} t_w}$. So when t_w is sufficiently large, the particle becomes shock wave, and the geometry can be approximated by shock wave geometry, which is given by glueing two black holes with mass M and mass $M+E$, as shown in figure 2. It is given by $u_L = u_R$ and $v_L = v_R + \alpha$, where α is defined by

$$\alpha = \frac{E}{u_w} \frac{dR}{dM} C(R, R) \quad (3.4)$$

with black hole mass M and $C(r, R) = \frac{e^{f'(R)r_*(r)}}{r-R}$. In 2d, α is given by $\alpha = \frac{E}{4M} e^{\frac{2\pi}{\beta} t_w}$. In higher dimensions, α is proportional to $G_N \cdot E \cdot e^{\frac{2\pi}{\beta} t_w}$.

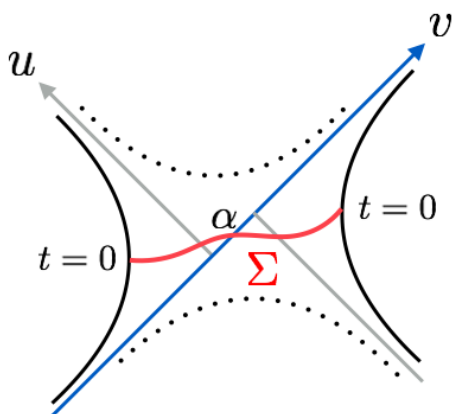


Figure 2. A diagram of the shock wave geometry. Black line are boundaries, dashed lines are singularities, and a blue line is the shock wave. Red surface connecting two boundaries is the extremal volume surface Σ . For 2d CFT, this surface can be understood as configuration of world sheet of a string ending on both boundaries.

According to the proposal in [29], inner product of TFD states is holographically given by the volume of codimension 1 extremal volume surface Σ , which connects $t = 0$ slices at two boundaries of spacetimes. This is given as

$$|\langle \Psi_{TFD}(t, \lambda + \delta\lambda) | \Psi_{TFD}(t, \lambda) \rangle| = e^{-n_d(\delta\lambda)^2 \text{Vol}(\Sigma)}, \quad (3.5)$$

where n_d is $\mathcal{O}(1)$ constant. The Fisher information metric is then

$$G_{\lambda\lambda} = n_d \text{Vol}(\Sigma). \quad (3.6)$$

The idea of this proposal stems from Janus geometry [47, 48] and the proposed duality between AdS/CFT and MERA [49–51]. We note that, in [40], the volume of maximal volume surface in the bulk was conjectured to be equal to the complexity of the corresponding boundary state. The volume of Σ is UV divergent, and we use the radial cutoff in AdS to regularize this volume.

Let's insert a homogeneous scalar operator W on left boundary at $t = -t_w$ and consider the perturbed TFD state at $t = 0$. The resulting holographic geometry is approximated by the shock wave geometry for large t_w . The volume of the surface connecting two boundaries at $t = 0$ is

$$\text{Vol}(\Sigma) = \int dt d\Omega_{d-1} \sqrt{(-f + \dot{r}^2 f^{-1}) r^{2(d-1)}}. \quad (3.7)$$

where $\dot{r} = \frac{dr}{dt}$. When the surface has extremal volume, the conserved quantity along the surface is

$$s_d = \frac{f(r) r^{d-1}}{\sqrt{-f(r) + \dot{r}^2 f(r)^{-1}}}. \quad (3.8)$$

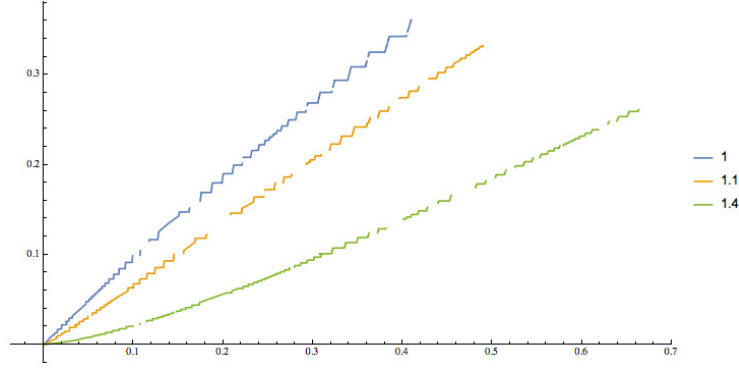


Figure 3. Numerical plot for growth of information metric versus $\alpha = \frac{E}{4M} e^{\frac{2\pi}{\beta} t_w}$ in AdS_3/CFT_2 . The numbers in the right are radiuses of BH horizon $R = \frac{2\pi}{\beta}$. The vertical ax is for $\frac{2\Delta\text{Vol}}{R^2}$, and horizontal ax is for α . We can confirm linear growth of information $\Delta\text{Vol} = f_1 \times \alpha + \mathcal{O}(\alpha^2)$ for small α .

