

Elliptic Quantum Group $U_{q,p}(B_N^{(1)})$ and Vertex Operators

Dedicated to Professor Akihiro Tsuchiya on his 70th birthday

By

Hitoshi KONNO* and Kazuyuki OSHIMA **

Abstract

Assuming the existence of the L -operators, we study the Hopf algebroid structure of $U_{q,p}(B_N^{(1)})$. As an application, we derive the type I and II vertex operators, which intertwine the $U_{q,p}(B_N^{(1)})$ -modules of generic level, by assuming some analytic properties of the L -operators. For the level-1 case, we construct their free field realizations and show that the results satisfy the desired commutation relations with coefficients given by the elliptic dynamical R -matrices of the $B_N^{(1)}$ type.

§ 1. Introduction

The algebra $U_{q,p}(\widehat{\mathfrak{g}})$ [20, 13, 6] is an elliptic analogue of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization[4] associated with the affine Lie algebra $\widehat{\mathfrak{g}}$. The $U_{q,p}(\widehat{\mathfrak{g}})$ is expected to give a realization of the face type elliptic quantum group [7, 12] equipped with the Hopf algebroid structure. In the previous works[13, 18, 23], we have constructed the L -operator of $U_{q,p}(A_N^{(1)})$ in terms of the elliptic currents, the generating functions of the Drinfeld generators of $U_{q,p}(A_N^{(1)})$. The L -operator satisfies the RLL -relation with the elliptic dynamical R -matrix of the $A_N^{(1)}$ type[10] and allows us to define the Hopf algebroid structure to $U_{q,p}(A_N^{(1)})$.

The elliptic quantum group $U_{q,p}(A_N^{(1)})$ equipped with the Hopf algebroid structure has proved to be quite useful in construction of both the finite and infinite dimensional

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*Department of Mathematics, Tokyo University of Marine Science and Technology, Etchujima, Koto, Tokyo 135-8533, Japan

e-mail: hkonno0@kaiyodai.ac.jp

**Center for General Education, Aichi Institute of Technology, Yakusa-cho, Toyota 470-0392, Japan

e-mail: oshima@aitech.ac.jp

representations as well as their intertwining operators, i.e. the vertex operators, in terms of the free fields. Such construction becomes a central tool in the algebraic analysis of the face type solvable lattice models associated with the vector representation of $\widehat{\mathfrak{g}}$ [10] in the spirit of Jimbo and Miwa[11]. See for example [28, 27, 19].

The purpose of this paper is to continue the above study to the case $U_{q,p}(B_N^{(1)})$. The $U_{q,p}(B_N^{(1)})$ itself has an interesting connection[6] to the deformation of Fateev-Lukyanov's WB_N algebra[26]. Assuming the existence of the L -operators in the elliptic algebra $U_{q,p}(B_N^{(1)})$, we give an Hopf algebroid structure of $U_{q,p}(B_N^{(1)})$. We then define the type I and II vertex operators as the intertwining operators of the $U_{q,p}(B_N^{(1)})$ -modules of generic level. By assuming some analytic properties of the L -operators and the half currents, which are expected to be defined recursively through the Gauss decomposition of the L -operators, we show that the components of the both types of vertex operators are constructed by applying certain half currents to the top component. For the level-1 case, we construct their free field realizations and show that the results satisfy the desired commutation relations with coefficients given by the elliptic dynamical R -matrices of the $B_N^{(1)}$ type. These results give elliptic and dynamical analogues of those obtained for $U_q(B_N^{(1)})$ in [14, 1].

This paper is organized as follows. In Sec.2, we define the elliptic algebra $U_{q,p}(B_N^{(1)})$ as a certain topological algebra. In particular, we introduce the orthonormal basis type elliptic bosons and define the elliptic currents $k_{\pm j}(z)$. The Sec.3 is devoted to a conjecture of the construction of the L -operators in terms of the half currents of $U_{q,p}(B_N^{(1)})$. In Sec.4, assuming the existence of the L -operators we introduce the H -Hopf algebroid structure to $U_{q,p}(B_N^{(1)})$. In Sec.5, we give the vector representation and the level-1 highest weight representation of $U_{q,p}(B_N^{(1)})$. In Sec.6, after giving a construction of the type I and II vertex operators at generic level, we present a free field realization of the level 1 vertex operators and show that they satisfy the desired commutation relations with the coefficients given by the elliptic dynamical R -matrices. In Appendix A, we summarize a connection of $U_{q,p}(B_N^{(1)})$ to the quasi-Hopf formulation $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ of the elliptic quantum group. In Appendix B, we give a list of conjectural expressions for the half currents of $U_{q,p}(B_N^{(1)})$.

§ 2. Elliptic Algebra $U_{q,p}(B_N^{(1)})$

In this section, we give a definition of the elliptic algebra $U_{q,p}(B_N^{(1)})$ associated with the affine Lie algebra $B_N^{(1)}$.

§ 2.1. Definition

Let $A = (a_{ij})$ $i, j \in \{0\} \cup I$, $I = \{1, \dots, N\}$ be the $B_N^{(1)}$ type generalized Cartan matrix[15]. We denote by $B = (b_{ij})$, $b_{ij} = d_i a_{ij}$ the symmetrization of A with $d_0 = \dots =$

$d_{N-1} = 1, d_N = 1/2$. Let $q = e^{\hbar} \in \mathbb{C}[[\hbar]]$ and set $q_i = q^{d_i}$. Let p be an indeterminate. We use the following notations.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$[n]_i! = [n]_i[n-1]_i \cdots [1]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i![m-n]_i!},$$

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n), \quad (x; q, t)_\infty = \prod_{n,m=0}^{\infty} (1 - xq^n t^m), \quad \Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty.$$

Let $\mathfrak{h} = \tilde{\mathfrak{h}} \oplus \mathbb{C}d$, $\tilde{\mathfrak{h}} = \bar{\mathfrak{h}} \oplus \mathbb{C}c$, $\bar{\mathfrak{h}} = \oplus_{i \in I} \mathbb{C}h_i$ be the Cartan subalgebra of $B_N^{(1)}$. Define $\delta, \Lambda_0, \alpha_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$(2.1) \quad \langle \alpha_i, h_j \rangle = a_{ji}, \quad \langle \delta, d \rangle = 1 = \langle \Lambda_0, c \rangle,$$

the other pairings are 0. We also define $\bar{\Lambda}_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$(2.2) \quad \langle \bar{\Lambda}_i, h_j \rangle = \delta_{i,j}$$

We set $\bar{\mathfrak{h}}^* = \oplus_{i \in I} \mathbb{C}\bar{\Lambda}_i$, $\tilde{\mathfrak{h}}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$, $\mathcal{Q} = \oplus_{i \in I} \mathbb{Z}\alpha_i$ and $\mathcal{P} = \oplus_{i \in I} \mathbb{Z}\bar{\Lambda}_i$. Let $\{\epsilon_j$ ($1 \leq j \leq N$) $\}$ be an orthonormal basis in $\mathbb{R}^N$ with the inner product $(\epsilon_j, \epsilon_k) = \delta_{j,k}$. We realize the simple roots by $\alpha_j = \epsilon_j - \epsilon_{j+1}$ ($1 \leq j \leq N-1$), $\alpha_N = \epsilon_N$ and the fundamental weights by $\bar{\Lambda}_j = \epsilon_1 + \cdots + \epsilon_j$ ($1 \leq j \leq N-1$), $\bar{\Lambda}_N = (\epsilon_1 + \cdots + \epsilon_N)/2$. We define $h_{\epsilon_j} \in \bar{\mathfrak{h}}$ ($j \in I$) by $\langle \epsilon_i, h_{\epsilon_j} \rangle = (\epsilon_i, \epsilon_j)$ and $h_\alpha \in \bar{\mathfrak{h}}$ for $\alpha = \sum_j c_j \epsilon_j$, $c_j \in \mathbb{C}$ by $h_\alpha = \sum_j c_j h_{\epsilon_j}$. We regard $\bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}^*$ as the Heisenberg algebra by

$$(2.3) \quad [h_{\epsilon_j}, \epsilon_k] = (\epsilon_j, \epsilon_k), \quad [h_{\epsilon_j}, h_{\epsilon_k}] = 0 = [\epsilon_j, \epsilon_k].$$

In particular, we have $[h_j, \alpha_k] = a_{jk}$. We also set $h^j = h_{\bar{\Lambda}_j}$.

Let $\{P_\alpha, Q_\beta\}$ ($\alpha, \beta \in \bar{\mathfrak{h}}^*$) be the Heisenberg algebra defined by the commutation relations

$$(2.4) \quad [P_{\epsilon_j}, Q_{\epsilon_k}] = (\epsilon_j, \epsilon_k), \quad [P_{\epsilon_j}, P_{\epsilon_k}] = 0 = [Q_{\epsilon_j}, Q_{\epsilon_k}],$$

where $P_\alpha = \sum_j c_j P_{\epsilon_j}$ for $\alpha = \sum_j c_j \epsilon_j$. We set $P_{\bar{\mathfrak{h}}} = \oplus_{j \in I} \mathbb{C}P_{\epsilon_j}$, $Q_{\bar{\mathfrak{h}}} = \oplus_{j \in I} \mathbb{C}Q_{\epsilon_j}$, $P_j = P_{\alpha_j^\vee}$, $P^j = P_{\bar{\Lambda}_j}$ and $Q_j = Q_{\alpha_j}$, $Q^j = Q_{\bar{\Lambda}_j^\vee}$.

For the abelian group $\mathcal{R}_Q = \sum_{j=1}^N \mathbb{Z}Q_{\alpha_j}$, we denote by $\mathbb{C}[\mathcal{R}_Q]$ the group algebra over $\mathbb{C}$ of $\mathcal{R}_Q$. We denote by e^α the element of $\mathbb{C}[\mathcal{R}_Q]$ corresponding to $\alpha \in \mathcal{R}_Q$. These e^α satisfy $e^\alpha e^\beta = e^{\alpha+\beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now let us consider to double the Cartan subalgebra : $H = \tilde{\mathfrak{h}} \oplus P_{\bar{\mathfrak{h}}} = \sum_j \mathbb{C}(P_{\epsilon_j} + h_{\bar{\epsilon}_j}) + \sum_j \mathbb{C}P_{\bar{\epsilon}_j} + \mathbb{C}c$. We denote its dual space by $H^* = \tilde{\mathfrak{h}}^* \oplus Q_{\bar{\mathfrak{h}}}$. We define the

paring by (2.2), $\langle Q_\alpha, P_\beta \rangle = (\alpha, \beta)$ and $\langle Q_\alpha, h_\beta \rangle = \langle Q_\alpha, c \rangle = \langle Q_\alpha, d \rangle = 0 = \langle \alpha, P_\beta \rangle = \langle \delta, P_\beta \rangle = \langle \Lambda_0, P_\beta \rangle$. We define $\mathbb{F} = \mathcal{M}_{H^*}$ to be the field of meromorphic functions on H^* .

Definition 2.1. [6] The elliptic algebra $U_{q,p}(B_N^{(1)})$ is a topological algebra over $\mathbb{F}[[p]]$ generated by $\mathcal{M}_{H^*}$, $e_{j,m}, f_{j,m}, \alpha_{j,n}^\vee, K_j^\pm$, ($j \in I, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}$), $\widehat{d}$ and the central element c . We assume $K_j^\pm$ are invertible and set

$$\begin{aligned} e_j(z) &= \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m}, \\ \psi_j^+(q^{-\frac{c}{2}} z) &= K_j^+ \exp \left(-(q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,-n}^\vee}{1-p^n} z^n \right) \exp \left((q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}^\vee}{1-p^n} z^{-n} \right), \\ \psi_j^-(q^{\frac{c}{2}} z) &= K_j^- \exp \left(-(q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,-n}^\vee}{1-p^n} z^n \right) \exp \left((q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,n}^\vee}{1-p^n} z^{-n} \right). \end{aligned}$$

We call $e_j(z), f_j(z), \psi_j^\pm(z)$ the elliptic currents. The defining relations are as follows. For $g(P), g(P+h) \in \mathcal{M}_{H^*}$,

$$(2.5) \quad g(P+h)e_j(z) = e_j(z)g(P+h), \quad g(P)e_j(z) = e_j(z)g(P - \langle Q_{\alpha_j}, P \rangle),$$

$$(2.6) \quad g(P+h)f_j(z) = f_j(z)g(P+h - \langle \alpha_j, P+h \rangle), \quad g(P)f_j(z) = f_j(z)g(P),$$

$$(2.7) \quad [g(P), \alpha_{i,m}^\vee] = [g(P+h), \alpha_{i,n}^\vee] = 0,$$

$$(2.8) \quad g(P)K_j^\pm = K_j^\pm g(P - \langle Q_{\alpha_j}, P \rangle),$$

$$(2.9) \quad g(P+h)K_j^\pm = K_j^\pm g(P+h - \langle Q_{\alpha_j}, P \rangle),$$

$$(2.10) \quad [\widehat{d}, g(P+h)] = [\widehat{d}, g(P)] = 0,$$

$$(2.11) \quad [\widehat{d}, \alpha_{j,n}^\vee] = n\alpha_{j,n}^\vee, \quad [\widehat{d}, e_j(z)] = -z \frac{\partial}{\partial z} e_j(z), \quad [\widehat{d}, f_j(z)] = -z \frac{\partial}{\partial z} f_j(z),$$

$$(2.12) \quad K_i^\pm e_j(z) = q_i^{\mp a_{ij}} e_j(z) K_i^\pm, \quad K_i^\pm f_j(z) = q_i^{\pm a_{ij}} f_j(z) K_i^\pm,$$

$$(2.13) \quad [\alpha_{i,m}^\vee, \alpha_{j,n}^\vee] = \delta_{m+n,0} \frac{[a_{ij}m]_i [cm]_j}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm},$$

$$(2.14) \quad [\alpha_{i,m}^\vee, e_j(z)] = \frac{[a_{ij}m]_i}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z),$$

$$(2.15) \quad [\alpha_{i,m}^\vee, f_j(z)] = -\frac{[a_{ij}m]_i}{m} z^m f_j(z),$$

$$(2.16) \quad z_1 \frac{(q^{b_{ij}} z_2 / z_1; p^*)_\infty}{(p^* q^{-b_{ij}} z_2 / z_1; p^*)_\infty} e_i(z_1) e_j(z_2) = -z_2 \frac{(q^{b_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{-b_{ij}} z_1 / z_2; p^*)_\infty} e_j(z_2) e_i(z_1),$$

$$(2.17) \quad z_1 \frac{(q^{-b_{ij}} z_2 / z_1; p)_\infty}{(pq^{b_{ij}} z_2 / z_1; p)_\infty} f_i(z_1) f_j(z_2) = -z_2 \frac{(q^{-b_{ij}} z_1 / z_2; p)_\infty}{(pq^{b_{ij}} z_1 / z_2; p)_\infty} f_j(z_2) f_i(z_1),$$

$$(2.18) \quad [e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c} z_1 / z_2) \psi_j^-(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi_j^+(q^{-\frac{c}{2}} z_2) \right),$$

$$\begin{aligned}
(2.19) \quad & \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(p^* q^2 z_{\sigma(k)} / z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p^*)_{\infty}} \\
& \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^* q^{b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}} \\
& \times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(s)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}) = 0,
\end{aligned}$$

$$\begin{aligned}
(2.20) \quad & \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(pq^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}}{(pq^2 z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}} \\
& \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(pq^{-b_{ij}} w / z_{\sigma(m)}; p)_{\infty}}{(pq^{b_{ij}} w / z_{\sigma(m)}; p)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(pq^{-b_{ij}} z_{\sigma(m)} / w; p)_{\infty}}{(pq^{b_{ij}} z_{\sigma(m)} / w; p)_{\infty}} \\
& \times f_i(z_{\sigma(1)}) \cdots f_i(z_{\sigma(s)}) f_j(w) f_i(z_{\sigma(s+1)}) \cdots f_i(z_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}),
\end{aligned}$$

where $p^* = pq^{-2c}$ and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. We also denote by $U'_{q,p}(\widehat{\mathfrak{g}})$ the subalgebra obtained by removing d .

We treat the relations (2.11), (2.14)-(2.20) as formal Laurent series in z, w and z_j 's. All the coefficients in z_j 's are well-defined in the p -adic topology.

§ 2.2. The orthonormal basis type elliptic bosons

It is convenient to introduce the simple root type generators $\alpha_{j,m}$ and $\alpha'_{j,m}$ defined by $\alpha_{j,m} = [d_j]_q \alpha_{j,m}^{\vee}$ and $\alpha'_{j,m} = \frac{1 - p^{*m}}{1 - p^m} q^{cm} \alpha_{j,m}$, ($j \in I, n \neq 0$). From (2.13), (2.14), (2.15), we have

$$(2.21) \quad [\alpha_{i,m}, \alpha_{j,n}] = \frac{[b_{ij}m]_q [cm]_q}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-km} \delta_{m+n,0},$$

$$(2.22) \quad [\alpha'_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m]_q [cm]_q}{m} \frac{1 - p^{*m}}{1 - p^m} q^{km} \delta_{m+n,0},$$

$$(2.23) \quad [\alpha_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m]_q [cm]_q}{m} \delta_{m+n,0},$$

$$(2.24) \quad [\alpha_{i,m}, e_j(z)] = \frac{[b_{ij}m]_q}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} z^m e_j(z),$$

$$(2.25) \quad [\alpha'_{i,m}, f_j(z)] = -\frac{[b_{ij}m]_q}{m} \frac{1 - p^{*m}}{1 - p^m} q^{cm} z^m f_j(z).$$

Let $\eta = -(2N - 1)/2$. Let us further introduce the orthonormal basis type elliptic bosons $\mathcal{E}_m^{\pm j}$ ($j \in \{0\} \cup I, m \in \mathbb{Z}_{\neq 0}$) [6] by

$$(2.26) \quad \mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{j-1} [km]_q \alpha_{k,m} \pm \sum_{k=j}^N [(\eta + k)m]_+ \alpha_{k,m} \right),$$

$$(2.27) \quad \mathcal{E}_m^0 = \frac{[\frac{m}{2}]_q}{[m]_q} (\mathcal{E}_m^{+N} + \mathcal{E}_m^{-N}).$$

Here we set

$$C_m = \frac{[\eta m]_q}{[m]_q^2 [2\eta m]_q}, \quad [m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.$$

Proposition 2.2. *For $1 \leq j, k \leq N$, the following commutation relations hold.*

$$\begin{aligned} [\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] &= \delta_{m+n,0} \frac{[cm]_q [\eta m]_q [2(\eta+1)m]_q}{m(q - q^{-1})^2 [m]_q^3 [2\eta m]_q [(\eta+1)m]_q} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \\ [\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] &= \mp \delta_{m+n,0} \\ &\quad \times \frac{q^{\pm jm} [cm]_q [\eta m]_q}{m [m]_q^3 (q - q^{-1})^2 [2\eta m]_q} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} \left(q^{\pm(\eta+j)m} [m]_q \pm q^{\mp(j-1)m} [\eta m]_+ \right), \\ [\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] &= \mp \operatorname{sgn}(k-j) \delta_{m+n,0} q^{\mp(\operatorname{sgn}(k-j)\eta+k-j)m} \frac{[cm]_q [\eta m]_q}{m(q - q^{-1}) [m]_q^2 [2\eta m]_q} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \\ [\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] &= \mp \delta_{m+n,0} q^{\pm(\eta+j+k)m} \frac{[cm]_q [\eta m]_q}{m(q - q^{-1}) [m]_q^2 [2\eta m]_q} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \end{aligned}$$

Here

$$\operatorname{sgn}(l-j) = \begin{cases} + & (l > j), \\ - & (l < j). \end{cases}$$

Then one can realize the simple root type $\alpha_{j,m}$ in terms of the orthonormal basis type $\mathcal{E}_m^{\pm j}$ as follows.

