

A note on Schmidt's built-up systems of fundamental sequences

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Introduction. Let $\langle F_\alpha \rangle_{\alpha \in \Delta}$ be a transfinite sequence of number theoretic functions indexed by an initial segment Δ of the second number class which satisfies the following conditions:

- (a) F_0 is strictly increasing,
- (b) if F_α is strictly increasing, $F_{\alpha+1}$ is also strictly increasing, $F_\alpha(0) \leq F_{\alpha+1}(0)$ and $F_\alpha(x) < F_{\alpha+1}(x)$ for $0 < x < \omega$,
- (c) $F_\alpha(x) = F_{\alpha[x]}(x)$ if α is a limit ordinal, where $\langle \alpha[x] \rangle_{x < \omega}$ is a fundamental sequence for α .

Schmidt[3] introduced the concept of built-up systems of fundamental sequences, and showed that, for the above sequence $\langle F_\alpha \rangle_{\alpha \in \Delta}$, each F_α is strictly increasing if the system of fundamental sequences used is built-up. However there are some standard systems of fundamental sequences in literatures, e.g. Ketonen and Solovay[1], which are not built-up in Schmidt's sense, but which determine a sequence of strictly increasing functions.

The purpose of this note is to extend the concept of built-up systems so that it can be applicable to wider classes of systems of sequences of ordinals.

In §1 we define (n)-built-up systems and quasi-(n)-built-up systems of sequences of ordinals. In §2 we show a theorem on a relation between (n)-built-up systems and $\langle F_\alpha \rangle_{\alpha \in \Delta}$, which corresponds to Theorem 1 in [3], and a theorem on a relation

between quasi-(n)-built-up systems and a sequence of number theoretic functions $\langle H_\alpha \rangle_{\alpha \in \Delta}$. In §3, we give an example of (1)-built-up system of fundamental sequences for Γ_0 . Finally, in §4, we extend the results in Schmidt[4], by using quasi-(n)-built-up systems of sequences of ordinals.

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§1. Preliminaries

Let Δ be an initial segment of second number class. We will use Greek letters $\alpha, \beta, \gamma, \dots$ for ordinal numbers in Δ . Let $P: \Delta \longrightarrow \Delta^\omega$ be an assignment of sequences of ordinals for Δ . We shall write $\alpha[i]$ for $(P(\alpha))(i)$ whenever $\alpha \in \Delta$ and $i < \omega$.

If P satisfies the following conditions (A)-(C):

(A) $\alpha[i] = 0$ if $\alpha = 0$ and $i < \omega$,

(B) $\alpha[i] = \beta$ if $\alpha = \beta + 1$ and $i < \omega$,

(C) $\alpha[i] < \alpha$ if α is a limit ordinal and $i < \omega$,

then we call P a system of sequences of ordinals for Δ .

Moreover, if a system P of sequences of ordinals satisfies the following conditions $(C)^+, (D)$:

$(C)^+$ $\alpha[i] < \alpha[i+1] < \alpha$ if α is a limit ordinal and $i < \omega$,

(D) $\lim_{i < \omega} \alpha[i] = \alpha$ if α is a limit ordinal,

then we call P a system of fundamental sequences for Δ .

In the following, we assume P is a system of sequences of

ordinals.

Definition 1.1. Let P be a system of sequences of ordinals for Δ . For each $n < \omega$, $\xrightarrow{1/n}$, $\xrightarrow{n}$, $\xRightarrow{n}$ are defined as follows:

- (1) $\alpha \xrightarrow{1/n} \beta$ iff $0 < \alpha$ and $\alpha[n] = \beta$,
- (2) $\alpha \xrightarrow{n} \beta$ iff there is a sequence $\gamma_0, \dots, \gamma_j$ ($0 < j < \omega$) such that $\gamma_0 = \alpha$, $\gamma_j = \beta$ and $\gamma_i \xrightarrow{1/n} \gamma_{i+1}$ ($0 \leq i < j$),
- (3) $\alpha \xRightarrow{n} \beta$ iff $\alpha \xrightarrow{n} \beta$ or $\alpha = \beta$.

For each $n < \omega$, P is (n)-built-up (and quasi-(n)-built-up), if $\alpha[i+1] \xrightarrow{n} \alpha[i]$ (and $\alpha[i+1] \xRightarrow{n} \alpha[i]$, respectively) for each limit ordinal and each $i < \omega$.