Therefore we conclude that the volume of extremal volume surface is

$$\text{Vol}(\Sigma) = \text{Vol}(S^{d-1}) \int dr r^{d-1} \sqrt{f(r)^{-1} \left(1 - \frac{1}{1 + \frac{f(r)r^{2(d-1)}}{s_d^2}} \right)}. \quad (3.9)$$

Let us define r_0 by $\dot{r}(r) = 0$ and corresponding time as t_0 . Then we can express α in terms of s_d as

$$\log \frac{\alpha}{2} = r_*(r_0) + t_0 + \int_{r_0}^R dt \frac{1}{|f(r)| \sqrt{1 + \frac{f(r)r^{2(d-1)}}{s_d^2}}} \left(1 - \sqrt{1 + \frac{f(r)r^{2(d-1)}}{s_d^2}} \right). \quad (3.10)$$

The numerical result is shown in figure 3, and we can conclude that

$$G_{\lambda\lambda}^{W:c} = f_1 \cdot \alpha + \mathcal{O}(\alpha^2). \quad (3.11)$$

where f_1 is N independent and positive, but has dependence on the temperature. For $d = 2$ at $R = 1, 1.1, 1.4$, we get $f_1 \approx 0.9, 0.7, 0.2$, respectively. This implies that the information metric for perturbed TFD states for external fields grows rapidly, proportional to α .

Although approximation by classical gravity is no longer applicable when $\alpha \gg 1$, we can estimate the behavior of inner product by sticking to classical computation. Numerical result in CFT_2 is

$$\frac{|\langle \Psi_{TFD}(\lambda + \delta\lambda, W) | \Psi_{TFD}(\lambda, W) \rangle|}{|\langle \Psi_{TFD}(\lambda + \delta\lambda) | \Psi_{TFD}(\lambda) \rangle|} \underset{\alpha \gg 1}{\sim} \left(\frac{1}{\alpha} \right)^{\frac{2\pi^2 n_d}{\beta^2} (\delta\lambda)^2} \propto e^{-\frac{4\pi^3 n_d t_w}{\beta^3} (\delta\lambda)^2}. \quad (3.12)$$

So we can observe the exponential decay of inner product.

4 Wilson loop

We can study two point functions of Wilson loops on different boundaries, using holographic prescription [52, 53]. We assume these loops are put on the great circles of S^{d-1} on different boundaries. It is expected that such two point functions should decay exponentially in time. In 2d CFT, the two point function is given by $\langle W_L W_R \rangle \sim e^{-(S_{NG}-reg)}$, using extremal value of Nambu-Goto action,

$$\begin{aligned} S_{NG} &= \frac{1}{2\pi\alpha'} \int dt d\Omega_1 \sqrt{(-f + \dot{r}^2 f^{-1}) r^2} \\ &= \frac{1}{2\pi\alpha'} \text{Vol}(S^1) \int dr \, r \sqrt{f(r)^{-1} \left(1 - \frac{1}{1 + \frac{f(r)r^2}{s_2^2}} \right)}. \end{aligned} \quad (4.1)$$

The result is same as the calculation of volume of extremal codimension 1 surface, up to constant factor. Therefore, two point function of Wilson loops decays as $\sim f_0 - f_1 \times \alpha + \dots$ for small α , as expected.

5 Conclusion

In this article, we estimated the time evolution of information metric and inner product, of thermofield double state perturbed by an operator W at $t = -t_w$. In large N theories, information metric can be separated into two terms, one is unperturbed information metric, and the other is proportional to commutator of V and $W(-t_w)$. Rapid growth of the later term indicates scrambling of W , and sensitivity of time evolution of thermofield double state or thermal mixed states to external environment or internal imperfections. Indeed, we calculated the information metric and confirmed its rapid growth proportional to $e^{\frac{2\pi t}{\beta}}$. This implies that decay of OTO correlator can be understood by growth of distance between different thermal states. We also studied two point function of Wilson loops and confirmed that it decays exponentially.

It is interesting to calculate Fisher information metrics and inner products for pure states or reduced density matrices on subregions. Our study focused only on small perturbations to original Hamiltonian, but we can also study Loschmidt echo for strong perturbations. In that case, the decay rate is expected to be independent of strength of perturbations, and the rate can be related to classical Lyapunov exponent of the system. Furthermore, it would be very interesting to explore physical quantities more, which capture scrambling, and have simple gravity correspondents.

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