Proposition 2.3. *For $1 \leq j \leq N-1$, we have*

$$(2.28) \quad \alpha_{j,m} = \pm [m]_q^2 (q - q^{-1}) (\mathcal{E}_m^{\pm j} - q^{\mp m} \mathcal{E}_m^{\pm(j+1)}),$$

$$(2.29) \quad \alpha_{N,m} = [m]_q (q^{m/2} - q^{-m/2}) (q^{-m/2} \mathcal{E}_m^{+N} - q^{m/2} \mathcal{E}_m^{-N}).$$

The following formulae are also useful.

Proposition 2.4. *For $1 \leq i, j \leq N$, the following commutation relations hold.*

$$(2.30) \quad [\alpha_{i,m}, \mathcal{E}_n^{\pm j}] = \pm \delta_{m+n,0} \frac{[cm]_q}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} (q^{\mp m} \delta_{i,j} - \delta_{i,j-1}),$$

$$(2.31) \quad [\mathcal{E}_m^{\pm i}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} e_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}),$$

$$(2.32) \quad [\mathcal{E}_m^{\pm i}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}).$$

§ 2.3. The elliptic currents $k_{\pm j}(z)$

Let us set

$$(2.33) \quad \psi_j(z) =: \exp \left\{ (q - q^{-1}) \sum_{m \neq 0} \frac{\alpha_{j,m}}{1 - p^m} p^m z^{-m} \right\} :.$$

Then the elliptic currents $\psi_j^{\pm}(z)$ in Definition 2.1 can be written as

$$(2.34) \quad \psi_j^+(q^{-\frac{c}{2}} z) = K_j^+ \psi_j(z), \quad \psi_j^-(q^{-\frac{c}{2}} z) = K_j^- \psi_j(pq^{-c} z).$$

Let us introduce the new elliptic currents $k_{\pm j}(z)$ ($j \in \{0\} \cup I$) associated with $\mathcal{E}_m^{\pm j}$ by

$$(2.35) \quad k_{\pm j}(z) =: \exp \left\{ \sum_{m \neq 0} \frac{[m]_q^2 (q - q^{-1})^2}{1 - p^m} p^m \mathcal{E}_m^{\pm j} z^{-m} \right\} :,$$

$$(2.36) \quad k_0(z) =: k_{-N}(q^{-1/2} z) \psi_N(q^{-1/2} z) := k_{+N}(q^{1/2} z) \psi_N(q^{1/2} z)^{-1} :.$$

Then from Proposition 2.3 we have the following decompositions.

Proposition 2.5. *For $1 \leq j \leq N - 1$, we have*

$$(2.37) \quad \psi_j(z) =: k_{+j}(z) k_{+(j+1)}(qz)^{-1} := k_{-j}(z)^{-1} k_{-(j+1)}(q^{-1} z) :,$$

$$(2.38) \quad \psi_N(z) =: k_{+N}(z) k_0(q^{-1/2} z)^{-1} := k_{-N}(z)^{-1} k_0(q^{1/2} z) :.$$

In addition, from Proposition 2.2 we obtain the following commutation relations.

Theorem 2.6. *The following commutation relations hold.*

$$\begin{aligned} k_{\pm j}(z_1) k_{\pm j}(z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} k_{\pm j}(z_2) k_{\pm j}(z_1), \quad (1 \leq j \leq N), \\ k_{+j}(q^j z_1) k_{+k}(q^k z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2} z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2} z)} k_{+k}(q^k z_2) k_{+j}(q^j z_1) \quad (1 \leq j < k \leq N), \\ k_{-j}(q^{-j} z_1) k_{-k}(q^{-k} z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2} z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2} z)} k_{-k}(q^{-k} z_2) k_{-j}(q^{-j} z_1) \\ &\quad (1 \leq k < j \leq N), \\ k_{+j}(q^j z_1) k_{-k}(q^{-k} \xi z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2} z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2} z)} k_{-k}(q^{-k} \xi z_2) k_{+j}(q^j z_1) \quad (j \neq k), \\ k_{+j}(q^j z_1) k_{-j}(q^{-j} \xi z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{2j-2} \xi^{-1} z) \Theta_p(q^{2j} \xi^{-1} z)}{\Theta_{p^*}(q^{2j} \xi^{-1} z) \Theta_p(q^{2j-2} \xi^{-1} z)} \frac{\Theta_{p^*}(q^{-2} z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2} z)} \\ &\quad \cdot k_{-j}(q^{-j} \xi z_2) k_{+j}(q^j z_1), \\ k_0(z_1) k_0(z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2} z) \Theta_p(q^2 z) \Theta_{p^*}(qz) \Theta_p(q^{-1} z)}{\Theta_{p^*}(q^2 z) \Theta_p(q^{-2} z) \Theta_{p^*}(q^{-1} z) \Theta_p(qz)} k_0(z_2) k_0(z_1), \end{aligned}$$

$$k_{+j}(q^j z_1) k_0(q^{N-1/2} z_2) = \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2} z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2} z)} k_0(q^{N-1/2} z_2) k_{+j}(q^j z_1) \quad (1 \leq j \leq N),$$

$$k_{-j}(\xi q^{-j} z_1) k_0(q^{N-1/2} z_2) = \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(z) \Theta_p(q^2 z)}{\Theta_{p^*}(q^2 z) \Theta_p(z)} k_0(q^{N-1/2} z_2) k_{-j}(\xi q^{-j} z_1) \quad (1 \leq j \leq N),$$

where $z = z_1/z_2$, and $\tilde{\rho}^+(z)$ is a function which appears associated with the elliptic dynamical R -matrices[21]. (See §3.1)

$$(2.39) \quad \tilde{\rho}^+(z) = \frac{\{\xi z\}^2 \{\xi^2 q^{-2} z\} \{q^2 z\}}{\{\xi^2 z\} \{z\} \{\xi q^2 z\} \{\xi q^{-2} z\}} \frac{\{p \xi^2 / z\} \{p / z\} \{p \xi q^2 / z\} \{p \xi q^{-2} / z\}}{\{p \xi / z\}^2 \{p \xi^2 q^{-2} / z\} \{p q^2 / z\}}$$

where $\xi = q^{-2n}$, $\{z\} = (z; p, \xi^2)_\infty$. We also set $\tilde{\rho}^{+*}(z) = \tilde{\rho}^+(z)|_{p \rightarrow p^*}$.

Proposition 2.7. *The following commutation relations hold.*

$$k_{\pm j}(z_1) e_j(z_2) = \frac{\Theta_{p^*}(q^{-c} z)}{\Theta_{p^*}(q^{-c \mp 2} z)} e_j(z_2) k_{\pm j}(z_1) \quad (1 \leq j \leq N),$$

$$k_{\pm j}(z_1) e_{j-1}(z_2) = \frac{\Theta_{p^*}(q^{-c \mp 1} z)}{\Theta_{p^*}(q^{-c \pm 1} z)} e_{j-1}(z_2) k_{\pm j}(z_1) \quad (2 \leq j \leq N),$$

$$k_{\pm j}(z_1) e_k(z_2) = e_k(z_2) k_{\pm j}(z_1) \quad (k \neq j, j-1),$$

$$k_{\pm j}(z_1) f_j(z_2) = \frac{\Theta_p(q^{\mp 2} z)}{\Theta_p(z)} f_j(z_2) k_{\pm j}(z_1) \quad (1 \leq j \leq N),$$

$$k_{\pm j}(z_1) f_{j-1}(z_2) = \frac{\Theta_p(q^{\pm 1} z)}{\Theta_p(q^{\mp 1} z)} f_{j-1}(z_2) k_{\pm j}(z_1) \quad (2 \leq j \leq N),$$

$$k_{\pm j}(z_1) f_k(z_2) = f_k(z_2) k_{\pm j}(z_1) \quad (k \neq j, j-1),$$

$$k_0(q^{N-1/2} z_1) e_N(z_2) = \frac{\Theta_{p^*}(q^{-c+N} z) \Theta_{p^*}(q^{-c+N-1} z)}{\Theta_{p^*}(q^{-c+N-2} z) \Theta_p(q^{-c+N+1} z)} e_N(z_2) k_0(q^{N-1/2} z_1),$$

$$k_0(q^{N-1/2} z_1) e_j(z_2) = e_j(z_2) k_0(q^{N-1/2} z_1) \quad (1 \leq j \leq N-1),$$

$$k_0(q^{N-1/2} z_1) f_N(z_2) = \frac{\Theta_p(q^{N-2} z) \Theta_p(q^{N+1} z)}{\Theta_p(q^N z) \Theta_p(q^{N-1} z)} f_N(z_2) k_0(q^{N-1/2} z_1),$$

$$k_0(q^{N-1/2} z_1) f_j(z_2) = f_j(z_2) k_0(q^{N-1/2} z_1) \quad (1 \leq j \leq N-1).$$

§ 2.4. The H -algebra $U_{q,p}(B_N^{(1)})$

Let $\mathcal{A}$ be a complex associative algebra, $\mathcal{H}$ be a finite dimensional commutative subalgebra of $\mathcal{A}$, and $\mathcal{M}_{\mathcal{H}^*}$ be the field of meromorphic functions on $\mathcal{H}^*$ the dual space of $\mathcal{H}$.

Definition 2.8 ($\mathcal{H}$ -algebra[5]). An $\mathcal{H}$ -algebra is an associative algebra $\mathcal{A}$ with 1, which is bigraded over $\mathcal{H}^*$, $\mathcal{A} = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} \mathcal{A}_{\alpha, \beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : \mathcal{M}_{\mathcal{H}^*} \rightarrow \mathcal{A}_{0,0}$ (the left and right moment maps), such that

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \quad a \in \mathcal{A}_{\alpha, \beta}, \quad \widehat{f} \in \mathcal{M}_{\mathcal{H}^*},$$

where T_α denotes the automorphism $(T_\alpha \hat{f})(\lambda) = \hat{f}(\lambda + \alpha)$ of $\mathcal{M}_{\mathcal{H}^*}$.

Proposition 2.9. $U = U_{q,p}(B_N^{(1)})$ is a H -algebra by

$$U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha, \beta}$$

$$U_{\alpha, \beta} = \left\{ x \in U \mid q^{P+h} x q^{-(P+h)} = q^{<\alpha, P+h>} x, \quad q^P x q^{-P} = q^{<\beta, P>} x \quad \forall P+h, P \in H \right\}$$

and $\mu_l, \mu_r : \mathbb{F} \rightarrow U_{0,0}$ defined by

$$\mu_l(\hat{f}) = \hat{f}(P+h, p) \in \mathbb{F}[[p]], \quad \mu_r(\hat{f}) = \hat{f}(P, p^*) \in \mathbb{F}[[p]].$$

We regard $T_\alpha = e^\alpha \in \mathbb{C}[\mathcal{R}_Q]$ as the shift operator $\mathcal{M}_{H^*} \rightarrow \mathcal{M}_{H^*}$

$$(T_\alpha \hat{f}) = e^\alpha \hat{f}(P, p^*) e^{-\alpha} = \hat{f}(P + <\alpha, P>, p^*).$$

Hereafter we abbreviate $f(P+h, p)$ and $f(P, p^*)$ as $f(P+h)$ and $f^*(P)$, respectively.

We also consider the H -algebra of the shift operators[5]

$$\mathcal{D} = \left\{ \sum_{\alpha} \hat{f}_\alpha T_\alpha \mid \hat{f}_\alpha \in M_{H^*}, \alpha \in \mathcal{R}_Q \right\}, \quad \mathcal{D}_{\alpha, \alpha} = \{ \hat{f} T_{-\alpha} \}, \quad \mathcal{D}_{\alpha, \beta} = 0 \quad (\alpha \neq \beta),$$

$$\mu_l^{\mathcal{D}}(\hat{f}) = \mu_r^{\mathcal{D}}(\hat{f}) = \hat{f} T_0 \quad \hat{f} \in M_{H^*}.$$

Then we have the H -algebra isomorphism

$$(2.40) \quad U \cong U \tilde{\otimes} \mathcal{D} \cong \mathcal{D} \tilde{\otimes} U.$$

§ 3. The L -operators and The Dynamical RLL -relations

We introduce the elliptic dynamical R -matrix of the $B_N^{(1)}$ type as a certain gauge transformation of Jimbo-Miwa-Okado's face type Boltzmann weight given in [10]. Then we propose a construction of the L -operator satisfying the RLL -relation by means of the elliptic currents of $U_{q,p}(B_N^{(1)})$.

Hereafter we regard q, p as a generic complex number satisfying $|q|, |p| < 1$ and set $p = q^{2r}$. We also use the following theta functions.

$$(3.1) \quad [u] = q^{\frac{u^2}{r} - u} \Theta_p(z), \quad [u]^* = q^{\frac{u^2}{r^*} - u} \Theta_{p^*}(z).$$

Here we set $z = q^{2u}$, $p^* = q^{2r^*} = pq^{-2c}$. Hence $r^* = r - c$.

§ 3.1. The elliptic dynamical R -matrix of the $B_N^{(1)}$ type

Let $\mathcal{I} = \{0, \pm 1, \pm 2, \dots, \pm N\}$. We fix the order $1 \prec 2 \prec \dots \prec N \prec 0 \prec -N \prec \dots \prec -2 \prec -1$. Let us consider the elliptic dynamical R -matrix of the $B_N^{(1)}$ type given by

$$\begin{aligned}
 (3.2) \quad R^+(u, s) &= \rho^+(u) \bar{R}^+(u, s), \\
 \bar{R}^+(u, s) &= \left\{ \sum_{\substack{j=1 \\ j \neq 0}}^{-1} E_{j,j} \otimes E_{j,j} + \sum_{\substack{1 \preceq j_1 \prec j_2 \preceq -1 \\ j_2 \neq -j_1}} (b(u, s_{j_1 j_2}) E_{j_1, j_1} \otimes E_{j_2, j_2} + \bar{b}(u) E_{j_2, j_2} \otimes E_{j_1, j_1}) \right. \\
 &\quad + \sum_{\substack{1 \preceq j_1 \prec j_2 \preceq -1 \\ j_2 \neq -j_1}} (c(u, s_{j_1 j_2}) E_{j_1, j_2} \otimes E_{j_2, j_1} + \bar{c}(u, s_{j_1 j_2}) E_{j_2, j_1} \otimes E_{j_1, j_2}) \\
 &\quad + \sum_{1 \preceq j_1 \prec j_2 \preceq -1} (d(u, s_{j_1}, s_{j_2}) E_{-j_2, j_1} \otimes E_{j_2, -j_1} + \bar{d}(u, s_{j_1}, s_{j_2}) E_{-j_1, j_2} \otimes E_{j_1, -j_2}) \\
 &\quad \left. + \sum_{j \in \{1, 2, \dots, -2, -1\}} e_j(u, s) E_{-j, j} \otimes E_{j, -j} \right\},
 \end{aligned}$$

where $s = P, P + h$, we set $s_{\pm j} \equiv \pm s_{\epsilon_j}$ for $1 \preceq j \preceq N$, $s_{ij} = s_i - s_j$, $s_0 = -\frac{1}{2}$, and

$$\begin{aligned}
 (3.3) \quad \rho^+(u) &= q^{-1} C(u, \xi)^{\frac{1}{2}} \rho_0^+(u), \\
 \rho_0^+(u) &= q^{-1} z^{\frac{1}{r}} \tilde{\rho}^+(u), \quad C(u, \xi) = \frac{\Theta_{\xi^2}(z)^2 \Theta_{\xi^2}(\xi q^2 z) \Theta_{\xi^2}(\xi q^{-2} z)}{\Theta_{\xi^2}(\xi z)^2 \Theta_{\xi^2}(q^2 z) \Theta_{\xi^2}(q^{-2} z)}, \\
 b(u, s) &= \frac{[s+1][s-1][u]}{[s]^2[u+1]}, \quad \bar{b}(u) = \frac{[u]}{[u+1]}, \\
 c(u, s) &= \frac{[1][s+u]}{[s][u+1]}, \quad \bar{c}(u, s) = \frac{[1][s-u]}{[s][u+1]},
 \end{aligned}$$

$$d(u, s_j, s_k) = G_{s_j} \frac{[u][1][s_j + s_k + 1 + \eta - u]}{[\eta - u][u+1][s_j + s_k + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m]}{[s_j - s_m + 1]} \prod_{m=1}^{k-1} \frac{[s_k - s_m + 1]}{[s_k - s_m]}$$

($j < k \preceq 0$)

$$d(u, s_{-k}, s_{-j}) = G_{s_{-j}} \frac{[u][1][s_{-j} + s_{-k} + 1 + \eta - u]}{[\eta - u][u+1][s_{-j} + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_{-j} + s_m]}{[s_{-j} + s_m + 1]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m + 1]}{[s_{-k} + s_m]}$$

($0 \preceq -k \prec -j$)

$$d(u, s_j, s_{-k}) = G_{s_j} G_{s_{-k}} \frac{[u][1][s_j + s_{-k} + 1 + \eta - u]}{[\eta - u][u+1][s_j + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m]}{[s_j - s_m + 1]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m]}{[s_{-k} + s_m + 1]}$$