Built-up systems in Schmidt's sense[3] is the same as (0)-built-up systems of fundamental sequences in our sense. Ketonen and Solovay[1] introduced the relation $\xrightarrow{n}$ for studying a standard system of fundamental sequences for ordinals up to ε_0 . Their system is (1)-built-up but not (0)-built-up (cf. Theorem 2.4 of [1]).

Proposition 1.1. Let P be quasi-(s)-built-up. If $s \leq m$, $n \leq m$ and $\alpha \xrightarrow{n} \beta$, then $\alpha \xrightarrow{m} \beta$.

(Proof) By induction on α . Case 1. $\alpha = 0$. This case is trivial because $\neg(0 \xrightarrow{n} \beta)$. Case 2. $\alpha = \gamma + 1$. If $\alpha \xrightarrow{n} \beta$, then $\gamma \xRightarrow{n} \beta$. So $\alpha = \gamma + 1 \xrightarrow{1/n} \gamma$ and $\gamma \xRightarrow{m} \beta$ by ind. hyp. So $\alpha \xrightarrow{m} \beta$. Case 3. α is limit. If $\alpha \xrightarrow{n} \beta$ then $\alpha[n] \xRightarrow{n} \beta$. Because P is quasi-(s)-built-up, $\alpha[m] \xRightarrow{s} \alpha[n]$. By ind. hyp., $\alpha[m] \xRightarrow{m} \alpha[n]$ and

$\alpha[n] \xRightarrow{m} \beta$. So, $\alpha \xrightarrow{m} \alpha[m] \xRightarrow{m} \alpha[n] \xRightarrow{m} \beta$.

Corollary 1.2. Let $n \leq m$. If P is (n) -built-up (and quasi- (n) -built-up), then P is (m) -built-up (and quasi- (m) -built-up, respectively).

§2. (n) -built-up systems and hierarchies of number theoretic functions

We say that a function $F: \omega \rightarrow \omega$ is strictly increasing after n if $F(x) < F(x+1)$ for $n \leq x < \omega$.

Let P be a system of sequences of ordinals for Δ . Suppose that $\langle F_\alpha \rangle_{\alpha \in \Delta}$ is any sequence of number theoretic functions satisfying the following conditions for $n < \omega$.

(a)_n F_0 is strictly increasing after n .

(b)_n If F_α is strictly increasing after n , then

$F_{\alpha+1}$ is also strictly increasing after n ,

$F_\alpha(n) \leq F_{\alpha+1}(n)$, and $F_\alpha(x) < F_{\alpha+1}(x)$ for $n < x < \omega$.

(c)_n $F_\alpha(x) = F_{\alpha[x]}(x)$ for $n \leq x < \omega$, if α is limit.

Remark that conditions $(a)_0$, $(b)_0$ and $(c)_0$ are the same as (a), (b) and (c) in Introduction.

Example 1. Let the fast growing hierarchy $\langle F_\alpha \rangle_{\alpha \in \Delta}$ define by

$F_0(x) = x+1$, $F_{\alpha+1}(x) = F_\alpha^{x+1}(x)$, where F_α^i is defined by

$F_\alpha^0(x) = x$, $F_\alpha^{i+1}(x) = F_\alpha^i(F_\alpha(x))$,

$F_\alpha(x) = F_{\alpha[x]}(x)$ if α is limit.

Then $\langle F_\alpha \rangle_{\alpha \in \Delta}$ satisfies $(a)_n$, $(b)_n$ and $(c)_n$.

Theorem 2.1. If $\langle F_\alpha \rangle_{\alpha \in \Delta}$ satisfies conditions $(a)_n$, $(b)_n$ and $(c)_n$, then the following hold for each $\alpha, \beta \in \Delta$.

(1) $\alpha \xrightarrow{n} \beta$ implies $F_\beta(n) \leq F_\alpha(n)$.

(2) If P is $(n+1)$ -built-up, then

(2.1) F_α is strictly increasing after n ,

(2.2) $\alpha \xrightarrow{m} \beta$ implies $F_\beta(s) \leq F_\alpha(s)$, $F_\beta(x) < F_\alpha(x)$
for $s < x < \omega$, where $s = \max(n+1, m)$.

Moreover, if P is (n) -built-up and $m \leq n$, then $\alpha \xrightarrow{m} \beta$ implies $F_\beta(n) \leq F_\alpha(n)$, $F_\beta(n+1) < F_\alpha(n+1)$.

(Proof) By induction on α . Case 1. $\alpha = 0$. (1) holds because $\neg(0 \xrightarrow{n} \beta)$. (2.1) is $(a)_n$, (2.2) holds because $\neg(0 \xrightarrow{n} \beta)$.

Case 2. $\alpha = \gamma + 1$. (1) If $\alpha \xrightarrow{n} \beta$ then $\gamma \xrightarrow{n} \beta$. $F_\beta(n) \leq F_\gamma(n) \leq F_\alpha(n)$ by ind.hyp. and $(b)_n$. (2.1) Since F_γ is strictly increasing after n , by ind. hyp., so is F_α by $(b)_n$. (2.2) Assume $\alpha \xrightarrow{m} \beta$. Then $\gamma \xrightarrow{m} \beta$. If $\beta = \gamma$, (2.2) holds by $(b)_n$. Hence, by ind. hyp. (2.2) holds for all $\alpha \xrightarrow{m} \beta$.

Case 3. α is limit. (1) If $\alpha \xrightarrow{n} \beta$ then $\alpha[n] \xrightarrow{n} \beta$. So

$$F_\beta(n) \leq F_{\alpha[n]}(n) = F_\alpha(n) \text{ by (1) of ind. hyp.}$$

(2) Let P be $(n+1)$ -built-up. (2.1) For $n \leq x < \omega$, $\alpha[x+1] \xrightarrow{n+1} \alpha[x]$. So,

$$\begin{aligned} F_\alpha(x+1) &= F_{\alpha[x+1]}(x+1) \geq F_{\alpha[x]}(x+1) \text{ by (2.2) of ind.hyp.} \\ &> F_{\alpha[x]}(x) \text{ by (2.1) of ind.hyp.} \\ &= F_\alpha(x). \end{aligned}$$

(2.2) If $\alpha \xrightarrow{m} \beta$ then $\alpha[m] \xrightarrow{m} \beta$. Let $s = \max(n+1, m)$. Then

$$F_\beta(s) \leq F_{\alpha[m]}(s) \leq F_{\alpha[s]}(s) = F_\alpha(s)$$

by $\alpha[m] \xrightarrow{n+1} \alpha[s]$ and ind.hyp.

$$F_\beta(x) \leq F_{\alpha[m]}(x) < F_{\alpha[x]}(x) = F_\alpha(x)$$

for $s < x < \omega$ by $\alpha[x] \xrightarrow{n+1} \alpha[m]$, and ind.hyp. Moreover if P is (n) -built-up and $m \leq n$, then $\alpha[n] \xrightarrow{n} \alpha[m]$ and $\alpha[n+1] \xrightarrow{n} \alpha[m]$,

$$F_\beta(n) \leq F_{\alpha[m]}(n) \leq F_{\alpha[n]}(n) = F_\alpha(n),$$

$$F_\beta(n+1) \leq F_{\alpha[m]}(n+1) < F_{\alpha[n+1]}(n+1) = F_\alpha(n+1), \text{ by ind.hyp.}$$

Corollary 2.2. If $\langle F_\alpha \rangle_{\alpha \in \Delta}$ satisfies (a), (b) and (c) (i.e.

$(a)_0, (b)_0$ and $(c)_0$), and P is (1) -built-up, then for each α , F_α is strictly increasing.

Next, we will introduce another hierarchies $\langle H_\alpha \rangle_{\alpha \in \Delta}$ of number theoretic functions which is constructed by the same way as $\langle F_\alpha \rangle_{\alpha \in \Delta}$ except $(c)_n$ (i.e. $F_\alpha(x) = F_{\alpha[x]}(x)$ if α is a limit ordinal).

We fix a system P of sequences of ordinals for Δ , and a function $f: \omega \rightarrow \omega$ which satisfies $n < f(n)$ and $f(x) < f(x+1)$ for $n \leq x < \omega$ (e.g. $f(x) = x + 1$).