($j \prec 0 \prec -k$),

$$\begin{aligned}
\bar{d}(u, s_j, s_k) &= G_{s_k} \frac{[u][1][s_j + s_k + 1 + \eta - u]}{[\eta - u][u + 1][s_j + s_k + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{m=1}^{k-1} \frac{[s_k - s_m]}{[s_k - s_m + 1]} \\
&\quad (j \prec k \preceq 0), \\
\bar{d}(u, s_{-k}, s_{-j}) &= G_{s_{-k}} \frac{[u][1][s_{-j} + s_{-k} + 1 + \eta - u]}{[\eta - u][u + 1][s_{-j} + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_{-j} + s_m + 1]}{[s_{-j} + s_m]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m]}{[s_{-k} + s_m + 1]} \\
&\quad (0 \preceq -k \prec -j), \\
\bar{d}(u, s_j, s_{-k}) &= \frac{[u][1][s_j + s_{-k} + 1 + \eta - u]}{[\eta - u][u + 1][s_j + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m + 1]}{[s_{-k} + s_m]} \\
&\quad (j \preceq 0 \preceq -k),
\end{aligned}$$

$$\begin{aligned}
e_j(u, s) &= \frac{[1][2s_j + 1 - u]}{[u + 1][2s_j + 1]} + \frac{[u][1][2s_j + 1 + \eta - u]}{[\eta - u][u + 1][2s_j + 1]} G_{s_j} \quad (j \neq 0), \\
e_0(u, s) &= \frac{[\eta + u][1][2\eta - u]}{[\eta - u][u + 1][2\eta]} - \frac{[u][1]}{[u + 1][2\eta]} H_s,
\end{aligned}$$

where for $k, -k = 0$ the product $\prod_{m=1}^{k-1}$ should be understood as $\prod_{m=1}^N$ etc. We also set

$$\begin{aligned}
G_{s_j} &= \frac{[s_j + 1]}{[s_j]} \prod_{\substack{k \in \mathcal{I} \\ \neq \pm j, 0}} \frac{[s_j - s_k + 1]}{[s_j - s_k]} = \frac{[s_j + 1]}{[s_j]} \prod_{\substack{m=1 \\ \neq |j|}}^N \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{\substack{m=1 \\ \neq |j|}}^N \frac{[s_j + s_m + 1]}{[s_j + s_m]}, \\
H_s &= \sum_{\substack{k \in \mathcal{I} \\ \neq 0}} \frac{[s_k + \frac{1}{2} + 2\eta]}{[s_k + \frac{1}{2}]} G_{s_k}.
\end{aligned}$$

Note that $G_{s_{j-1}} = G_{s_{-j}}^{-1}$, $G_{s_{-j-1}} = G_{s_j}^{-1}$, and

$$(3.4) \quad \rho^+(u)\rho^+(-u) = 1, \quad \rho^+(\eta - u) = \rho^+(u) \frac{[u][\eta - u + 1]}{[u + 1][\eta - u]}.$$

In later sections we also need the R matrix $R^{*+}(u, s) = R^+(u, s)|_{r \mapsto r^*}$.

The matrix $R^+(u, s)$ in (3.2) is related to Jimbo-Miwa-Okado's face type Botzmann weight [10] by the gauge transformation. For $j \in \mathcal{I}$, we set $\hat{j} = \epsilon_j$ for $1 \preceq j \preceq N$, $\hat{j} = -\epsilon_{|j|}$ for $-N \preceq j \preceq -1$ and $\hat{0} = 0$. Let us define $F(s, s + \hat{j}) = \left(\frac{G_{s_j}}{G_{s_j}(\hat{j})} \right)^{\frac{1}{2}}$ with

$$G_{s_j}(\hat{j}) = \begin{cases} \prod_{m=1}^{j-1} \frac{[s_j - s_m + 1]}{[s_j - s_m]} & \text{if } j \prec 0 \\ \frac{[s_j + 1]}{[s_j]} \prod_{\substack{m=1 \\ \neq |j|}}^N \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{m=|j|+1}^N \frac{[s_j + s_m + 1]}{[s_j + s_m]} & \text{if } 0 \prec j \end{cases}$$

For $a \in \tilde{\mathfrak{h}}^*$, $\rho = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_N$ and $a_j = \langle a + \rho, \hat{j} \rangle$, we identify a_j with s_j . Then we have

$$(3.5) \quad R^+(u, s)_{kl}^{ij} = W \left(\begin{array}{cc} a & a + \hat{i} \\ a + \hat{l} & a + \hat{i} + \hat{j} \end{array} \middle| u \right) \quad (i + j = k + l),$$

$$(3.5) \quad W \left(\begin{array}{cc} a & a + \hat{i} \\ a + \hat{l} & a + \hat{i} + \hat{j} \end{array} \middle| u \right) = \rho^+(u) \frac{[\eta][1]}{[\eta - u][u + 1]} \frac{F(a, a + \hat{i})F(a + \hat{i}, a + \hat{i} + \hat{j})}{F(a, a + \hat{l})F(a + \hat{l}, a + \hat{l} + \hat{k})}$$

$$\times W_{JMO} \left(\begin{array}{cc} a & a + \hat{i} \\ a + \hat{l} & a + \hat{i} + \hat{j} \end{array} \middle| u \right)$$

One can derive the following relations from (2.10)-(2.13b) in [10].

1) Crossing symmetry

$$(3.6) \quad W \left(\begin{array}{cc} a & b \\ d & c \end{array} \middle| u \right) = \frac{F(b, c)F(c, b)}{F(a, d)F(d, a)} \left(\frac{G_b G_d}{G_a G_c} \right)^{\frac{1}{2}} W \left(\begin{array}{cc} d & a \\ c & b \end{array} \middle| \eta - u \right).$$

2) Reflection symmetry

$$W \left(\begin{array}{cc} a & b \\ d & c \end{array} \middle| u \right) = \left(\frac{F(a, b)F(b, c)}{F(a, d)F(d, c)} \right)^2 W \left(\begin{array}{cc} a & d \\ b & c \end{array} \middle| u \right).$$

3) Unitarity

$$\sum_g W \left(\begin{array}{cc} a & g \\ d & c \end{array} \middle| u \right) W \left(\begin{array}{cc} a & b \\ g & c \end{array} \middle| -u \right) = \delta_{bd}.$$

4) 2nd inversion relation

$$\sum_g \left(\frac{G_a G_g}{G_b G_d} \right)^{\frac{1}{2}} W \left(\begin{array}{cc} a & b \\ d & g \end{array} \middle| \eta - u \right) \left(\frac{G_c G_g}{G_b G_d} \right)^{\frac{1}{2}} W \left(\begin{array}{cc} c & d \\ b & g \end{array} \middle| \eta + u \right) = \delta_{ac}.$$

Here

$$(3.7) \quad G_a = \varepsilon(a) \prod_{j=1}^N [a_j] \prod_{1 \leq i < j \leq N} [a_i - a_j][a_i + a_j]$$

and $\varepsilon(a)$ is a sign factor such that $\varepsilon(a + \hat{j})/\varepsilon(a) = 1$.

Remark. The choice of the gauge (3.5) and the resultant R -matrix (3.2) is convenient to discuss the RLL -relations in the next sections, because it allows the L -operator $\hat{L}^-(u)$ to be related to $\hat{L}^+(u)$ simply by $\hat{L}^-(u) = \hat{L}^+(u + r - \frac{c}{2})$, i.e. one needs no extra modifications follow from Proposition 4.3 in [12] (4.8). Note that p in [12] is p^*

in the present paper. See also [13]. One drawback is that one needs to introduce a set of extra generators and a central extension to the group algebra $\mathbb{C}[\mathcal{R}_Q]$ in order to remove constant gauge factors such as $q^{1/r}$ and q^{1/r^*} in a realization of the proper modified elliptic currents, which will be discussed in the next section. See Remark below Proposition 3.10 in [18]. However in order to avoid such unessential complications, we hereafter treat the whole formulas up to those constant gauge factors.

§ 3.2. Modified elliptic currents

Since our elliptic R -matrix is given by the theta functions (3.1) accompanied by the fractional power of z , we need to introduce the following modifications of the elliptic currents.

$$\begin{aligned} E_j(u) &= e_j(z) z^{-\frac{P_{\alpha_j}-1}{r^*}} \quad (1 \leq j \leq N-1), \\ E_N(u) &= e_N(z) z^{-\frac{P_{\alpha_N}-1/2}{r^*}}, \\ F_j(u) &= f_j(z) z^{\frac{P_{\alpha_j}+h_{\alpha_j}-1}{r}} \quad (1 \leq j \leq N-1), \\ F_N(u) &= f_N(z) z^{\frac{P_{\alpha_N}+h_{\alpha_N}-1/2}{r}}, \\ H_j^\pm(z) &= \psi_j^\pm(z) (K_j^\pm)^{-1} e^{-Q_{\alpha_j}} (q^{\pm(r-\frac{c}{2})} z)^{-\frac{r-r^*}{rr^*}(P_{\alpha_j}-1)+\frac{1}{r}h_{\alpha_j}}, \end{aligned}$$

and

$$\begin{aligned} K_{+j}^+(u) &= k_{+j}(q^j z) e^{-Q_{\epsilon_j}} (q^j z q^{-r})^{-\frac{r-r^*}{rr^*}P_{\epsilon_j}+\frac{1}{r}h_{\epsilon_j}}, \\ K_{-j}^+(u) &= k_{-j}(q^{-j} \xi z) e^{Q_{\epsilon_j}} (q^{-j} \xi z q^{-r})^{\frac{r-r^*}{rr^*}P_{\epsilon_j}-\frac{1}{r}h_{\epsilon_j}}, \\ K_0^+(u) &= k_0(\xi^{1/2} z), \\ K_{\pm j}^-(u) &= K_{\pm j}^+ \left(u + r - \frac{c}{2} \right). \end{aligned}$$

for $1 \leq j \leq N$. We also set

$$\tilde{d} = \hat{d} + \frac{1}{2r^*} \sum_{j=1}^N (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^N ((P + h)_j + 2) (P + h)^j.$$

Proposition 3.1. *For $1 \leq j \leq N-1$, we have*

$$\begin{aligned} H_j^\pm(u) &= : K_{+j}^\pm \left(u + \frac{c}{4} - \frac{j}{2} \right) K_{+(j+1)}^\pm \left(u + \frac{c}{4} - \frac{j}{2} \right)^{-1} :, \\ &= : K_{-j}^\pm \left(u + \frac{c}{4} + \frac{j}{2} + \eta \right)^{-1} K_{-(j+1)}^\pm \left(u + \frac{c}{4} + \frac{j}{2} + \eta \right) :, \\ \text{and} \quad H_N^\pm(u) &= : K_{+N}^\pm \left(u + \frac{c}{4} - \frac{N}{2} \right) K_0^\pm \left(u + \frac{c}{4} - \frac{N}{2} \right)^{-1} :, \\ &= : K_{-N}^\pm \left(u + \frac{c}{4} + \frac{N}{2} + \eta \right)^{-1} K_0^\pm \left(u + \frac{c}{4} + \frac{N}{2} + \eta \right) :. \end{aligned}$$

Then one can rewrite the formulas in Theorem 2.6 and Proposition 2.7 as follows.

Proposition 3.2. *The following commutation relations hold.*

$$\begin{aligned}
K_{\pm j}^+(u_1)K_{\pm j}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} K_{\pm j}^+(u_2)K_{\pm j}^+(u_1), \\
K_{+j}^+(u_1)K_{+l}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 - 1]^*[u_1 - u_2]}{[u_1 - u_2]^*[u_1 - u_2 - 1]} K_{+l}^+(u_2)K_{+j}^+(u_1) \quad (1 \leq j < l \leq 0), \\
K_{-j}^+(u_1)K_{-l}^+(u_2) &= \frac{\rho^+(u_1 - u_2)}{\rho^{+*}(u_1 - u_2)} \frac{[u_1 - u_2]^*[u_1 - u_2 - 1]}{[u_1 - u_2 - 1]^*[u_1 - u_2]} K_{-l}^+(u_2)K_{-j}^+(u_1) \quad (1 \leq j < l \leq 0), \\
K_{+j}^+(u_1)K_{-j}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 + \eta + j - 1]^*[u_1 - u_2 - 1]^*}{[u_1 - u_2 + \eta + j]^*[u_1 - u_2]^*} \\
&\quad \times \frac{[u_1 - u_2 + \eta + j][u_1 - u_2]}{[u_1 - u_2 + \eta + j - 1][u_1 - u_2 - 1]} K_{-l}^+(u_2)K_{+j}^+(u_1), \\
K_{+j}^+(u_1)K_{-l}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 - 1]^*[u_1 - u_2]}{[u_1 - u_2]^*[u_1 - u_2 - 1]} K_{-l}^+(u_2)K_{+j}^+(u_1) \quad (1 \leq j, l \leq N, j \neq l), \\
K_0^+(u_1)K_0^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 - 1]^*[u_1 - u_2 + \frac{1}{2}]^*}{[u_1 - u_2 + 1]^*[u_1 - u_2 - \frac{1}{2}]^*} \\
&\quad \times \frac{[u_1 - u_2 + 1][u_1 - u_2 - \frac{1}{2}]}{[u_1 - u_2 - 1][u_1 - u_2 + \frac{1}{2}]} K_0^+(u_2)K_0^+(u_1).
\end{aligned}$$

Proposition 3.3. *The following commutation relations hold.*

$$\begin{aligned}
K_{+j}^+(u_1)E_j(u_2) &= \frac{[u_1 - u_2 + \frac{j-c}{2}]^*}{[u_1 - u_2 + \frac{j-c}{2} - 1]^*} E_j(u_2)K_{+j}^+(u_1) \quad (1 \leq j \leq N), \\
K_{+j}^+(u_1)E_{j-1}(u_2) &= \frac{[u_1 - u_2 + \frac{j-1-c}{2}]^*}{[u_1 - u_2 + \frac{j-1-c}{2} + 1]^*} E_{j-1}(u_2)K_{+j}^+(u_1) \quad (2 \leq j \leq N), \\
K_j^+(u_1)E_l(u_2) &= E_l(u_2)K_j^+(u_1) \quad (l \neq j, j-1), \\
K_{-j}^+(u_1)E_j(u_2) &= \frac{[u_1 - u_2 - \frac{j+c}{2} - \eta]^*}{[u_1 - u_2 - \frac{j+c}{2} - \eta + 1]^*} E_j(u_2)K_{-j}^+(u_1) \quad (1 \leq j \leq N), \\
K_{-j}^+(u_1)E_{j-1}(u_2) &= \frac{[u_1 - u_2 - \frac{j-1+c}{2} - \eta]^*}{[u_1 - u_2 - \frac{j-1+c}{2} - \eta - 1]^*} E_{j-1}(u_2)K_{-j}^+(u_1) \quad (2 \leq j \leq N), \\
K_{-j}^+(u_1)E_l(u_2) &= E_l(u_2)K_{-j}^+(u_1) \quad (l \neq j, j-1), \\
K_{+j}^+(u_1)F_j(u_2) &= \frac{[u_1 - u_2 + \frac{j}{2} - 1]}{[u_1 - u_2 + \frac{j}{2}]} F_j(u_2)K_{+j}^+(u_1) \quad (1 \leq j \leq N), \\
K_{+j}^+(u_1)F_{j-1}(u_2) &= \frac{[u_1 - u_2 + \frac{j+1}{2}]}{[u_1 - u_2 + \frac{j+1}{2} - 1]} F_{j-1}(u_2)K_{+j}^+(u_1) \quad (2 \leq j \leq N), \\
K_j^+(u_1)F_l(u_2) &= F_l(u_2)K_j^+(u_1) \quad (l \neq j, j-1),
\end{aligned}$$

$$\begin{aligned}
 K_{-j}^+(u_1)F_j(u_2) &= \frac{[u_1 - u_2 - \frac{j}{2} - \eta + 1]}{[u_1 - u_2 - \frac{j}{2} - \eta]} F_j(u_2) K_{-j}^+(u_1) \quad (1 \leq j \leq N), \\
 K_{-j}^+(u_1)F_{j-1}(u_2) &= \frac{[u_1 - u_2 - \frac{j+1}{2} - \eta]}{[u_1 - u_2 - \frac{j+1}{2} - \eta + 1]} F_{j-1}(u_2) K_{-j}^+(u_1) \quad (2 \leq j \leq N), \\
 K_{-j}^+(u_1)F_l(u_2) &= F_l(u_2) K_{-j}^+(u_1) \quad (l \neq j, j-1),
 \end{aligned}$$

and

$$\begin{aligned}
 K_0^+(u_1)E_N(u_2) &= \frac{[u_1 - u_2 + \frac{N-c}{2}]^*}{[u_1 - u_2 + \frac{N-c}{2} - 1]^*} \frac{[u_1 - u_2 + \frac{N-c-1}{2}]^*}{[u_1 - u_2 + \frac{N-c+1}{2}]^*} E_N(u_2) K_0^+(u_1), \\
 K_0^+(u_1)E_j(u_2) &= E_j(u_2) K_0^+(u_1) \quad (j \neq N, 0), \\
 K_0^+(u_1)F_N(u_2) &= \frac{[u_1 - u_2 + \frac{N}{2} - 1]}{[u_1 - u_2 + \frac{N}{2}]} \frac{[u_1 - u_2 + \frac{N+1}{2}]}{[u_1 - u_2 + \frac{N-1}{2}]} F_N(u_2) K_0^+(u_1), \\
 K_0^+(u_1)F_j(u_2) &= F_j(u_2) K_0^+(u_1) \quad (j \neq N, 0).
 \end{aligned}$$

In addition, the defining relations (2.5)–(2.20) of $U_{q,p}(B_N^{(1)})$ can be rewritten as follows in the sense of analytic continuation.