Suppose $\langle H_\alpha \rangle_{\alpha \in \Delta}$ is any sequence of number theoretic functions satisfying the following conditions for n .

(a)_n and (b)_n are the same as the case of $\langle F_\alpha \rangle_{\alpha \in \Delta}$,

(c)_n^{*} $H_\alpha(x) = H_{\alpha[x]}(f(x))$ for $n \leq x < \omega$, if α is limit.

Example 2. Let the Hardy hierarchy $\langle H_\alpha \rangle_{\alpha \in \Delta}$ define by

$$H_0(x) = x, \quad H_\alpha(x) = H_{\alpha[x]}(x+1) \text{ if } \alpha > 0.$$

Then $\langle H_\alpha \rangle_{\alpha \in \Delta}$ satisfies (a)_n, (b)_n and (c)_n^{*} (where we take $x+1$ as $f(x)$).

We can prove a theorem which is a relation between quasi-(n)-built-up systems and $\langle H_\alpha \rangle_{\alpha \in \Delta}$.

Theorem 2.3. If $\langle H_\alpha \rangle_{\alpha \in \Delta}$ satisfies conditions (a)_n, (b)_n and (c)_n^{*} and P is quasi-(f(n+1))-built-up, then for each $\alpha \in \Delta$,

(1) H_α is strictly increasing after n,

(2) $\alpha \xrightarrow{m} \beta$ implies $H_\beta(x) < H_\alpha(x)$ for $\max(n+1, m) \leq x < \omega$.

In addition, $\alpha \xrightarrow{n} \beta$ implies $H_\beta(n) \leq H_\alpha(n)$.

Moreover, if P is quasi-(f(n))-built-up and $m < n$, then

$\alpha \xrightarrow{m} \beta$ implies $H_\beta(n) \leq H_\alpha(n)$.

(Proof) By induction on α . Assume that P is quasi-(f(n+1))-built-up. Case 1. $\alpha=0$. (1) is (a)_n. (2) holds because $1(0 \xrightarrow{m} \beta)$.

Case 2. $\alpha=\gamma+1$. (1) Since H_γ is strictly increasing after n by ind.hyp., so is H_α by (b)_n. (2) Assume $\alpha \xrightarrow{m} \beta$. Then $\gamma \xrightarrow{m} \beta$. If $\beta = \gamma$, (2) holds by (b)_n. Hence, by ind. hyp., (2) holds for all $\alpha \xrightarrow{m} \beta$.

Case 3. α is limit. (1) For $n \leq x < \omega$, $\alpha[x+1] \xrightarrow{f(n+1)} \alpha[x]$, $f(n+1) \leq f(x+1)$,

$$\begin{aligned} H_\alpha(x+1) &= H_{\alpha[x+1]}(f(x+1)) \geq H_{\alpha[x]}(f(x+1)) \quad \text{by (2) of ind.hyp.} \\ &> H_{\alpha[x]}(f(x)) \quad \text{by (1) of ind.hyp.} \\ &= H_\alpha(x). \end{aligned}$$

(2) If $\alpha \xrightarrow{m} \beta$ then $\alpha[m] \xrightarrow{m} \beta$. For $\max(n+1, m) \leq x < \omega$, by ind. hyp. and $\alpha[x] \xrightarrow{f(n+1)} \alpha[m]$, $f(n+1) \leq f(x)$, so

$$H_\beta(x) \leq H_{\alpha[m]}(x) < H_{\alpha[m]}(f(x)) \leq H_{\alpha[x]}(f(x)) = H_\alpha(x).$$

If $\alpha \xrightarrow{n} \beta$ then $\alpha[n] \xrightarrow{n} \beta$. By ind. hyp.,

$$H_\beta(n) \leq H_{\alpha[n]}(n) < H_{\alpha[n]}(f(n)) = H_\alpha(n).$$

Moreover if P is quasi- $(f(n))$ -built-up, $\alpha \xrightarrow{m} \beta$ and $m < n$ then $\alpha[m] \xrightarrow{m} \beta$ and $\alpha[n] \xrightarrow{f(n)} \alpha[m]$, by ind. hyp.,

$$H_\beta(n) \leq H_{\alpha[m]}(n) < H_{\alpha[m]}(f(n)) \leq H_{\alpha[n]}(f(n)) = H_\alpha(n).$$

§3. A (1)-built-up system of fundamental sequences for Γ_0 .

We give a system P of fundamental sequences for Γ_0 (for details about Γ_0 see Schütte[5]), by modifying the system in §3 of Schmidt[3]. If we restrict P to ordinals below ε_0 , P corresponds to a standard system of fundamental sequences below ε_0 (e.g. Ketonen and Solovay[1]). Then we can prove that P is (1)-built-up (cf. Theorem 2.4 in [1]).

All ordinals below Γ_0 can be generated from 0 by the two functions ν and κ defined by

$$\nu(\alpha, \beta) = \omega^\alpha + \beta,$$

$$\kappa(0, \beta) = \varepsilon_\beta = \text{the } \beta\text{-th inaccessible of } \nu,$$

for $\gamma > 0$, $\kappa(\gamma, \beta) = \text{the } \beta\text{-th ordinal which is inaccessible for all } \lambda \delta. \kappa(\alpha, \delta) \text{ such that } \alpha < \gamma$.

If $\gamma < \Gamma_0$, γ is a limit ordinal, then there is exactly one pair $(\alpha, \beta) \in \Gamma_0^2$ and one $\rho \in \{\kappa, \nu\}$ such that $\alpha, \beta < \gamma$ and $\gamma = \rho(\alpha, \beta)$. We will write $\gamma =_{\text{nf}} \rho(\alpha, \beta)$ ($\rho(\alpha, \beta)$ is the normal form of γ). We define a system P of fundamental sequences for all limit ordinals $\gamma < \Gamma_0$ by induction on γ .

(1) $\gamma =_{\text{nf}} \nu(\alpha, \beta) = \omega^\alpha + \beta$. β is not a successor, α is not 0.

$$(1.1) \text{ If } \gamma = \omega^{\delta+1}, \text{ then } \gamma[i] = \begin{cases} \omega^\delta \cdot (i+1) & \text{if } \delta \geq \varepsilon_0, \\ \omega^\delta \cdot i & \text{if } \delta < \varepsilon_0. \end{cases}$$

$$(1.2) \text{ If } \gamma = \omega^\alpha, \alpha \text{ is limit, then } \gamma[i] = \omega^{\alpha[i]}.$$

$$(1.3) \text{ If } \gamma = \omega^\alpha + \beta, \beta \text{ is limit, then } \gamma[i] = \omega^\alpha + \beta[i].$$

(2) $\gamma =_{\text{nf}} \kappa(\alpha, \beta)$.

(2.1) If $\gamma = \kappa(\alpha, \beta)$, β is limit, then $\gamma[i] = \kappa(\alpha, \beta[i])$.

(2.2) If $\gamma = \kappa(0, 0) = \varepsilon_0$, then $\gamma[0] = \omega$, $\gamma[i+1] = \omega^{\gamma[i]}$.

(2.3) If $\gamma = \kappa(0, \eta+1) = \varepsilon_{\eta+1}$, then $\gamma[0] = \varepsilon_{\eta}+1$, $\gamma[i+1] = \omega^{\gamma[i]}$.

(2.4) If $\gamma = \kappa(\delta+1, 0)$, then $\gamma[0] = \kappa(\delta, 0)$, $\gamma[i+1] = \kappa(\delta, \gamma[i])$.

(2.5) If $\gamma = \kappa(\delta+1, \eta+1)$, then

$$\gamma[0] = \kappa(\delta+1, \eta)+1, \gamma[i+1] = \kappa(\delta, \gamma[i]).$$

(2.6) If $\gamma = \kappa(\alpha, 0)$, α is limit, then $\gamma[i] = \kappa(\alpha[i], 0)$.

(2.7) If $\gamma = \kappa(\alpha, \eta+1)$, α is limit, then $\gamma[i] = \kappa(\alpha[i], \kappa(\alpha, \eta)+1)$.

Theorem 3.1. P is a (1)-built-up system of fundamental sequences for Γ_0 .

Lemma 3.2. Let $m < \omega$ and $\alpha, \beta, \gamma, \sigma, \tau < \Gamma_0$.