Proposition 3.4. *We have the following relations.*

$$(3.8) \quad [h_i, \alpha_{j,n}] = 0, \quad [h_i, E_j(u)] = a_{ij} E_j(u), \quad [h_i, F_j(u)] = -a_{ij} F_j(u),$$

$$(3.9) \quad [\hat{d}, h_i] = 0, \quad [\hat{d}, \alpha_{i,n}] = n\alpha_{i,n},$$

$$(3.10) \quad [\hat{d}, E_i(u)] = \left(-z \frac{\partial}{\partial z} + \frac{1}{r^*}\right) E_i(u), \quad [\hat{d}, F_i(u)] = \left(-z \frac{\partial}{\partial z} + \frac{1}{r}\right) F_i(u),$$

$$(3.11) \quad \left[u - v - \frac{b_{ij}}{2}\right]^* E_i(u) E_j(v) = \left[u - v + \frac{b_{ij}}{2}\right]^* E_j(v) E_i(u),$$

$$(3.12) \quad \left[u - v + \frac{b_{ij}}{2}\right] F_i(u) F_j(v) = \left[u - v - \frac{b_{ij}}{2}\right] F_j(v) F_i(u),$$

$$(3.13) \quad [E_i(u), F_j(v)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c} \frac{z}{w}) H_i^-(q^{c/2} w) - \delta(q^c \frac{z}{w}) H_i^+(q^{-c/2} w) \right),$$

$$\begin{aligned}
 (3.14) \quad & \sum_{\sigma \in S_a} z_{\sigma(1)}^{-\frac{1}{r^*}} \prod_{1 \leq k < m \leq a} \frac{(p^* q^2 z_{\sigma(m)}/z_{\sigma(k)}; p^*)_{\infty}}{(p^* q^{-2} z_{\sigma(m)}/z_{\sigma(k)}; p^*)_{\infty}} \\
 & \times \sum_{s=0}^a (-)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{s+1 \leq m \leq a-1} \left(\frac{w}{z_{\sigma(m)}} \right)^{\frac{1}{r^*}} \prod_{a \leq m \leq s} \left(\frac{z_{\sigma(m)}}{w} \right)^{\frac{1}{r^*}} \\
 & \times \prod_{1 \leq m \leq s} \frac{(p^* q^{b_{ij}} w/z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-b_{ij}} w/z_{\sigma(m)}; p^*)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{b_{ij}} z_{\sigma(m)}/w; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_{\sigma(m)}/w; p^*)_{\infty}} \\
 & \times E_i(u_{\sigma(1)}) \cdots E_i(u_{\sigma(s)}) E_j(v) E_i(u_{\sigma(s+1)}) \cdots E_i(u_{\sigma(a)}) = 0 \quad (i \neq j, \ a = 1 - a_{ij}),
 \end{aligned}$$

$$\begin{aligned}
(3.15) \quad & \sum_{\sigma \in S_a} z_{\sigma(1)}^{\frac{1}{r}} \prod_{1 \leq k < m \leq a} \frac{(pq^{-2} z_{\sigma(m)}/z_{\sigma(k)}; p)_{\infty}}{(pq^2 z_{\sigma(m)}/z_{\sigma(k)}; p)_{\infty}} \\
& \times \sum_{s=0}^a (-)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{s+1 \leq m \leq a-1} \left(\frac{w}{z_{\sigma(m)}} \right)^{-\frac{1}{r}} \prod_{a \leq m \leq s} \left(\frac{z_{\sigma(m)}}{w} \right)^{-\frac{1}{r}} \\
& \times \prod_{1 \leq m \leq s} \frac{(pq^{-b_{ij}} w/z_{\sigma(m)}; p)_{\infty}}{(pq^{b_{ij}} w/z_{\sigma(m)}; p)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(pq^{-b_{ij}} z_{\sigma(m)}/w; p)_{\infty}}{(pq^{b_{ij}} z_{\sigma(m)}/w; p)_{\infty}} \\
& \times F_i(u_{\sigma(1)}) \cdots F_i(u_{\sigma(s)}) F_j(v) F_i(u_{\sigma(s+1)}) \cdots F_i(u_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}).
\end{aligned}$$

§ 3.3. The half currents and the L -operators

We next introduce the half currents $E_{i,j}^{\pm}(u), F_{j,i}^{\pm}(u)$ ($1 \preceq i \prec j \preceq -1$) and propose a construction of the L -operators of $U_{q,p}(B_N^{(1)})$.

Definition 3.5. For $1 \preceq j \preceq -2$, we define the basic half currents $E_{j+1,j}^+(u), F_{j,j+1}^+(u)$ as follows.

$$\begin{aligned}
F_{j,j+1}^+(u) &:= a_{j,j+1} \oint_C \frac{dz'_j}{2\pi i z'_j} F_j(u'_j) \frac{[u - u'_j + (P+h)_{j,j+1} + \frac{j}{2} - 1][1]}{[u - u'_j + \frac{j}{2}][(P+h)_{j,j+1} - 1]}, \\
E_{j+1,j}^+(u) &:= a_{j+1,j}^* \oint_{C^*} \frac{dz'_j}{2\pi i z'_j} E_j(u'_j) \frac{[u - u'_j + \frac{j-c}{2} + 1 - P_{j,j+1}]^*[1]^*}{[u - u'_j + \frac{j-c}{2}]^*[P_{j,j+1} - 1]^*},
\end{aligned}$$

for $1 \preceq j \preceq N$ and

$$\begin{aligned}
F_{-(j+1),-j}^+(u) &:= a_{-(j+1),-j} \oint_C \frac{dz'_j}{2\pi i z'_j} F_j(u'_j) \frac{[u - u'_j + (P+h)_{-(j+1),-j} - \frac{j}{2} - \eta - 1][1]}{[u - u'_j - \frac{j}{2} - \eta][(P+h)_{-(j+1),-j} - 1]}, \\
E_{-j,-(j+1)}^+(u) &:= a_{-j,-(j+1)}^* \oint_{C^*} \frac{dz'_j}{2\pi i z'_j} E_j(u'_j) \frac{[u - u'_j - \frac{j+c}{2} - \eta + 1 - P_{-(j+1),-j} - \delta_{j,N}]^*[1]^*}{[u - u'_j - \frac{j+c}{2} - \eta]^*[P_{-(j+1),-j} - 1 + \delta_{j,N}]^*},
\end{aligned}$$

for $-N \preceq -j \preceq -1$, where $N+1 \equiv 0 \equiv -N-1$. We also define

$$E_{j+1,j}^-(u) := E_{j+1,j}^+(u + r - \frac{c}{2}), \quad F_{j,j+1}^-(u) := F_{j,j+1}^+(u + r - \frac{c}{2}).$$

By using Propositions 3.2-3.4 one can derive the following relations.

Proposition 3.6. *For $1 \preceq j \preceq -2$, the following equations hold.*

$$\begin{aligned}
 K_{j+1}^+(u_1)^{-1} E_{j+1,j}^+(u_2) K_{j+1}^+(u_1) &= E_{j+1,j}^+(u_2) \frac{1}{\bar{b}^*(u)} - E_{j+1,j}^+(u_1) \frac{c^*(u, P_{j,j+1})}{\bar{b}^*(u)}, \\
 K_{j+1}^+(u_1) F_{j,j+1}^+(u_2) K_{j+1}^+(u_1)^{-1} &= \frac{1}{\bar{b}(u)} F_{j,j+1}^+(u_2) - \frac{\bar{c}(u, P_{j,j+1} + h_{j,j+1})}{\bar{b}(u)} F_{j,j+1}^+(u_1), \\
 \frac{[1-u]^*}{[u]^*} E_{j+1,j}^+(u_1) E_{j+1,j}^+(u_2) &+ \frac{[1+u]^*}{[u]^*} E_{j+1,j}^+(u_2) E_{j+1,j}^+(u_1) \\
 &= E_{j+1,j}^+(u_1)^2 \frac{[1]^* [P_{j,j+1} - 2 + u]^*}{[P_{j,j+1} - 2]^* [u]^*} + E_{j+1,j}^+(u_2)^2 \frac{[1]^* [P_{j,j+1} - 2 - u]^*}{[P_{j,j+1} - 2]^* [u]^*}, \\
 \frac{[1+u]}{[u]} F_{j,j+1}^+(u_1) F_{j,j+1}^+(u_2) &+ \frac{[1-u]}{[u]} F_{j,j+1}^+(u_2) F_{j,j+1}^+(u_1) \\
 &= F_{j,j+1}^+(u_1)^2 \frac{[1][P_{j,j+1} + h_{j,j+1} - 2 - u]}{[P_{j,j+1} + h_{j,j+1} - 2][u]} + F_{j,j+1}^+(u_2)^2 \frac{[1][P_{j,j+1} + h_{j,j+1} - 2 + u]}{[P_{j,j+1} + h_{j,j+1} - 2][u]}, \\
 [E_{j+1,j}^+(u_1), F_{j,j+1}^+(u_2)] & \\
 &= K_j^+(u_2) \frac{\bar{c}^*(u, P_{j,j+1})}{\bar{b}^*(u)} K_{j+1}^+(u_2)^{-1} - K_{j+1}^+(u_1)^{-1} \frac{\bar{c}(u, (P+h)_{j,j+1})}{\bar{b}(u)} K_j^+(u_1).
 \end{aligned}$$

Proposition 3.7. *For $1 \preceq j \preceq -2, j \neq N, 0$ the relations in Proposition 3.6 and those for $K_j^+(u), K_{j+1}^+(u)$ in Proposition 3.2 coincide with the following RLL-relation of the $U_{q,p}(\widehat{\mathfrak{gl}}_2)$ type.*

$$\begin{aligned}
 (3.16) \quad R_{j,j+1}^{+(12)}(u_1 - u_2, P + h) \widehat{L}_{j,j+1}^{+(1)}(u_1) \widehat{L}_{j,j+1}^{+(2)}(u_2) \\
 = \widehat{L}_{j,j+1}^{+(2)}(u_2) \widehat{L}_{j,j+1}^{+(1)}(u_1) R_{j,j+1}^{*+(12)}(u_1 - u_2, P - h^{(1)} - h^{(2)}),
 \end{aligned}$$

where

$$\begin{aligned}
 R_{j,j+1}^+(u, s) &= \rho^+(u) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u, s_{j,j+1}) & c(u, s_{j,j+1}) & 0 \\ 0 & \bar{c}(u, s_{j,j+1}) & \bar{b}(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
 \widehat{L}_{j,j+1}(u) &= \begin{pmatrix} 1 & F_{j,j+1}^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_j^+(u) & 0 \\ 0 & K_{j+1}^+(u) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E_{j+1,j}^+(u) & 1 \end{pmatrix}.
 \end{aligned}$$

Definition 3.8. By means of the basic half currents $E_{j+1,j}^\pm(u), F_{j,j+1}^\pm(u)$ ($1 \preceq j \preceq -2$), we define the other half currents $E_{i,j}^\pm(u), F_{j,i}^\pm(u)$ ($1 \preceq i \prec j \preceq -1, j = i + 1$) by requiring the following conditions.

1) The half currents $E_{j,i}^{\pm}(u)$ and $F_{i,j}^{\pm}(u)$ ($i \prec j$) have the following series expansions.

$$E_{j,i}^{+}(u) = \left(\sum_{n \in \mathbb{Z}_{\geq 0}} E_{j,i;-n}^{+} z^n + \sum_{n \in \mathbb{Z}_{> 0}} E_{j,i;n}^{+} p^n z^{-n} \right) z^{-\frac{P_{i,j}-1+\delta_{i,0}-\delta_{j,-i}}{r^{*}}},$$

$$F_{i,j}^{+}(u) = \left(\sum_{n \in \mathbb{Z}_{\geq 0}} F_{i,j;-n}^{+} z^n + \sum_{n \in \mathbb{Z}_{> 0}} F_{i,j;n}^{+} p^n z^{-n} \right) z^{\frac{(P+h)_{i,j}-1+\delta_{i,0}-\delta_{j,-i}}{r}},$$

where $E_{j,i;\pm n}^{+}, F_{i,j;\pm n}^{+} \in U_{q,p}(B_N^{(1)})$ and

(3.17)

$$E_{j,i}^{-}(u) := E_{j,i}^{+}(u + r - \frac{c}{2}), F_{i,j}^{-}(u) := F_{i,j}^{+}(u + r - \frac{c}{2}), K_j^{-}(u) = K_j^{+}(u + r - \frac{c}{2}).$$

2) Let us set

$$(3.18) \quad \widehat{L}^{+}(u) = \begin{pmatrix} 1 & F_{1,2}^{+}(u) & F_{1,3}^{+}(u) & \cdots & F_{1,-1}^{+}(u) \\ 0 & 1 & F_{2,3}^{+}(u) & \cdots & F_{2,-1}^{+}(u) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F_{-2,-1}^{+}(u) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} K_1^{+}(u) & 0 & \cdots & 0 \\ 0 & K_2^{+}(u) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_{-1}^{+}(u) \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E_{2,1}^{+}(u) & 1 & \ddots & & \vdots \\ E_{3,1}^{+}(u) & E_{3,2}^{+}(u) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E_{-1,1}^{+}(u) & E_{-1,2}^{+}(u) & \cdots & E_{-1,-2}^{+}(u) & 1 \end{pmatrix}.$$

Due to 1), the matrix products in (3.18) are well defined in the p -adic topology. Then $\widehat{L}^{+}(u)$ and $\widehat{L}^{-}(u) = \widehat{L}^{+}(u + r - c/2)$ satisfy the following RLL -relations.

$$(3.19) \quad R^{\pm(12)}(u, P + h) \widehat{L}^{\pm(1)}(u_1) \widehat{L}^{\pm(2)}(u_2) \\ = \widehat{L}^{\pm(2)}(u_2) \widehat{L}^{\pm(1)}(u_1) R^{*\pm(12)}(u, P - h^{(1)} - h^{(2)}),$$

$$(3.20) \quad R^{\pm(12)}(u \pm \frac{c}{2}, P + h) \widehat{L}^{\pm}(u_1) \widehat{L}^{\mp}(u_2) \\ = \widehat{L}^{\mp}(u_2) \widehat{L}^{\pm}(u_1) R^{*\pm(12)}(u \mp \frac{c}{2}, P - h^{(1)} - h^{(2)}).$$

Here

$$R^{-}(u, s) = R^{+}(u + r - \frac{c}{2}, s).$$

Note also that $r^{*} + \frac{c}{2} = r - \frac{c}{2}$.

Remark. Since $\rho^+(u)/\rho^{+*}(u) = \rho_0^+(u)/\rho_0^{+*}(u)$, the *RLL*-relations (3.16), (3.19), (3.20) remain unchanged even if one uses $\rho_0^+(u)$ and $\rho_0^{+*}(u)$ in (3.3) instead of $\rho^+(u)$ and $\rho^{+*}(u)$, respectively. See Sec.6

Conjecture 3.9. *The RLL-relation (3.19) and (3.20) determines the half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ ($1 \preceq i \prec j \preceq -1, j \neq i+1$) recursively and uniquely from the basic ones in Definition 3.5.*

In fact, the half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ with $1 \preceq i \prec j \prec 0$ or $0 \prec i \prec j \preceq -1$ are determined recursively by the basic ones in the same way as for $U_{q,p}(A_N^{(1)})$ case [2, 24, 25]. As for the other half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ with $1 \preceq i \preceq 0 \prec j \preceq -1$ or $1 \preceq i \prec 0 \preceq j \preceq -1$, we have observed that the combinations $E_{j,i}^+(u+c/4) - E_{j,i}^-(u-c/4)$ and $F_{i,j}^+(u-c/4) - F_{i,j}^-(u+c/4)$ satisfy a system of linear equations with the operator valued coefficients given by the *total* elliptic currents respectively. In addition, the half currents $E_{-1,1}^\pm(u)$ (reps. $F_{1,-1}^\pm(u)$) is determined by all the other half currents $E_{j,i}^\pm(u)$ (resp. $F_{i,j}^\pm(u)$) $i \prec j$. An explicit expression for the half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ ($1 \preceq i \prec j \prec 0$ or $0 \prec i \prec j \preceq -1$) and conjectural expressions for the other half currents are given in Appendix B.

The existence of the operator $\widehat{L}^+(u)$ satisfying (3.19)-(3.20), and hence the existence and the uniqueness of the half currents, can also be seen in the following argument. Consider the elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ realized by the Chevalley generators equipped with the quasi-Hopf algebra structure[12]. See Appendix A. Note that $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$ is isomorphic to the Drinfeld-Jimbo's quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ [3, 9] as an associative algebra. In addition, we have shown the isomorphism[6]

$$U_{q,p}(\widehat{\mathfrak{g}})/pU_{q,p}(\widehat{\mathfrak{g}}) \cong (\mathcal{M}_{H^*} \otimes_{\mathbb{C}} U_q(\widehat{\mathfrak{g}}))\sharp\mathbb{C}[\mathcal{R}_Q]$$

where $U_q(\widehat{\mathfrak{g}})$ is the quantum affine algebra in the Drinfeld realization. Furthermore in [13] Appendix A, we have obtained a realization of $U_{q,p}(\widehat{\mathfrak{g}})$ in terms of the Drinfeld generators in $U_q(\widehat{\mathfrak{g}})$ and a Heisenberg algebra $\mathbb{C}[P_{\alpha_j}, e^{Q_{\alpha_j}} (j \in I)]^1$. Note that such realization is well-defined in the p -adic topology. Hence applying the isomorphism between the Drinfeld-Jimbo realization of $U_q(\widehat{\mathfrak{g}})$ in terms of the Chevalley generators and the Drinfeld realization of the same algebra in terms of the Drinfeld generators, one can expect the isomorphism

$$(3.21) \quad U_{q,p}(\widehat{\mathfrak{g}}) \cong \mathcal{B}_{q,\lambda(r^*,P)}(\widehat{\mathfrak{g}})\sharp\mathbb{C}[\mathcal{R}_Q]$$

as an associative algebra. Here $\lambda(r^*, P)$ is given in Appendix A. In fact one can derive the same *RLL*-relations as (3.19)-(3.20) by using the universal *R*-matrix of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$.

¹The Heisenberg generators $P_{\alpha_j}, Q_{\alpha_j}$ are related to P_j, Q_j in [13] by $P_{\alpha_j} = d_j P_j, Q_{\alpha_j} = -2Q_j$, respectively.

There the $\mathcal{L}^\pm(u)$ operators are the elements in $\text{End}_{\mathbb{C}} V \otimes \mathcal{B}_{q, \lambda(r^*, P)}(\widehat{\mathfrak{g}}) \# \mathbb{C}[\mathcal{R}_Q]$ and satisfy $\mathcal{L}^-(u) = \mathcal{L}^+(u + r^* + c/2)$. Then by assuming the Gauss decomposition such as (3.18) in $\mathcal{L}^\pm(u)$ and denoting their Gauss coordinates by $\mathcal{E}_{j,i}^+(u), \mathcal{F}_{i,j}^+(u), \mathcal{K}_j^+(u)$ one can show that for $1 \leq j \leq N-1$, the

$$(3.22) \quad \mathcal{E}_j(u) := -\frac{1}{a_{j+1,j}^*[1]^*} \left(\mathcal{E}_{j+1,j}^+(u - \frac{j}{2} + \frac{c}{2}) - \mathcal{E}_{j+1,j}^-(u - \frac{j}{2}) \right),$$

$$(3.23) \quad \mathcal{F}_j(u) := \frac{1}{a_{j,j+1}[1]} \left(\mathcal{F}_{j+1,j}^+(u - \frac{j}{2}) - \mathcal{F}_{j+1,j}^-(u - \frac{j}{2} + \frac{c}{2}) \right)$$

with

$$\frac{a_{j+1,j}^* a_{j,j+1}[1]}{q - q^{-1}} = 1$$

satisfy the same relations as the elliptic currents $E_j(u)$ and $F_j(u)$ in Proposition 3.4 [2, 24, 25]. Note that the formulas in Definition 3.5 gives a solution to (3.22)-(3.23). However we have not yet succeeded to confirm similar formulas for the $j = N$ case due to a difficulty of extracting the relations for these half currents from the *RLL*-relations.

§ 4. Hopf Algebroid Structure

In this section, we introduce an *H*-Hopf algebroid structure into the elliptic algebra $U_{q,p}(B_N^{(1)})$ and formulate it as an elliptic quantum group.

§ 4.1. Definition of the *H*-Hopf algebroid

Let us recall some basic facts on the *H*-Hopf algebroid following the works of Etingof and Varchenko[5] and of Koelink and Rosengren [17].

Definition 4.1 (*H*-bialgebroid). An *H*-bialgebroid is an *H*-algebra A equipped with two *H*-algebra homomorphisms $\Delta : A \rightarrow A \widetilde{\otimes} A$ (the comultiplication) and $\varepsilon : A \rightarrow \mathcal{D}$ (the counit) such that

$$\begin{aligned} (\Delta \widetilde{\otimes} \text{id}) \circ \Delta &= (\text{id} \widetilde{\otimes} \Delta) \circ \Delta, \\ (\varepsilon \widetilde{\otimes} \text{id}) \circ \Delta &= \text{id} = (\text{id} \widetilde{\otimes} \varepsilon) \circ \Delta, \end{aligned}$$

under the identification (2.40).

Definition 4.2 (*H*-Hopf algebroid). An *H*-Hopf algebroid is an *H*-bialgebroid A equipped with a $\mathbb{C}$ -linear map $S : A \rightarrow A$ (the antipode), such that

$$\begin{aligned} S(\mu_r(\widehat{f})a) &= S(a)\mu_l(\widehat{f}), \quad S(a\mu_l(\widehat{f})) = \mu_r(\widehat{f})S(a), \quad \forall a \in A, \widehat{f} \in \mathcal{M}_{\mathfrak{h}^*}, \\ m \circ (\text{id} \widetilde{\otimes} S) \circ \Delta(a) &= \mu_l(\varepsilon(a)1), \quad \forall a \in A, \\ m \circ (S \widetilde{\otimes} \text{id}) \circ \Delta(a) &= \mu_r(T_\alpha(\varepsilon(a)1)), \quad \forall a \in A_{\alpha\beta}, \end{aligned}$$

where $m : A \tilde{\otimes} A \rightarrow A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in \mathcal{M}_{H^*}$.

The H -algebra $\mathcal{D}$ is an H -Hopf algebroid with $\Delta_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D} \tilde{\otimes} \mathcal{D}$, $\varepsilon_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$, $S_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$ defined by

$$\begin{aligned} \Delta_{\mathcal{D}}(\hat{f}T_{-\alpha}) &= \hat{f}T_{-\alpha} \tilde{\otimes} T_{-\alpha}, \\ \varepsilon_{\mathcal{D}} &= \text{id}, \quad S_{\mathcal{D}}(\hat{f}T_{-\alpha}) = T_{\alpha}\hat{f} = (T_{\alpha}\hat{f})T_{\alpha}. \end{aligned}$$

§ 4.2. The H -Hopf algebroid $U_{q,p}(B_N^{(1)})$

Now let us consider the H -Hopf algebroid structure on $U = U_{q,p}(B_N^{(1)})$. Let us consider the generating function of the L -operator matrix elements $L_{i,j}^+(u)$. We define two H -algebra homomorphisms, the co-unit $\varepsilon : U \rightarrow \mathcal{D}$ and the co-multiplication $\Delta : U \rightarrow U \tilde{\otimes} U$ by

$$(4.1) \quad \varepsilon(L_{i,j}^+(u)) = \delta_{i,j} T_{Q_{\epsilon_i}} \quad (n \in \mathbb{Z}), \quad \varepsilon(e^Q) = e^Q,$$

$$(4.2) \quad \varepsilon(\mu_l(\hat{f})) = \varepsilon(\mu_r(\hat{f})) = \hat{f}T_0,$$

$$(4.3) \quad \Delta(L_{i,j}^+(u)) = \sum_k L_{i,k}^+(u) \tilde{\otimes} L_{k,j}^+(z),$$

$$(4.4) \quad \Delta(e^Q) = e^Q \tilde{\otimes} e^Q,$$

$$(4.5) \quad \Delta(\mu_l(\hat{f})) = \mu_l(\hat{f}) \tilde{\otimes} 1, \quad \Delta(\mu_r(\hat{f})) = 1 \tilde{\otimes} \mu_r(\hat{f}).$$

In fact, one can check that Δ preserves the RLL -relations (3.19)-(3.20).

Lemma 4.3. *The maps ε and Δ satisfy*

$$(4.6) \quad (\Delta \tilde{\otimes} \text{id}) \circ \Delta = (\text{id} \tilde{\otimes} \Delta) \circ \Delta,$$

$$(4.7) \quad (\varepsilon \tilde{\otimes} \text{id}) \circ \Delta = \text{id} = (\text{id} \tilde{\otimes} \varepsilon) \circ \Delta.$$

Combining this with the H -algebra structure, the set $(U, \Delta, \mathcal{M}_{H^*}, \mu_l, \mu_r, \varepsilon)$ is an H -bialgebroid.

From (4.3), one can derive the following coproduct formulas for the basic half currents.

Proposition 4.4. *For $1 \preceq j \preceq -1$, we obtain*

$$\begin{aligned} \Delta(K_j^+(u)) &= K_{j+1}^+(u) \tilde{\otimes} K_{j+1}^+(u) \\ &\times \left(1 + 1 \tilde{\otimes} l_{j+1,j+1}^{+'}(u) + l_{j+1,j+1}^{+'}(u) \tilde{\otimes} 1 + l_{j+1,j+1}^{+'}(u) \tilde{\otimes} l_{j+1,j+1}^{+'}(u) \right) \left(1 + \Delta(l_{j+1,j+1}^{+'}(u)) \right)^{-1} \\ &= \left(1 + \Delta(l_{j+1,j+1}^{+''}(u)) \right)^{-1} \left(1 + 1 \tilde{\otimes} l_{j+1,j+1}^{+''}(u) + l_{j+1,j+1}^{+''}(u) \tilde{\otimes} 1 + l_{j+1,j+1}^{+''}(u) \tilde{\otimes} l_{j+1,j+1}^{+''}(u) \right) \\ &\quad \times K_{j+1}^+(u) \tilde{\otimes} K_{j+1}^+(u), \end{aligned}$$

for $1 \preceq j \preceq -2$, we obtain

$$\begin{aligned}
\Delta(E_{j+1,j}^+(u)) &= \left(1 + \Delta(l_{j+1,j+1}^{+'}(u))\right) \\
&\quad \times \left(1 + 1 \widetilde{\otimes} l_{j+1,j+1}^{+'}(u) + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} 1 + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} l_{j+1,j+1}^{+'}(u)\right)^{-1} \\
&\quad \times \left(1 \widetilde{\otimes} E_{j+1,j}^+(u) + E_{j+1,j}^+(u) \widetilde{\otimes} K_{j+1}^+(u)^{-1} K_j^+(u) + E_{j+1,j}^+(u) \widetilde{\otimes} l_{j,j}^{+'}(u) \right. \\
&\quad \left. + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} E_{j+1,j}^+(u) + 1 \widetilde{\otimes} l_{j+1,j}^{+'}(u) + l_{j+1,j}^{+'}(u) \widetilde{\otimes} K_{j+1}^+(u)^{-1} K_j^+(u) \right. \\
&\quad \left. + l_{j+1,j}^{+'}(u) \widetilde{\otimes} l_{j,j}^{+'}(u) + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} l_{j+1,j}^{+'}(u) + \sum_{\substack{1 \preceq k \preceq -1 \\ \neq j, j+1}} \widehat{L}_{j+1,k}^{+'}(u) \widetilde{\otimes} \widehat{L}_{k,j}^{+'}(u)\right) \\
&\quad - \Delta(l_{j+1,j}^{+'}(u)), \\
\Delta(F_{j,j+1}^+(u)) &= \left(K_j^+(u) K_{j+1}^+(u)^{-1} \widetilde{\otimes} F_{j,j+1}^+(u) + F_{j,j+1}^+(u) \widetilde{\otimes} 1 + F_{j,j+1}^+(u) \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) \right. \\
&\quad \left. + l_{j,j}^{+''}(u) \widetilde{\otimes} F_{j,j+1}^+(u) + l_{j,j+1}^{+''}(u) \widetilde{\otimes} 1 + K_j^+(u) K_{j+1}^+(u)^{-1} \widetilde{\otimes} l_{j,j+1}^{+''}(u) \right. \\
&\quad \left. + l_{j,j+1}^{+''}(u) \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) + l_{j,j}^{+''}(u) \widetilde{\otimes} l_{j,j+1}^{+''}(u) + \sum_{\substack{1 \preceq k \preceq -1 \\ \neq j, j+1}} \widehat{L}_{j,k}^{+''}(u) \widetilde{\otimes} \widehat{L}_{k,j+1}^{+''}(u)\right) \\
&\quad \times \left(1 + 1 \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) + l_{j+1,j+1}^{+''}(u) \widetilde{\otimes} 1 + l_{j+1,j+1}^{+''}(u) \widetilde{\otimes} l_{j+1,j+1}^{+''}(u)\right)^{-1} \\
&\quad \times \left(1 + \Delta(l_{j+1,j+1}^{+''}(u))\right) - \Delta(l_{j,j+1}^{+''}(u)).
\end{aligned}$$

Here $l_{k,l}^{+'}(u) = K_{j+1}^+(u)^{-1} l_{k,l}^+(u)$, $l_{k,l}^{+''}(u) = l_{k,l}^+(u) K_{j+1}^+(u)^{-1}$ and

$$\begin{aligned}
l_{j,j}^+(u) &= \sum_{j \prec k} F_{j,k}^+(u) K_k^+(u) E_{k,j}^+(u), \\
l_{i,j}^+(u) &= \sum_{j \prec k} F_{i,k}^+(u) K_k^+(u) E_{k,j}^+(u), \quad (i \prec j) \\
l_{j,i}^+(u) &= \sum_{j \prec k} F_{j,k}^+(u) K_k^+(u) E_{k,i}^+(u).
\end{aligned}$$

Now let us define formally an algebra antihomomorphism (the antipode) $S : U \rightarrow U$ by

$$\begin{aligned}
S(L^+(z)) &= L^+(z)^{-1}, \\
S(e^Q) &= e^{-Q}, \quad S(\mu_r(\widehat{f})) = \mu_l(\widehat{f}), \quad S(\mu_l(\widehat{f})) = \mu_r(\widehat{f}).
\end{aligned}$$

Then we have the following Lemma.

Lemma 4.5. *The map S satisfies the antipode axioms*

$$\begin{aligned} m \circ (\text{id} \otimes S) \circ \Delta(x) &= \mu_l(\varepsilon(x)1), \quad \forall x \in U, \\ m \circ (S \otimes \text{id}) \circ \Delta(x) &= \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in (U)_{\alpha\beta}. \end{aligned}$$

From Lemmas 4.3 and 4.5, we have

Theorem 4.6. *The H -algebra $U_{q,p}(B_N^{(1)})$ equipped with (Δ, ε, S) is an H -Hopf algebroid.*

Definition 4.7. We call the H -Hopf algebroid $(U_{q,p}(B_N^{(1)}), H, \mathcal{M}_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S)$ the elliptic quantum group $U_{q,p}(B_N^{(1)})$.

§ 5. Representations

§ 5.1. Dynamical representations

Let us consider a vector space $\widehat{V}$ over $\mathbb{F} = \mathcal{M}_{H^*}$, which is H -diagonalizable, i.e.

$$\begin{aligned} \widehat{V} &= \bigoplus_{\lambda, \mu \in H^*} \widehat{V}_{\lambda, \mu}, \\ \widehat{V}_{\lambda, \mu} &= \{v \in \widehat{V} \mid q^{P+h} \cdot v = q^{<\lambda, P+h>} v, \quad q^P \cdot v = q^{<\mu, P>} v \quad \forall P+h, P \in H\}. \end{aligned}$$

Let us define the H -algebra $\mathcal{D}_{H, \widehat{V}}$ of the $\mathbb{C}$ -linear operators on $\widehat{V}$ by

$$\begin{aligned} \mathcal{D}_{H, \widehat{V}} &= \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H, \widehat{V}})_{\alpha, \beta}, \\ (\mathcal{D}_{H, \widehat{V}})_{\alpha, \beta} &= \left\{ X \in \text{End}_{\mathbb{C}} \widehat{V} \left| \begin{array}{l} f(P+h)X = Xf(P+h+<\alpha, P+h>), \\ f(P)X = Xf(P+<\beta, P>) \\ f(P), f(P+h) \in \mathbb{F}, \quad X \cdot \widehat{V}_{\lambda, \mu} \subseteq \widehat{V}_{\lambda+\alpha, \mu+\beta} \end{array} \right. \right\}, \\ \mu_l^{\mathcal{D}_{H, \widehat{V}}}(\widehat{f})v &= f(<\lambda, P+h>, p)v, \quad \mu_r^{\mathcal{D}_{H, \widehat{V}}}(\widehat{f})v = f(<\mu, P>, p^*)v, \quad \widehat{f} \in \mathbb{F}, \quad v \in \widehat{V}_{\lambda, \mu}. \end{aligned}$$

Definition 5.1. [5, 17, 23] We define a dynamical representation of $U_{q,p}(B_N^{(1)})$ on $\widehat{V}$ to be an H -algebra homomorphism $\pi : U_{q,p}(B_N^{(1)}) \rightarrow \mathcal{D}_{H, \widehat{V}}$. By the action of $U_{q,p}(B_N^{(1)})$ we regard $\widehat{V}$ as a $U_{q,p}(B_N^{(1)})$ -module.

Definition 5.2. For $k \in \mathbb{C}$, we say that a $U_{q,p}(B_N^{(1)})$ -module has level k if c act as the scalar k on it.

Definition 5.3. Let $\mathcal{H}, \mathcal{N}_+, \mathcal{N}_-$ be the subalgebras of $U_{q,p}(B_N^{(1)})$ generated by $c, d, K_i^\pm$ ($i \in I$), by $\alpha_{i,n}^\vee$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$) $f_{i,n}$ ($i \in I, n \in \mathbb{Z}_{>0}$) and by $\alpha_{i,-n}^\vee$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $f_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$), respectively.

Definition 5.4. For $k \in \mathbb{C}$, $\lambda \in \mathfrak{h}^*$ and $\mu \in H^*$, a (dynamical) $U_{q,p}(B_N^{(1)})$ -module $\widehat{V}(\lambda, \mu)$ is called the level- k highest weight module with the highest weight (λ, μ) , if there exists a vector $v \in \widehat{V}(\lambda, \mu)$ such that

$$\begin{aligned} \widehat{V}(\lambda, \mu) &= U_{q,p}(B_N^{(1)}) \cdot v, \quad \mathcal{N}_+ \cdot v = 0, \\ c \cdot v &= kv, \quad f(P) \cdot v = f(< \mu, P >)v, \quad f(P+h) \cdot v = f(< \lambda, P+h >)v. \end{aligned}$$

§ 5.2. Finite dimensional dynamical representation

We here give an elliptic and dynamical analogue of the evaluation representation associated with the vector representation of $B_N^{(1)}$. Let us consider $\widehat{V} = \bigoplus_{1 \preceq m \preceq -1} \mathbb{F}v_m \otimes 1$ and $\widehat{V}_z = \widehat{V}[z, z^{-1}]$. Here $e^{Q_\alpha} \in \mathbb{C}[\mathcal{R}_Q]$ acts on $f(P_\beta)v \otimes 1$ by $e^{Q_\alpha}(f(P_\beta)v \otimes 1) = f(P_\beta - (\alpha, \beta))v \otimes 1$.

Theorem 5.5. Let $E_{j,k}$ ($1 \preceq j, k \preceq -1$) denote the matrix units such that $E_{j,k}v_l = \delta_{k,l}v_j$. The following gives the $(2N+1)$ -dimensional dynamical representation of $U_{q,p}(B_N^{(1)})$ on $\widehat{V}_z$.

$$\begin{aligned} \pi_z(e_j(w)) &= \left(\frac{(pq^2; p)_\infty}{(p; p)_\infty} E_{j,j+1} \delta\left(q^j \frac{z}{w}\right) + \frac{(pq^2; p)_\infty}{(p; p)_\infty} E_{-j-1, -j} \delta\left(q^{-j} \xi \frac{z}{w}\right) \right) e^{-Q_{\alpha_j}}, \\ \pi_z(f_j(w)) &= \frac{(pq^{-2}; p)_\infty}{(p; p)_\infty} E_{j+1, j} \delta\left(q^j \frac{z}{w}\right) + \frac{(pq^{-2}; p)_\infty}{(p; p)_\infty} E_{-j, -j-1} \delta\left(q^{-j} \xi \frac{z}{w}\right), \\ \pi_z(\psi_j^-(w, p)) &= q^{\pi(h_j)} e^{-Q_{\alpha_j}} \frac{\Theta_p(q^{j-2h_j^+} \frac{z}{w})}{\Theta_p(q^j \frac{z}{w})} \frac{\Theta_p(q^{-j-2h_j^-} \xi \frac{z}{w})}{\Theta_p(q^{-j} \xi \frac{z}{w})}, \\ \pi_z(\psi_j^+(w, p)) &= q^{-\pi(h_j)} e^{-Q_{\alpha_j}} \frac{\Theta_p(q^{-j+2h_j^+} \frac{w}{z})}{\Theta_p(q^{-j} \frac{w}{z})} \frac{\Theta_p(q^{j+2h_j^-} \xi^{-1} \frac{w}{z})}{\Theta_p(q^j \xi^{-1} \frac{w}{z})}, \quad (1 \leq j \leq N-1) \\ \pi_z(e_N(w)) &= \frac{(pq^2; p)_\infty}{(p; p)_\infty} [2]_N E_{N,0} \delta\left(q^N \frac{z}{w}\right) + \frac{(pq; p)_\infty (pq^{-2}; p)_\infty}{(p; p)_\infty (pq^{-1}; p)_\infty} E_{0,-N} \delta\left(q^{N-1} \frac{z}{w}\right), \\ \pi_z(f_N(w)) &= \frac{(pq^2; p)_\infty (pq^{-1}; p)_\infty}{(pq; p)_\infty (p; p)_\infty} [2]_N E_{-N,0} \delta\left(q^{N-1} \frac{z}{w}\right) + \frac{(pq^{-2}; p)_\infty}{(p; p)_\infty} E_{0,N} \delta\left(q^N \frac{z}{w}\right), \\ \pi_z(\psi_N^-(w)) &= q_N^{\pi(h_N)} e^{-Q_{\alpha_N}} \frac{\Theta_p(q^{N-\frac{1}{2}-\frac{3}{2}h_N^+} \frac{z}{w})}{\Theta_p(q^{N-\frac{1}{2}+\frac{1}{2}h_N^+} \frac{z}{w})} \frac{\Theta_p(q^{N-\frac{1}{2}-\frac{3}{2}h_N^-} \xi \frac{z}{w})}{\Theta_p(q^{N-\frac{1}{2}+\frac{1}{2}h_N^-} \xi \frac{z}{w})}, \\ \pi_z(\psi_N^+(w)) &= q_N^{-\pi(h_N)} e^{-Q_{\alpha_N}} \frac{\Theta_p(q^{-(N-\frac{1}{2})+\frac{3}{2}h_N^+} \frac{w}{z})}{\Theta_p(q^{-(N-\frac{1}{2})-\frac{1}{2}h_N^+} \frac{w}{z})} \frac{\Theta_p(q^{-(N-\frac{1}{2})+\frac{3}{2}h_N^-} \xi \frac{w}{z})}{\Theta_p(q^{-(N-\frac{1}{2})-\frac{1}{2}h_N^-} \xi \frac{w}{z})}. \end{aligned}$$

Here $\pi(h_j) = h_j^+ + h_j^-$, $h_j^+ = E_{j,j} - E_{j+1,j+1}$, $h_j^- = E_{-j-1, -j-1} - E_{-j, -j}$, $\pi(h_N) = 2(E_{N,N} - E_{-N, -N})$, $h_N^+ = E_{N,N} - E_{0,0}$, $h_N^- = E_{0,0} - E_{-N, -N}$.