(1) If $\omega^\alpha + \beta > \beta$ and $\beta \xrightarrow{1} \gamma$, then $\omega^\alpha \cdot m + \beta \xrightarrow{1} \omega^\alpha \cdot m + \gamma$.

(2) For each α such that $1 < \alpha$, $\alpha \xrightarrow{1} 1$.

(3) If $\alpha \xrightarrow{1} \beta$ and there is no γ such that $\alpha \xrightarrow{1} \varepsilon_\gamma \xrightarrow{1} \beta$, then $\omega^\alpha \xrightarrow{1} \omega^\beta$.

(4) If $\beta \xrightarrow{1} \gamma$ and there is no δ such that $\beta \xrightarrow{1} \kappa(\alpha+1, \delta) \xrightarrow{1} \gamma$, then $\kappa(\alpha, \beta) \xrightarrow{1} \kappa(\alpha, \gamma)$.

(5) If $\alpha \xrightarrow{1} \beta$, then $\kappa(\alpha, 0) \xrightarrow{1} \kappa(\beta, 0)$.

(6) If $\alpha \xrightarrow{1} \beta$, $\alpha < \sigma$ and $\gamma = \kappa(\sigma, \tau)$ for some τ , then $\kappa(\alpha, \gamma+1) \xrightarrow{1} \kappa(\beta, \gamma+1)$.

(7) Let α be limit, $\alpha =_{\text{nf}} \rho(\sigma, \tau)$, $\eta < \Gamma_0$. Then either the following condition (a) or (b) holds.

(a) there is an m such that all of the $\alpha[i]$ for $m < i$ are in the range of $\lambda\xi.\kappa(\eta, \xi)$.

(b) there is a $\xi < \Gamma_0$ such that $\kappa(\eta, \xi) \leq \alpha[i] < \kappa(\eta, \xi+1)$ for all i or $\alpha[i] < \kappa(\eta, 0)$ for all $i < \omega$.

(Proof) (1) For $m = 1$, by induction on β . For $m > 1$, by induction on m . (2), (3) By induction on α . (4) By induction on α with subsidiary induction on β . (5), (6) By induction on α . (7) By induction on α ; if $\rho = \nu$ then (b) holds.

(Proof of Theorem 3.1) By induction on γ , we will show that

$$\gamma[i+1] \xrightarrow{1} \gamma[i].$$

(1.1) $\gamma = \omega^{\delta+1}$. By Lemma 3.2(2), $\gamma[i+1] = \gamma[i] + \omega^\delta \xrightarrow{1} \gamma[i] + 1 \xrightarrow{1} \gamma[i]$.

(1.2) $\gamma = \omega^\alpha$, α is limit. By Lemma 3.2(3)(7) and ind.hyp.,

$$\gamma[i+1] = \omega^{\alpha[i+1]} \xrightarrow{1} \omega^{\alpha[i]} = \gamma[i].$$

(1.3) $\gamma = \omega^\alpha + \beta$, β is limit. By Lemma 3.2(1) and ind.hyp.,

$$\gamma[i+1] = \omega^{\alpha+\beta[i+1]} \xrightarrow{1} \omega^{\alpha+\beta[i]} = \gamma[i].$$

(2.1) $\gamma = \kappa(\alpha, \beta)$, β is limit. $\gamma[i] = \kappa(\alpha, \beta[i])$. Now (a) of Lemma

3.2(7) cannot hold for $\beta = \inf \kappa(\alpha+1, \tau)$. (For then $\gamma = \lim_{i < \omega} \gamma[i] = \lim_{i < \omega} \kappa(\alpha, \beta[i]) = \lim_{i < \omega} \beta[i] = \beta$, it is contradiction.) Hence (b)

must hold. Therefore by Lemma 3.2(4) and ind.hyp.,

$$\gamma[i+1] = \kappa(\alpha, \beta[i+1]) \xrightarrow{1} \kappa(\alpha, \beta[i]) = \gamma[i].$$

(2.2) $\gamma = \varepsilon_0$. By Lemma 3.2(3) and induction on i ,

$$\gamma[1] = \omega^\omega \xrightarrow{1} \omega = \gamma[0], \gamma[i+2] = \omega^{\gamma[i+1]} \xrightarrow{1} \omega^{\gamma[i]} = \gamma[i+1].$$