Theorem 5.6. *In terms of the half currents the dynamical representation $(\pi_z, \widehat{V}_z)$ is given as follows.*

$$\begin{aligned}
& \pi_z(K_{+j}^+(v)) \\
&= \rho^+(v-u) \left\{ \frac{[v-u]}{[v-u+1]} \sum_{1 \leq k \leq j-1} E_{k,k} + E_{j,j} + \frac{[v-u-1]}{[v-u]} \sum_{j+1 \leq k \leq -j-1} E_{k,k} \right. \\
&\quad \left. + \frac{[v-u-1][v-u+j+\eta-1]}{[v-u][v-u+j+\eta]} E_{-j,-j} \right\} e^{-Q_{\varepsilon_j}}, \\
& \pi_z(K_{-j}^+(v)) \\
&= \rho^+(v-u) \left\{ \frac{[v-u]}{[v-u+1]} \sum_{1 \leq k \neq j \leq -(j+1)} E_{k,k} + \frac{[v-u][v-u-j-\eta]}{[v-u+1][v-u-j+\frac{2N-1}{2}+1]} E_{j,j} \right. \\
&\quad \left. + \frac{[v-u-1]}{[v-u]} \sum_{-(j-1) \leq k \leq -1} E_{k,k} + E_{-j,-j} \right\} e^{Q_{\varepsilon_j}}, \\
& \pi_z(K_0^+(v)) = \rho^+(v-u) \\
&\quad \times \left\{ \frac{[v-u]}{[v-u+1]} \sum_{1 \leq k \leq N} E_{k,k} + \frac{[v-u-1]}{[v-u]} \sum_{1 \leq k \leq N} E_{-k,-k} + \frac{[v-u+\frac{1}{2}][v-u-1]}{[v-u-\frac{1}{2}][v-u+1]} E_{0,0} \right\}.
\end{aligned}$$

For $1 \leq j \prec l \leq N+1 \equiv 0$,

$$\begin{aligned}
& \pi_z(E_{l,j}^+(v)) = e^{Q_{\varepsilon_l}} \left\{ -E_{j,l} \frac{[v-u-P_{j,l}][1]}{[v-u][P_{j,l}]} \right. \\
&\quad \left. + E_{-l,-j} \frac{[v-u+l-1+\eta-P_{j,l}][1]}{[v-u+l-1+\eta][P_{j,l}]} \prod_{m=j+1}^{l-1} \frac{[P_{j,m}+1]}{[P_{j,m}]} \right\} e^{-Q_{\varepsilon_j}}, \\
& \pi_z(F_{j,l}^+(v)) = E_{1,j} \frac{[v-u+P_{j,l}][1]}{[v-u][P_{j,l}]} - E_{-j,-l} \frac{[v-u+l-1+\eta+P_{j,l}][1]}{[v-u+l-1+\eta][P_{j,l}]} \prod_{m=j+1}^{l-1} \frac{[P_{j,m}-1]}{[P_{j,m}]}, \\
& \pi_z(E_{-j,-l}^+(v)) = e^{-Q_{\varepsilon_j}} \left\{ -E_{-l,-j} \frac{[v-u-P_{-l,-j}][1]}{[v-u][P_{-l,-j}]} \right. \\
&\quad \left. + E_{j,l} \frac{[v-u-j-\eta-P_{-l,-j}][1]}{[v-u-j-\eta][P_{-l,-j}]} \prod_{m=j+1}^{l-1} \frac{[P_{-l,-m}+1]}{[P_{-l,-m}]} \right\} e^{Q_{\varepsilon_l}}, \\
& \pi_z(F_{-l,-j}^+(v)) = E_{-j,-l} \frac{[v-u+P_{-l,-j}][1]}{[v-u][P_{-l,-j}]} \\
&\quad - E_{l,j} \frac{[v-u-j-\eta+P_{-l,-j}+\delta_{l,N+1}][1]}{[v-u-j-\eta][P_{-l,-j}+\delta_{l,N+1}]} \prod_{m=j+1}^{l-1} \frac{[P_{-l,-m}-1+\delta_{l,N+1}]}{[P_{-l,-m}+\delta_{l,N+1}]}
\end{aligned}$$

i) $j \prec k \prec N$,

$$\begin{aligned}\pi_z(E_{-k,j}^+(v)) &= e^{-Q_{\varepsilon_k}} \left\{ -E_{j,-k} \frac{[v-u-P_{j,-k}][1]}{[v-u][P_{j,-k}]} \right. \\ &\quad \left. + E_{k,-j} \frac{[v-u-k-\eta-P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j+1]}{[P_j]} \prod_{m=k+1}^N \frac{[P_{j,-m}+1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}+1]}{[P_{j,m}]} \right\} e^{-Q_{\varepsilon_j}}, \\ \pi_z(F_{j,-k}^+(v)) &= E_{-k,j} \frac{[v-u+P_{j,-k}][1]}{[v-u][P_{j,-k}]} \\ &\quad - E_{-j,k} \frac{[v-u-k-\eta+P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j-1]}{[P_j]} \prod_{m=k+1}^N \frac{[P_{j,-m}-1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}-1]}{[P_{j,m}]}.\end{aligned}$$

ii) $j = k \prec N$,

$$\begin{aligned}\pi_z(E_{-j,j}^+(v)) &= e^{-Q_{\varepsilon_j}} E_{j,-j} \left\{ -\frac{[v-u-2P_j-1][1][v-u-j-\eta+1]}{[v-u][2P_j+1][v-u-j-\eta]} \right. \\ &\quad \left. + G_{P_j} \frac{[v-u-2P_j-j-\lambda][1]}{[v-u-j-\lambda][2P_j+1]} \prod_{m=1}^{j-1} \left(\frac{[P_{j,-m}]}{[P_{j,-m}+1]} \frac{[P_{j,m}]}{[P_{j,m}+1]} \right) \right\} e^{-Q_{\varepsilon_j}}, \\ \pi_z(F_{j,-j}^+(v)) &= E_{-j,j} \left\{ \frac{[v-u+2P_j-1][1][v-u-j-\eta+1]}{[v-u][2P_j-1][v-u-j-\eta]} \right. \\ &\quad \left. - G_{-P_j} \frac{[v-u+2P_j-j-\eta][1]}{[v-u-j-\eta][2P_j-1]} \prod_{m=1}^{j-1} \left(\frac{[P_{j,-m}]}{[P_{j,-m}-1]} \frac{[P_{j,m}]}{[P_{j,m}-1]} \right) \right\}.\end{aligned}$$

iii) $k \prec j \preceq N+1 \equiv 0$,

$$\begin{aligned}\pi_z(E_{-k,j}^+(v)) &= e^{-Q_{\varepsilon_k}} \left\{ -E_{j,-k} \frac{[v-u-P_{j,-k}][1]}{[v-u][P_{j,-k}]} \right. \\ &\quad \left. + E_{k,-j} \frac{[v-u-k-\eta-P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j+1]}{[P_j]} \prod_{\substack{m=k+1 \\ \neq j}}^N \frac{[P_{j,-m}+1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}+1]}{[P_{j,m}]} \right\} e^{-Q_{\varepsilon_j}}, \\ \pi_z(F_{j,-k}^+(v)) &= E_{-k,j} \frac{[v-u+P_{j,-k}][1]}{[v-u][P_{j,-k}]} \\ &\quad - E_{-j,k} \frac{[v-u-k-\eta+P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j-1]}{[P_j]} \prod_{\substack{m=k+1 \\ \neq j}}^N \frac{[P_{j,-m}-1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}-1]}{[P_{j,m}]}.\end{aligned}$$

In addition, it is also worth to remark the following formulas.

$$\begin{aligned}
\pi_z(H_j^+(v)) &= \left\{ \frac{[v-u-\frac{j}{2}-1]}{[v-u-\frac{j}{2}]} E_{j,j} + \frac{[v-u-\frac{j}{2}-1]}{[v-u-\frac{j}{2}]} E_{j+1,j+1} \right. \\
&\quad \left. + \frac{[v-u+\eta+\frac{j}{2}+1]}{[v-u+\eta+\frac{j}{2}]} E_{-j-1,-j-1} + \frac{[v-u+\eta+\frac{j}{2}-1]}{[v-u+\eta+\frac{j}{2}]} E_{-j,-j} \right\} e^{-Q_{\alpha_j}} \\
&\quad (1 \leq j \leq N-1), \\
\pi_z(H_N^+(v)) &= \left\{ \frac{[v-u-\frac{N}{2}+1]}{[v-u-\frac{N}{2}]} E_{N,N} + \frac{[v-u-\frac{N}{2}-1][v-u-\frac{N}{2}-\frac{1}{2}]}{[v-u-\frac{N}{2}][v-u-\frac{N}{2}+\frac{1}{2}]} E_{0,0} \right. \\
&\quad \left. + \frac{[v-u-\frac{N}{2}-\frac{1}{2}]}{[v-u-\frac{N}{2}+\frac{1}{2}]} E_{-N,-N} \right\} e^{-Q_{\alpha_N}}, \\
\pi_z(H_j^-(v)) &= \left\{ \frac{[u-v+\frac{j}{2}-1]}{[u-v+\frac{j}{2}]} E_{j,j} + \frac{[u-v+\frac{j}{2}+1]}{[u-v+\frac{j}{2}]} E_{j+1,j+1} \right. \\
&\quad \left. + \frac{[u-v-\eta-\frac{j}{2}-1]}{[u-v-\eta-\frac{j}{2}]} E_{-j-1,-j-1} + \frac{[u-v-\eta-\frac{j}{2}+1]}{[u-v-\eta-\frac{j}{2}]} E_{-j,-j} \right\} e^{-Q_{\alpha_j}} \\
&\quad (1 \leq j \leq N-1), \\
\pi_z(H_N^-(v)) &= \left\{ \frac{[u-v+\frac{N}{2}-1]}{[u-v+\frac{N}{2}]} E_{N,N} + \frac{[u-v+\frac{N}{2}+\frac{1}{2}][u-v+\frac{N}{2}-1]}{[u-v+\frac{N}{2}-\frac{1}{2}][u-v+\frac{N}{2}]} E_{0,0} \right. \\
&\quad \left. + \frac{[u-v+\frac{N}{2}+\frac{1}{2}]}{[u-v+\frac{N}{2}-\frac{1}{2}]} E_{-N,-N} \right\} e^{-Q_{\alpha_N}}.
\end{aligned}$$

Remark. The statements in this theorem and the next one remain unchanged when one uses $\rho_0^+(u)$ and $\rho_0^{+*}(u)$ in (3.3) instead of $\rho^+(u)$ and $\rho^{+*}(u)$, respectively. See Sec.6

Combining the formulas in Theorem 5.6, it is not so hard to show the following.

Corollary 5.7. *For $1 \preceq i, j, k, l \preceq -1$, we have*

$$\pi_z(\widehat{L}_{i,j}^+(v))_{k,l} = R^+(v-u, P)_{ik}^{jl}.$$

Proof. For example, for $-N \preceq -j \preceq -1$, we obtain

$$\begin{aligned}
\pi_z(L_{-j,-j}^+(v)) &= \pi_z \left(K_{-j}^+(v) + \sum_{-j \preceq -k \preceq -1} F_{-j,-k}^+(v) K_{-k}^+(v) E_{-k,-j}^+(v) \right) \\
&= \rho^+(v-u) \left\{ \bar{b}(v-u) \sum_{1 \preceq k \neq j \preceq -(j+1)} E_{k,k} + \sum_{-(j-1) \preceq -k \preceq -1} b(v-u, P_{-j,-k}) E_{-k,-k} \right. \\
&\quad \left. + \bar{d}(v-u, P_j, P_{-j}) E_{j,j} + E_{-j,-j} \right\} e^{Q_{\varepsilon_j}}.
\end{aligned}$$

Here we used the identity

$$\bar{d}(u, P_j, P_{-j}) = \frac{[u]}{[u+1]} \left(\frac{[u-j-\eta]}{[u-j+1-\eta]} - \sum_{-(j-1) \preceq -k \preceq -1} \frac{[u-k-\eta+P_{-j,-k}][u-k-\eta-P_{-j,-k}][1]^2}{[u-k-\eta][u-k+1-\eta][P_{-j,-k}]^2} \prod_{m=k+1}^{j-1} \frac{[P_{-j,-m}-1][P_{-j,-m}+1]}{[P_{-j,-m}]^2} \right).$$

In addition, we have

$$\begin{aligned} \pi_z(L_{0,0}^+(v)) &= \pi_z \left(K_0^+(v) + \sum_{-N \preceq -k \preceq -1} F_{0,-k}^+(v) K_{-k}^+(v) E_{-k,0}^+(v) \right) \\ &= \rho^+(v-u) \left\{ \bar{b}(v-u) \sum_{1 \preceq k \preceq N} E_{k,k} + e_0(v-u, P) E_{0,0} \right. \\ &\quad \left. + \sum_{-N \preceq -k \preceq -1} b(v-u, P_{0,-k}) E_{-k,-k} \right\} e^{Q_{\varepsilon_j}}. \end{aligned}$$

This is due to the identity

$$\begin{aligned} e_0(u, P) &= \frac{[u-1][u+\frac{1}{2}]}{[u+1][u-\frac{1}{2}]} - \frac{[u]}{[u+1]} \sum_{-N \preceq -k \preceq -1} \frac{[u-k-\eta+\frac{1}{2}+P_k][u-k-\eta+\frac{1}{2}-P_k][1]^2}{[u-k-\eta][u-k+1-\eta][P_k+\frac{1}{2}][P_k-\frac{1}{2}]}. \end{aligned}$$

Furthermore, the coefficient of $E_{1,-1}$ in

$$\begin{aligned} \pi_z(\widehat{L}_{-1,1}^+(v)) &= \pi_z(K_{-1}^+(v) E_{-1,1}^+(v)) \\ &= \rho^+(v-u) E_{1,-1} \left\{ -\frac{[1][v-u-2P_1-1]}{[v-u+1][2P_1+1]} + G_{P_1} \frac{[v-u][v-u-2P_1-1-\lambda][1]}{[v-u+1][v-u-\lambda][2P_1+1]} \right\} e^{-Q_{\varepsilon_1}} \end{aligned}$$

coincides with $e_1(v-u, P)$, and for $k \prec j \preceq N$ the coefficient of $E_{k,-j}$ in

$$\begin{aligned} \pi_z(L_{-k,j}^+(v)) &= \pi_z \left(K_{-k}^+(v) E_{-k,j}^+(v) + \sum_{-(k-1) \preceq -l \preceq -1} F_{-k,-l}^+(v) K_{-l}^+(v) E_{-l,j}^+(v) \right) \\ &= \rho^+(v-u) \left\{ \bar{c}(v-u, P_{j,-k}) E_{j,-k} \right. \\ &\quad \left. + E_{k,-j} G_{P_j} \frac{[v-u][v-u-1-\eta-P_{j,-k}][1]}{[v-u+1][v-u-\eta][P_{j,-k}+1]} \prod_{m=1}^{j-1} \frac{[P_{j,m}]}{[P_{j,m}+1]} \prod_{m=1}^{k-1} \frac{[P_{-k,-m}-1]}{[P_{-k,-m}]} \right\} e^{-Q_{\varepsilon_j}} \end{aligned}$$

coincides with $\bar{d}(v-u, P_k, P_j)$. □

§ 5.3. The level-1 representation

Next we consider level-1 representation of $U_{q,p}(B_N^{(1)})$. We follow the work [6]. Let e^{α_i} ($i \in I$) be the generators of the group algebra $\mathbb{C}[\mathcal{Q}]$ with the following central extension.