(2.3) $\gamma = \varepsilon_{\eta+1}$. By Lemma 3.2(2), $\gamma[1] = \omega^{\varepsilon_\eta+1} \xrightarrow{1} \omega^{\varepsilon_\eta \cdot 2} \xrightarrow{1} \varepsilon_{\eta+1} = \gamma[0]$. By Lemma 3.2(3) and induction on i ,

$$\gamma[i+2] = \omega^{\gamma[i+1]} \xrightarrow{1} \omega^{\gamma[i]} = \gamma[i+1].$$

(2.4) $\gamma = \kappa(\delta+1, 0)$. $\gamma[0] = \kappa(\delta, 0)$, $\gamma[i+1] = \kappa(\delta, \gamma[i])$. Now $\gamma[i] <$

$\kappa(\delta+1,0)$. By Lemma 3.2(4) and induction on i ,

$$\gamma[1] = \kappa(\delta, \gamma[0]) \xrightarrow{1} \kappa(\delta, 0) = \gamma[0],$$

$$\gamma[i+2] = \kappa(\delta, \gamma[i+1]) \xrightarrow{1} \kappa(\delta, \gamma[i]) = \gamma[i+1].$$

(2.5) $\gamma = \kappa(\delta+1, \eta+1)$. By Lemma 3.2(6), $\gamma[1] = \kappa(\delta, \kappa(\delta+1, \eta)+1) \xrightarrow{1} \kappa(0, \kappa(\delta+1, \eta)+1) \xrightarrow{1} \omega^{\kappa(0, \kappa(\delta+1, \eta))+1} = \omega^{\kappa(\delta+1, \eta)+1} \xrightarrow{1} \omega^{\kappa(\delta+1, \eta)} \cdot 2 \xrightarrow{1} \kappa(\delta+1, \eta)+1 = \gamma[0]$. Now $\kappa(\delta+1, \eta) < \gamma[i] < \kappa(\delta+1, \eta+1)$. By Lemma 3.2(4) and induction on i ,

$$\gamma[i+2] = \kappa(\delta, \gamma[i+1]) \xrightarrow{1} \kappa(\delta, \gamma[i]) = \gamma[i+1].$$

(2.6) $\gamma = \kappa(\alpha, 0)$, α is limit. By Lemma 3.2(5),

$$\gamma[i+1] = \kappa(\alpha[i+1], 0) \xrightarrow{1} \kappa(\alpha[i], 0) = \gamma[i].$$

(2.7) $\gamma = \kappa(\alpha, \eta+1)$, α is limit. By Lemma 3.2(6) and ind.hyp.,

$$\gamma[i+1] = \kappa(\alpha[i+1], \kappa(\alpha, \eta)+1) \xrightarrow{1} \kappa(\alpha[i], \kappa(\alpha, \eta)+1) = \gamma[i].$$

§3. (n)-built-upness and Bachmann's property B[n]

In this section, we extend the theorems in Schmidt[4] by using (n)-built-up system. In the following, we assume that P is a system of fundamental sequences for Δ .

Definition 4.1. P has property B[n] iff if $\alpha[i] < \mu \leq \alpha[i+1]$ then $\alpha[i] \leq \mu[n]$ for each limit $\alpha \in \Delta$, $i < \omega$ and $\mu \in \Delta$.

Theorem 4.1. P has property B[n] iff P is (n)-built-up.

(Proof) Let P have B[n], $\alpha \in \Delta$ be limit. Then for $\alpha[i] < \mu \leq \alpha[i+1]$, $\mu \xrightarrow{n} \alpha[i]$ by ind. on μ , in particular, $\alpha[i+1] \xrightarrow{n} \alpha[i]$. Let P be (n)-built-up, $\alpha \in \Delta$ be limit and $\alpha[i] < \mu \leq \alpha[i+1]$. We define $\langle \alpha_k \rangle_{k < \omega}$ inductively as follows: $\alpha_0 = \alpha[i+1]$,

$$\alpha_{k+1} = \begin{cases} \mu & \text{if } \alpha_k = \mu, \\ (\alpha_k)[j] & \text{if } \alpha_k \text{ is limit } > \mu, \text{ where } j \text{ is the least} \\ & \text{such that } (\alpha_k)[j] > \mu, \\ \beta & \text{if } \alpha_k = \beta + 1. \end{cases}$$

Because P is (n) -built-up, we can prove $\alpha \geq \mu$ and $\alpha_k \xrightarrow{n} \alpha[i]$ by ind. on k . Then $\langle \alpha_k \rangle_{k < \omega}$ is non-ascending sequence, hence there is an $m < \omega$ such that $\alpha_k = \mu$ for $k \geq m$. Hence $\mu \xrightarrow{n} \alpha[i]$, in particular, $\alpha[i] \leq \mu[n]$.

Theorem 4.2. For any $n < \omega$, there is no assignment of fundamental sequences with property $B[n]$ for the whole of the second number class.

(Proof) Let Ω be the first uncountable ordinal. Assume there is such a system for fixed $n < \omega$. Then $\lambda\alpha.\alpha[n]$ is a regressive function (i.e. $\alpha[n] < \alpha$ for $0 < \alpha < \Omega$). and therefore there is an $A \subset \Omega$ of order type Ω and a $\beta \in \Omega$ such that $\alpha[n] = \beta$ for all $\alpha \in A$ (cf. Levy[2]). Let $\{\alpha_i : i < \omega\}$ be the first ω elements of A , $\alpha = \lim_{n < \omega} \alpha_i$ and $\langle \alpha[i] \rangle_{i < \omega}$ be the fundamental sequence for α . Since $\alpha > \beta$, there is an $i < \omega$ such that $\beta < \alpha[i] < \alpha$. Let $m < \omega$ be the number such that $\alpha[i] < \alpha_m < \alpha$, and p be the greatest number such that $\alpha[p] < \alpha_m$. Then $\alpha[p] < \alpha_m \leq \alpha[p+1]$, so $\beta < \alpha[p] \leq \alpha_m[n]$. But $\alpha_m[n] = \beta$ because $\alpha_m \in A$. Contradiction.

Finally, we define properties A and $C[n]$ to extend Remark of [4] as follows:

P has property A iff if $\beta < \alpha < \Delta$, then $\alpha \xrightarrow{n} \beta$ for some $n < \omega$.

P has property $C[n]$ iff if $\langle F_\alpha \rangle_{\alpha < \Delta}$ is a sequence of number theoretic functions satisfies the properties (a)_n, (b)_n and (c), and $\beta < \gamma \in \Delta$, then F_β is dominated by F_γ (i.e. there is an m such that for all $n \geq m$, $F_\beta(n) < F_\gamma(n)$).

We can prove the following theorem which is an extension of Remark of [4].

Theorem 4.3. If P is (n) -built-up for some $n < \omega$, then P has properties A and $C[n]$.

(Proof) Assume P is (n) -built-up. First we prove that P has A by ind. on α . Case 1. $\alpha = 0$. Trivial. Case 2. $\alpha = \gamma + 1$. $\gamma \xrightarrow{m} \beta$ for some $m < \omega$ by ind. hyp. Then $\alpha \xrightarrow{m} \gamma \xrightarrow{m} \beta$. Case 3. α is limit. $\alpha[i] > \beta$ for some i . Then $\alpha[i] \xrightarrow{k} \beta$ for some k by ind. hyp. Let $r = \max(n, k, i)$. By using Proposition 1.1, $\alpha \xrightarrow{r} \alpha[r] \xrightarrow{r} \alpha[i] \xrightarrow{r} \beta$. Next we prove that P has $C[n]$. Because P has A , $\gamma \xrightarrow{r} \beta$ for some r . If we put $m = \max(n, r) + 1$, then by Theorem 1.3, $F_\beta(x) < F_\alpha(x)$ for $m \leq x < \omega$.

References.

- [1] J.Ketonen and R.Solovay, Rapidly growing Ramsey functions, Ann. of Math. 113(1981), 267-314.
- [2] A.Levy, Basic Set Theory, Springer-Verlag, 1979.
- [3] D.Schmidt, Built-up systems of fundamental sequences and hierarchies of number theoretic functions, Arch. math. Logik

18(1976),47-53.

[4] D.Schmidt, Postscript to "Built-up systems of fundamental sequences and hierarchies of number theoretic functions",
ibid. 18(1977),145-146.

[5] K.Schütte, Proof Theory, Springer-Verlag, 1977.

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