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} e^{\alpha_j} e^{\alpha_i}$$

Let us consider the Neveu-Schwartz (NS) fermion $\{\Psi_n | n \in \mathbb{Z} + \frac{1}{2}\}$ and the Ramond (R) fermion $\{\Psi_n | n \in \mathbb{Z}\}$ satisfying the following anti-commutation relations.

$$\{\Psi_m, \Psi_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$$

with $\mathcal{N} = 1/(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$. We define

$$\mathcal{F}^{NS} = \mathbb{C}[\Psi_{-\frac{1}{2}}, \Psi_{-\frac{3}{2}} \cdots], \quad \tilde{\mathcal{F}}^R = \mathbb{C}[\Psi_{-1}, \Psi_{-2}, \dots]$$

and their submodules $\mathcal{F}_{even}^{NS,R}$ (reps. $\mathcal{F}_{odd}^{NS,R}$) generated by the even (reps. odd) number of Ψ_{-m} 's. Due to the zero-mode Ψ_0 we have two degenerate vacuum states 1 and $\Psi_0 1$. We hence consider the extended space

$$\hat{\mathcal{F}}^R = \tilde{\mathcal{F}}^R \otimes \mathbb{C}^2$$

and realize the R -fermions by

$$\hat{\Psi}_m = \Psi_m \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (m \in \mathbb{Z}_{\neq 0}), \quad \hat{\Psi}_0 = \mathcal{N}^{\frac{1}{2}}(1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

Note that $\{\hat{\Psi}_m, \hat{\Psi}_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$. We set

$$\mathcal{F}^R = \mathcal{F}_{even}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{F}_{odd}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of Ψ_m on $\mathcal{F}^{NS}$ is given by

$$\Psi_{-m} \cdot u = \Psi_{-m} u, \quad \Psi_m \cdot u = \{\Psi_m, u\} \quad (m \in \frac{1}{2} + \mathbb{Z}_{\geq 0}),$$

where $u \in \mathcal{F}^{NS}$, whereas $\hat{\Psi}_m$ acts on $\mathcal{F}^R$ as

$$\begin{aligned} \hat{\Psi}_{-m} \cdot u \otimes v &= \Psi_{-m} u \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), & \hat{\Psi}_0 \cdot u \otimes v &= u \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v, \\ \hat{\Psi}_m \cdot u \otimes v &= \{\Psi_m, u\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \end{aligned}$$

where $u \in \tilde{\mathcal{F}}^R$, $v \in \mathbb{C}^2$. We define the fermion fields $\Psi^{NS}(z)$ and $\Psi^R(z)$ by

$$\Psi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \Psi_n z^{-n}, \quad \Psi^R(z) = \sum_{n \in \mathbb{Z}} \hat{\Psi}_n z^{-n}.$$

Then we have the operator product expansions

$$\begin{aligned} \Psi(z)\Psi(w) &= : \Psi(z)\Psi(w) : + \langle \Psi(z)\Psi(w) \rangle, \\ \langle \Psi(z)\Psi(w) \rangle &= \begin{cases} \frac{(zw)^{1/2}(z-w)}{(z-qw)(z-q^{-1}w)} & \text{for NS} \\ \mathcal{N} \frac{(z-w)(z+w)}{(z-qw)(z-q^{-1}w)} & \text{for R.} \end{cases} \end{aligned}$$

Now we define

$$\begin{aligned} W(\Lambda_0) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1}, \\ W(\Lambda_1) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1} \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0], \\ W(\Lambda_N) &= \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}] e^{\bar{\Lambda}_N} \cong \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_N} \oplus \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1 + \bar{\Lambda}_N}, \end{aligned}$$

where $\mathcal{Q}_0$ denotes the sublattice of $\mathcal{Q}$ generated by the long roots. For generic $\mu \in \mathfrak{h}^*$ and $a = 0, 1, N$, we set

$$\hat{V}(\Lambda_a + \mu, \mu) = \mathcal{F}_{\alpha,1} \otimes_{\mathbb{C}} (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_{\mathcal{Q}}].$$

Then we have the following decomposition.

$$\hat{V}(\Lambda_a + \mu, \mu) = \bigoplus_{\gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}} \bigoplus_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta}} \mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu),$$

where

$$\begin{aligned} \mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^{\gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^{\gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_N, \gamma, \kappa}(\Lambda_N, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_N + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_N - \alpha_N, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_N + \bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}. \end{aligned}$$

Theorem 5.8. [6] *The three spaces $\hat{V}(\Lambda_a + \mu, \mu)$ ($a = 0, 1, N$) give the level-1 irreducible $U_{q,p}(\hat{B}_l^{(1)})$ -modules with the highest weight $(\Lambda_a + \mu, \mu)$, where the highest weight vectors are given by $1 \otimes 1 \otimes 1 \otimes e^{Q_{\bar{\mu}}}$ for $\hat{V}(\Lambda_0 + \mu, \mu)$, $1 \otimes 1 \otimes e^{\bar{\Lambda}_1} \otimes e^{Q_{\bar{\mu}}}$ for $\hat{V}(\Lambda_1 + \mu, \mu)$ and $1 \otimes 1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e^{\bar{\Lambda}_N} \otimes e^{Q_{\bar{\mu}}}$ for $\hat{V}(\Lambda_N + \mu, \mu)$, respectively. The action*

of the elliptic currents on $\widehat{V}(\Lambda_a + \mu, \mu)$ ($a = 0, 1, N$) is given as follows.

$$\begin{aligned}
 E_j(v) &= : \exp \left\{ - \sum_{n \neq 0} \frac{1}{[n]_q} \alpha_{j,n} w^{-n} \right\} : e^{\alpha_j} w^{h_{\alpha_j}} e^{-Q_{\alpha_j}} w^{-\frac{P_{\alpha_j}-1}{r^*}}, \\
 E_N(v) &= \frac{1}{\mathcal{N}^{\frac{1}{2}}} : \exp \left\{ - \sum_{n \neq 0} \frac{1}{[n]_q} \alpha_{N,n} w^{-n} \right\} : \Psi(w) e^{\alpha_N} w^{h_{\alpha_N} + \frac{1}{2}} e^{-Q_{\alpha_N}} w^{-\frac{P_{\alpha_N}-1/2}{r^*}}, \\
 F_j(v) &= : \exp \left\{ \sum_{n \neq 0} \frac{1}{[n]_q} \frac{1-p^{*n}}{1-p^n} \alpha_{j,n} (q^{-1}w)^{-n} \right\} : e^{-\alpha_j} w^{-h_{\alpha_j}} w^{\frac{P_{\alpha_j}+h_{\alpha_j}-1}{r}}, \\
 F_N(v) &= \frac{1}{\mathcal{N}^{\frac{1}{2}}} : \exp \left\{ \sum_{n \neq 0} \frac{1}{[n]_q} \frac{1-p^{*n}}{1-p^n} \alpha_{N,n} (q^{-1}w)^{-n} \right\} : \Psi(w) e^{-\alpha_N} w^{-h_{\alpha_N} + \frac{1}{2}} w^{\frac{P_{\alpha_N}+h_{\alpha_N}-1/2}{r}},
 \end{aligned}$$

$w = q^{2v}$, ($1 \leq j \leq N-1$) together with $H_j^\pm(v)$, $K_j^\pm(v)$ in Sec. 3.2.

§ 6. Vertex Operators of $U_{q,p}(B_N^{(1)})$

In this section we discuss the type I and II vertex operators of the $U_{q,p}(B_N^{(1)})$ -modules. Through this section, we use $\rho_0^+(u)$ in (3.3) as the prefactor of the R -matrix i.e.

$$R^+(u, s) = \rho_0^+(u) \bar{R}^+(u, s).$$

In addition, we often use the following component form of the RLL -relation (3.19).

$$\begin{aligned}
 & \sum_{i', j'} R^+(u, P + h)_{i' j}^{i' j'} \widehat{L}_{i', i''}^+(u_1) \widehat{L}_{j', j''}^+(u_2) \\
 (6.1) \quad &= \sum_{i', j'} \widehat{L}_{j, j'}^+(u_2) \widehat{L}_{i, i'}^+(u_1) R^{*+}(u, P - (\pi(h))_{i', i''} - (\pi(h))_{j', j''})_{i' j'}^{i'' j''}.
 \end{aligned}$$

Here the components of the R^{*+} -matrix is evaluated in the same way as (3.2). For example, the $(j_1, j_2), (j_1, j_2)$ ($j_1 \prec j_2$) component is given by

$$b^*(u, P_{j_1, j_2} - (\pi(h_{j_1, j_2}))_{j_1, j_1} - (\pi(h_{j_1, j_2}))_{j_2, j_2}),$$

where $P_{j_1, j_2} = P_{\epsilon_{j_1}} - P_{\epsilon_{j_2}}$, $h_{j_1, j_2} = h_{\epsilon_{j_1}} - h_{\epsilon_{j_2}}$ and $\pi(h_{\epsilon_j}) = E_{j, j} - E_{-j, -j}$ ($1 \leq j \leq N$).

§ 6.1. Definition

The type I and II vertex operators are the intertwiners of the $U_{q,p}(B_N^{(1)})$ -modules of the form

$$(6.2) \quad \widehat{\Phi}(u) : \widehat{V}(\lambda, \mu) \rightarrow \widehat{V}(\lambda', \mu) \otimes \widehat{V}_z,$$

$$(6.3) \quad \widehat{\Psi}^*(u) : \widehat{V}_z \otimes \widehat{V}(\lambda, \mu) \rightarrow \widehat{V}(\lambda, \mu'),$$

where $\lambda, \lambda' \in \mathfrak{h}^*$, $\mu, \mu' \in H^*$, $z = q^{2u}$. The $\widehat{V}_z$ denote the $(2N+1)$ -dimensional dynamical evaluation module of $U_{q,p}(B_N^{(1)})$ given in Theorem 5.5 and 5.6, and $\widehat{V}(\lambda, \mu)$ denote the level- k highest weight $U_{q,p}(B_N^{(1)})$ -module with highest weight (λ, μ) . The level-1 case is given in Theorem 5.8. The vertex operators satisfy the intertwining relations with respect to the comultiplication Δ given in (4.3)

$$(6.4) \quad \Delta(x)\widehat{\Phi}(u) = \widehat{\Phi}(u)x,$$

$$(6.5) \quad x\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\Delta(x), \quad \forall x \in U_{q,p}(B_N^{(1)}).$$

These intertwining relations are equivalent to the following relations[22]:

$$(6.6) \quad \widehat{\Phi}^{(23)}(u_2)\widehat{L}^{+(12)}(u_1) = R^{+(13)}(u_1 - u_2, P + h)\widehat{L}^{+(12)}(u_1)\widehat{\Phi}^{(23)}(u_2),$$

$$(6.7) \quad \widehat{L}^{+(13)}(u_1)\widehat{\Psi}^{*(23)}(u_2) = \widehat{\Psi}^{*(23)}(u_2)\widehat{L}^{+(13)}(u_1)R^{+*(12)}(u_1 - u_2, P - h^{(1)} - h^{(2)}).$$

The relation (6.6) (resp. (6.7)) should be understood on $\widehat{V}_{z_1} \otimes \widehat{V}(\lambda, \mu)$ (resp. $\widehat{V}_{z_1} \otimes \widehat{V}_{z_2} \otimes \widehat{V}(\lambda, \mu)$). These relations are also expected[13] from the quasi-Hopf algebra formulation of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ [12] by using the connection given in Appendix A.

We define the components of the vertex operators by

$$(6.8) \quad \widehat{\Phi}(u + \frac{1}{2}) = \sum_{1 \preceq m \preceq -1} \Phi_m(u) \otimes v_m, \quad \widehat{\Psi}^*(u)(v_m \otimes u) = \Psi_m^*\left(u - \frac{c}{2}\right) u,$$

where $v_m \in \widehat{V}_z$, $u \in \widehat{V}(\lambda, \mu)$, and the matrix elements of the L -operator $\widehat{L}^+(u)$ by

$$(6.9) \quad \widehat{L}^+(u)v_m = \sum_{1 \preceq k \preceq -1} L_{k,m}^+(u)v_k.$$

Using these and Corollary 5.7, the intertwining relations (6.6), (6.7) are rewritten as follows:

$$(6.10) \quad \Phi_m(u_2)L_{k,j}^+(u_1) = \sum_{1 \preceq m', k' \preceq -1} R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{km}^{k'm'} L_{k',j}^+(u_1)\Phi_{m'}(u_2),$$

$$(6.11) \quad L_{k,j}^+(u_1)\Psi_m^*(u_2) = \sum_{1 \preceq j', m' \preceq -1} \Psi_{m'}^*(u_2)L_{k,j'}^+(u_1)R^{+*}(u_1 - u_2 - \frac{c}{2}, P - h^{(1)} - h^{(2)})_{j'm'}^{jm}.$$

Proposition 6.1. *Let the half currents $E_{l,k}^+(u)$ and $F_{k,l}^+(u)$ ($1 \preceq k \prec l \preceq -1$) take their form as given in Definition 3.5 and Appendix B. Assume that the top components $\Phi_{-1}(u)$ and $\Psi_{-1}^*(u)$ satisfy the following conditions:*

- i) $K_{-1}^+(u_1)\Phi_{-1}(u_2)$ does not have a pole at $u_1 - u_2 = -\frac{3}{2}$
- ii) $\Psi_{-1}(u_2)K_{-1}^+(u_1)$ does not have a pole at $u_1 - u_2 = \frac{c-2}{2} + r^*$.

Then the sufficient conditions for (6.10) and (6.11) are given as follows. For the type I,

$$(6.12) \quad \Phi_k(u_2) = F_{k,-1}^+(u_2 - \frac{1}{2})\Phi_{-1}(u_2) \quad (1 \preceq k \preceq -2),$$

and

$$(6.13) \quad \Phi_{-1}(u_2)K_{-1}^+(u_1) = \rho_0^+(u_1 - u_2 + \frac{1}{2})K_{-1}^+(u_1)\Phi_{-1}(u_2),$$

$$(6.14) \quad [\Phi_{-1}(u_2), P_l] = 0, \quad [\Phi_{-1}(u_2), E_l(u_1)] = 0. \quad (1 \leq l \leq N),$$

$$(6.15) \quad \Phi_{-1}(u_2)(P + h)_{k,-1} = ((P + h)_{k,-1} - 1)\Phi_{-1}(u_2),$$

$$(6.16) \quad \Phi_{-1}(u_2)F_1(u_1) = \frac{[u_2 - u_1 - \eta]}{[u_2 - u_1 - \eta - 1]}F_1(u_2)\Phi_{-1}(u_1),$$

$$(6.17) \quad \Phi_{-1}(u_2)F_j(u_1) = F_j(u_1)\Phi_{-1}(u_2) \quad (2 \leq j \leq N).$$

For the type II,

$$(6.18) \quad \Psi_k^*(u_2) = \Psi_{-1}^*(u_2)E_{-1,k}^+(u_2 + \frac{c}{2} + r^*) \quad (1 \preceq k \preceq -2),$$

and

$$(6.19) \quad K_{-1}^+(u_1)\Psi_{-1}^*(u_2) = \Psi_{-1}^*(u_2)K_{-1}^+(u_1)\rho_0^{+*}(u_1 - u_2 - \frac{c}{2}),$$

$$(6.20) \quad [\Psi_{-1}^*(u_2), (P + h)_l] = 0, \quad [\Psi_{-1}^*(u_2), F_l(u_1)] = 0 \quad (1 \leq l \leq N),$$

$$(6.21) \quad P_{j,-1}\Psi_{-1}^*(u_2) = \Psi_{-1}^*(u_2)(P_{j,-1} + 1) \quad (j \prec -1),$$

$$(6.22) \quad E_1(u_1)\Psi_{-1}^*(u_2) = \frac{[u_2 - u_1 - \eta + \frac{1}{2}]^*}{[u_2 - u_1 - \eta - \frac{1}{2}]^*}\Psi_{-1}^*(u_2)E_1(u_1),$$

$$(6.23) \quad E_j(u_1)\Psi_{-1}^*(u_2) = \Psi_{-1}^*(u_2)E_j(u_1) \quad (2 \leq j \leq N).$$

Proof. We consider the type I case only. The type II case can be proved similarly. From the component $k = m (\neq 0)$ in (6.10), we have

$$(6.24) \quad \Phi_m(u_2)L_{m,j}^+(u_1) = \rho_0^+(u_1 - u_2 + \frac{1}{2})L_{m,j}^+(u_1)\Phi_m(u_2).$$

In particular, the component $m = j = -1$ of (6.24) is

$$(6.25) \quad \Phi_{-1}(u_2)K_{-1}^+(u_1) = \rho_0^+(u_1 - u_2 + \frac{1}{2})K_{-1}^+(u_1)\Phi_{-1}(u_2).$$

Note the formula

$$\rho_0^+(u) = \frac{[u+1]}{\varphi(u)},$$

$$\varphi(u) = q^{-1}z^{\frac{1}{r}}[u-1] \frac{\{\xi^2 z\}\{z\}\{\xi q^2 z\}\{\xi q^{-2} z\}}{\{\xi z\}^2\{\xi^2 q^2 z\}\{q^{-2} z\}} \frac{\{p\xi/z\}^2\{p\xi^2 q^2/z\}\{pq^{-2}/z\}}{\{p\xi^2/z\}\{p/z\}\{p\xi q^2/z\}\{p\xi q^{-2}/z\}}.$$

Then from the assumption i), (6.25) implies that $\Phi_{-1}(u_2)K_{-1}^+(u_1)$ has a zero at $u_1 - u_2 = -\frac{3}{2}$. We will check these points for the level-1 representation.

In addition, from the component $m = -1 \succ j$ of (6.24), and putting the definition $L_{-1,j}^+(u) = K_{-1}^+(u)E_{-1,j}^+(u)$, we have

$$(6.26) \quad \Phi_{-1}(u_2)E_{-1,j}^+(u_1) = E_{-1,j}^+(u_1)\Phi_{-1}(u_2).$$

From the conjectural expressions for $E_{-1,j}^+(u)$ in Appendix B, the sufficient conditions for (6.26) are

$$(6.27) \quad [\Phi_{-1}(u_2), P_l] = 0, \quad [\Phi_{-1}(u_2), E_l(u_1)] = 0. \quad (1 \leq l \leq N)$$

Next, the component $k \neq \pm m (\neq 0)$, $k \prec m$ in (6.10) is

$$\begin{aligned} \Phi_m(u_2)L_{k,j}^+(u_1) &= \rho_0^+(u_1 - u_2 + \frac{1}{2}) \left\{ b(u_1 - u_2 + \frac{1}{2}, (P+h)_{k,m})L_{k,j}^+(u_1)\Phi_m(u_2) \right. \\ &\quad \left. + c(u_1 - u_2 + \frac{1}{2}, (P+h)_{k,m})L_{m,j}^+(u_1)\Phi_k(u_2) \right\}. \end{aligned}$$

Then putting the definition $L_{-1,-1}^+(u) = K_{-1}^+(u)$ and $L_{k,-1}^+(u) = F_{k,-1}^+(u)K_{-1}^+(u)$ in the case $k \prec m = j = -1$ of (6.28), we have

$$\begin{aligned} (6.28) \quad &\Phi_{-1}(u_2)F_{k,-1}^+(u_1)K_{-1}^+(u_1) \\ &= \rho_0^+(u_1 - u_2 + \frac{1}{2}) \left\{ \frac{[(P+h)_{k,-1}+1][(P+h)_{k,-1}-1][u_1 - u_2 + \frac{1}{2}]}{[(P+h)_{k,-1}]^2[u_1 - u_2 + \frac{3}{2}]} \right. \\ &\quad \left. \times F_{k,-1}^+(u_1)K_{-1}^+(u_1)\Phi_{-1}(u_2) + \frac{[1][(P+h)_{k,-1}+u_1 - u_2 + \frac{1}{2}]}{[P+h)_{k,-1}][u_1 - u_2 + \frac{3}{2}]} K_{-1}^+(u_1)\Phi_k(u_2) \right\}. \end{aligned}$$

Putting $u_1 - u_2 = -\frac{3}{2}$, the left hand side of (6.28) vanishes. Then we obtain for $2 \preceq k \preceq -2$,

$$\begin{aligned} (6.29) \quad \Phi_k(u_2) &= K_{-1}^+(u_1)^{-1} \frac{[(P+h)_{k,-1}+1]}{[(P+h)_{k,-1}]} F_{k,-1}^+(u_1)K_{-1}^+(u_1)\Phi_{-1}(u_2) \\ &= F_{k,-1}^+(u_1+1)\Phi_{-1}(u_2) = F_{k,-1}^+(u_2 - \frac{1}{2})\Phi_{-1}(u_2). \end{aligned}$$

The remaining component $\Phi_1(u)$ is also obtained from $\Phi_{-1}(u)$ as follows. From

the component $m = j = -1$ in (6.10), we have

$$\begin{aligned} \Phi_{-1}(u_2) F_{1,-1}^+(u_1) K_{-1}^+(u_1) &= R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{1,-1}^{1,-1} F_{1,-1}^+(u_1) K_{-1}^+(u_1) \Phi_{-1}(u_2) \\ &\quad + R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{1,-1}^{-11} K_{-1}^+(u_1) \Phi_1(u_2) \\ &\quad + \sum_{2 \leq l \leq -2} R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{1,-1}^{-ll} F_{l,-1}^+(u_1) K_{-1}^+(u_1) F_{l,-1}(u_2) \Phi_{-1}(u_2). \end{aligned}$$

Using (6.29) and the component $(i, j) = (1, -1)$, $(i'', j'') = (-1, -1)$ of (6.1), we obtain

$$\begin{aligned} \Phi_{-1}(u_2) F_{1,-1}^+(u_1) K_{-1}^+(u_1) &= R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{1,-1}^{-11} K_{-1}^+(u_1) \Phi_1(u_2) \\ &\quad + \rho_0^+(u_1 - u_2 + \frac{1}{2}) K_{-1}^+(u_2) F_{1,-1}^+(u_1) K_{-1}^+(u_2)^{-1} K_{-1}^+(u_1) \Phi_{-1}(u_2) \\ &\quad - R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{1,-1}^{-11} K_{-1}^+(u_1) F_{1,-1}^+(u_2) \Phi_{-1}(u_2). \end{aligned}$$

Then again setting $u_1 - u_2 = -\frac{3}{2}$, the left hand side and the second term in right hand side vanish. Then we obtain

$$(6.30) \quad \Phi_1(u_2) = F_{1,-1}^+(u_2 - \frac{1}{2}) \Phi_{-1}(u_2).$$

Combining (6.29) and (6.30), we obtain (6.12).

Furthermore, substituting (6.12) into (6.28), we obtain the sufficient conditions (6.15)-(6.17). $\square$

Lemma 6.2. For $1 \preceq k \preceq -2$, we have

$$\begin{aligned} \Phi_{-1}(u_2) F_{k,-1}(u_1 - \frac{1}{2}) &= K_{-1}^+(u_2 - \frac{1}{2}) F_{k,-1}^+(u_1 - \frac{1}{2}) K_{-1}^+(u_2 - \frac{1}{2})^{-1} \Phi_{-1}(u_2), \\ E_{-1,k}^+(u_1 + \frac{c}{2}) \Psi_{-1}^*(u_2) &= \Psi_{-1}^*(u_2) K_{-1}^+(u_2 + \frac{c}{2})^{-1} E_{-1,k}^+(u_1 + \frac{c}{2}) K_{-1}^+(u_2 + \frac{c}{2}). \end{aligned}$$

Proof. From the component $m = -1$, $j = -1$ in the intertwining relation (6.10), we have

$$\begin{aligned} \Phi_{-1}(u_2) F_{k,-1}^+(u_1) K_{-1}^+(u_1) \\ = \rho_0^+(u_1 - u_2 + \frac{1}{2}) b(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1}) F_{k,-1}^+(u_1) K_{-1}^+(u_1) \Phi_{-1}(u_2) \\ + \rho_0^+(u_1 - u_2 + \frac{1}{2}) c(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1}) K_{-1}^+(u_1) \Phi_k(u_2). \end{aligned}$$

Using (6.12), we have

$$\begin{aligned} \Phi_{-1}(u_2) F_{k,-1}^+(u_1) K_{-1}^+(u_1) \\ = \rho_0^+(u_1 - u_2 + \frac{1}{2}) b(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1}) F_{k,-1}^+(u_1) K_{-1}^+(u_1) \Phi_{-1}(u_2) \\ + \rho_0^+(u_1 - u_2 + \frac{1}{2}) c(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1}) K_{-1}^+(u_1) F_{k,-1}^+(u_2 - \frac{1}{2}) \Phi_{-1}(u_2). \end{aligned}$$

From (6.25), we have

$$\begin{aligned} & \Phi_{-1}(u_2)F_{k,-1}^+(u_1 - \frac{1}{2}) \\ &= b(u_1 - u_2, (P+h)_{k,-1})F_{k,-1}^+(u_1 - \frac{1}{2})\Phi_{-1}(u_2) \\ & \quad + c(u_1 - u_2, (P+h)_{k,-1})K_{-1}^+(u_1 - \frac{1}{2})F_{k,-1}^+(u_2 - \frac{1}{2})K_{-1}(u_1 - \frac{1}{2})^{-1}\Phi_{-1}(u_2). \end{aligned}$$

Then it is sufficient to show

$$\begin{aligned} & b(u_1 - u_2, (P+h)_{k,-1})F_{k,-1}^+(u_1) + c(u_1 - u_2, (P+h)_{k,-1})K_{-1}^+(u_1)F_{k,-1}^+(u_2)K_{-1}^+(u_1)^{-1} \\ &= K_{-1}^+(u_2)F_{k,-1}^+(u_1)K_{-1}^+(u_2)^{-1}. \end{aligned}$$

This is nothing but the component $(i, j) = (-1, -1)$, $(i'', j'') = (k, -1)$ of (6.1). $\square$

§ 6.2. Level-one vertex operators and commutation relations

Next we consider a free field realization of the vertex operators fixing the representation level $c = 1$.

From the sufficient conditions obtained in Proposition 6.1, we can determine the free field realizations of vertex operators as follows:

Proposition 6.3. *The top components of the type I and type II vertex operators $\Phi_{-1}(u)$ and $\Psi_{-1}^*(u)$ are realized in terms of the free field by*

$$(6.31) \quad \Phi_{-1}(u) =: \exp \left\{ \sum_{m \neq 0} (q^m - q^{-m}) \frac{1 - p^{*m}}{1 - p^m} \mathcal{E}_m^{-1} (q^{-3}\xi z)^{-m} \right\} : e^{\epsilon_1} (q^{-1}\xi z)^{h_{\epsilon_1}} (q^{-1}\xi z)^{-\frac{1}{r}(P+h)_{\epsilon_1}},$$

$$(6.32) \quad \Psi_{-1}^*(u) =: \exp \left\{ - \sum_{m \neq 0} (q^m - q^{-m}) \mathcal{E}_m^{-1} (q^{-1}\xi z)^{-m} \right\} : e^{-\epsilon_1} e^{Q_{\epsilon_1}} (\xi z)^{-h_{\epsilon_1}} (\xi z)^{\frac{1}{r^*}P_{\epsilon_1}}.$$

These realizations satisfy the assumptions i) and ii) in Proposition 6.1.

Proof. By straightforward calculations, we can show that (6.31) (resp. (6.32)) satisfies the sufficient conditions (6.25), (6.27), and (6.15)-(6.17) (resp. (6.19)-(6.23)). $\square$

Theorem 6.4. *The realizations of the type I $\Phi_j(u)$ and the type II $\Psi_j^*(u)$ vertex operators given by Proposition 6.3, (6.12) and (6.18) satisfy the following commutation*

relations:

$$(6.33) \quad \Phi_{j_2}(u_2)\Phi_{j_1}(u_1) = \sum_{j'_1, j'_2=1}^{-1} R(u_1 - u_2, P + h)_{j_1 j_2}^{j'_1 j'_2} \Phi_{j'_1}(u_1)\Phi_{j'_2}(u_2),$$

$$(6.34) \quad \Psi_{j_1}^*(u_1)\Psi_{j_2}^*(u_2) = \sum_{j'_1, j'_2=1}^{-1} \Psi_{j'_2}^*(u_2)\Psi_{j'_1}^*(u_1)R^*(u_1 - u_2, P - h^{(1)} - h^{(2)})_{j'_1 j'_2}^{j_1 j_2},$$

$$(6.35) \quad \Phi_j(u_1)\Psi_k^*(u_2) = \chi(u_1 - u_2)\Psi_k^*(u_2)\Phi_j(u_1).$$

Here we set

$$(6.36) \quad R(u, P) = \mu(u)\bar{R}^+(u, P), \quad R^*(u, P) = \mu^*(u)\bar{R}^{+*}(u, P)$$

with

$$(6.37) \quad \mu(u) = z^{-1+\frac{1}{r}} \frac{\{p\xi^2 q^{-2}z\}\{p\xi z\}\{\xi z\}\{q^2 z\}}{\{p\xi q^{-2}z\}\{pz\}\{\xi^2 z\}\{\xi q^2 z\}} \frac{\{p\xi q^{-2}/z\}\{p/z\}\{\xi^2/z\}\{\xi q^2/z\}}{\{p\xi^2 q^{-2}/z\}\{p\xi/z\}\{\xi/z\}\{q^2/z\}},$$

$$(6.38) \quad \mu^*(u) = \mu(u)|_{r \rightarrow r^*}$$

and

$$(6.39) \quad \chi(u) = \frac{\Theta_{\xi^2}(z)\Theta_{\xi^2}(q^{-2}\xi z)}{\Theta_{\xi^2}(\xi z)\Theta_{\xi^2}(q^{-2}\xi^2 z)}.$$

Proof. Let us show the commutation relation of the type I vertex operators (6.33). For $j_1 = j_2 = -1$, the equation

$$(6.40) \quad \Phi_{-1}(u_2)\Phi_{-1}(u_1) = \mu(u_1 - u_2)\Phi_{-1}(u_1)\Phi_{-1}(u_2)$$

can be shown by straightforward calculation with the use of the free field realization (6.31).

For $1 \preceq j_1, j_2 \preceq -2$, using (6.12), (6.40) and Lemma 6.2, the equation (6.33) is reduced to the following equation:

$$(6.41) \quad F_{j_2, -1}^+(u_2)K_{-1}^+(u_2)F_{j_1, -1}^+(u_1)K_{-1}^+(u_2)^{-1} \\ = \sum_{j'_1, j'_2=1}^{-1} \bar{R}^+(u, P + h)_{j_1 j_2}^{j'_1 j'_2} F_{j'_1, -1}^+(u_1)K_{-1}^+(u_1)F_{j'_2, -1}^+(u_2)K_{-1}^+(u_1)^{-1},$$

where $u = u_1 - u_2$. From the component $(i, j) = (-1, -1)$, $(i'', j'') = (j_1, j_2)$ of (6.1), we have

$$\sum_{j'_1, j'_2=1}^{-1} R^+(u, P + h)_{j_1 j_2}^{j'_1 j'_2} F_{j'_1, -1}^+(u_1)K_{-1}^+(u_1)F_{j'_2, -1}^+(u_2)K_{-1}^+(u_2) \\ = F_{j_2, -1}^+(u_2)K_{-1}^+(u_2)F_{j_1, -1}^+(u_1)K_{-1}^+(u_1)\tilde{\rho}^{*+}(u).$$

Multiplying the above equation by

$$(6.42) \quad \tilde{\rho}^+(u)^{-1} K_{-1}^+(u_2)^{-1} K_{-1}^+(u_1)^{-1} = K_{-1}^+(u_1)^{-1} K_{-1}^+(u_2)^{-1} \tilde{\rho}^{*+}(u)^{-1}$$

from the right, we obtain the desired equation (6.41).

For $j_1 = -1$, $1 \preceq j_2 \preceq -2$, using (6.12), (6.40) and Lemma 6.2, the equation (6.33) is reduced to the following equation of the half currents:

$$(6.43) \quad F_{j_2, -1}^+(u_2) \\ = \bar{R}^+(u, P+h)_{-1j_2}^{-1-1} + \sum_{j'_2=1}^{-2} \bar{R}^+(u, P+h)_{-1j_2}^{-1j'_2} K_{-1}^+(u_1 - \frac{1}{2}) F_{j'_2, -1}^+(u_2 - \frac{1}{2}) K_{-1}^+(u_1 - \frac{1}{2})^{-1} \\ + \sum_{j'_1=1}^{-2} \bar{R}^+(u, P+h)_{-1j_2}^{j'_1-1} F_{j'_1, -1}^+(u_1 - \frac{1}{2}) \\ + \sum_{j'_1, j'_2=1}^{-2} \bar{R}^+(u, P+h)_{-1j_2}^{j'_1 j'_2} F_{j'_1, -1}^+(u_1 - \frac{1}{2}) K_{-1}^+(u_1 - \frac{1}{2}) F_{j'_2, -1}^+(u_2 - \frac{1}{2}) K_{-1}^+(u_1 - \frac{1}{2})^{-1}.$$

From the component $(i, j) = (-1, -1)$, $(i'', j'') = (-1, j_2)$ of (6.1), we have

$$R^+(u, P+h)_{-1j_2}^{-1-1} K_{-1}^+(u_1) K_{-1}^+(u_2) + \sum_{j'=1}^{-2} R^+(u, P+h)_{-1j_2}^{-1j'} K_{-1}^+(u_1) F_{j', -1}^+(u_2) K_{-1}^+(u_2) \\ + \sum_{i'=1}^{-2} R^+(u, P+h)_{-1j_2}^{i'-1} F_{i', -1}^+(u_1) K_{-1}^+(u_1) K_{-1}^+(u_2) \\ + \sum_{i', j'=1}^{-2} R^+(u, P+h)_{-1j_2}^{i' j'} F_{i', -1}^+(u_1) K_{-1}^+(u_1) F_{j', -1}^+(u_2) K_{-1}^+(u_2) \\ = F_{j_2, -1}^+(u_2) K_{-1}^+(u_2) K_{-1}^+(u_1) \rho^{*+}(u).$$

Multiplying the above equation by (6.42) from the right, we obtain the desired equation (6.43). The case $1 \preceq j_1 \preceq -2$, $j_2 = -1$ can be proved in the same manner.

Similarly, one can prove the commutation relation of the type II vertex operators (6.34).

Next, let us consider the relation (6.35). The case $j = k = -1$ is a direct consequence from Proposition 6.3. The cases $j = -1$ or $k = -1$ can be shown as follows: consider the case $k = -1$ for instance. By (6.12),

$$(6.44) \quad \Phi_j(u_1) \Psi_{-1}^*(u_2) = F_{j, -1}^+(u_1 - \frac{1}{2}) \Phi_{-1}(u_1) \Psi_{-1}^*(u_2) \\ = F_{j, -1}^+(u_1 - \frac{1}{2}) \chi(u_1 - u_2) \Psi_{-1}^*(u_2) \Phi_{-1}(u_1) \\ = \chi(u_1 - u_2) \Psi_{-1}^*(u_2) F_{j, -1}^+(u_1 - \frac{1}{2}) \Phi_{-1}(u_1) \\ = \chi(u_1 - u_2) \Psi_{-1}^*(u_2) \Phi_j(u_1).$$

Then the general case is proved as follows. Since both $\widehat{\Phi}(u_1)\widehat{\Psi}^*(u_2)$ and $(\widehat{\Psi}^*(u_2) \otimes \text{id})(\text{id} \otimes \widehat{\Phi}(u_1))$ commute with $\Delta(x)$ ($\forall x \in U_{q,p}(B_N^{(1)})$), they act as scalars on the irreducible module $V_{z_2} \otimes \widehat{V}(\lambda)$. In order to compare the scalars, we will see their actions on $v_{-1} \otimes |\lambda\rangle \in V_{z_2} \otimes \widehat{V}(\lambda)$.

$$\begin{aligned}
 (6.45) \quad \widehat{\Phi}(u_1)\widehat{\Psi}^*(u_2)(v_{-1} \otimes |\lambda\rangle) &= \widehat{\Phi}(u_1)\Psi_{-1}^*(u_2 - \frac{c}{2})|\lambda\rangle \\
 &= \sum_j \Phi_j(u_1)\Psi_{-1}^*(u_2 - \frac{c}{2})|\lambda\rangle \otimes v_j \\
 &= \chi(u_1 - u_2 + \frac{1}{2}) \sum_j \Psi_{-1}^*(u_2 - \frac{c}{2})\Phi_j(u_1 + \frac{1}{2})|\lambda\rangle \otimes v_j.
 \end{aligned}$$

Here the last equality follows from (6.44). On the other hand

$$\begin{aligned}
 (6.46) \quad (\widehat{\Psi}^*(u_2) \otimes \text{id})(\text{id} \otimes \widehat{\Phi}(u_1))(v_{-1} \otimes |\lambda\rangle) &= (\widehat{\Psi}^*(u_2) \otimes \text{id})(v_{-1} \otimes \sum_j \Phi_j(u_1)|\lambda\rangle \otimes v_j) \\
 &= \Psi_{-1}^*(u_2 - \frac{c}{2}) \sum_j \Phi_j(u_1 + \frac{1}{2})|\lambda\rangle \otimes v_j.
 \end{aligned}$$

Comparing (6.45) and (6.46), we get

$$(6.47) \quad \widehat{\Phi}(u_1)\widehat{\Psi}^*(u_2) = \chi(u_1 - u_2 + \frac{1+c}{2})(\widehat{\Psi}^*(u_2) \otimes \text{id})(\text{id} \otimes \widehat{\Phi}(u_1)).$$

Hence comparing the components of the both sides, and changing variables $u_1 + \frac{1}{2} \rightarrow u_1$, $u_2 - \frac{c}{2} \rightarrow u_2$, we obtain

$$(6.48) \quad \Phi_j(u_1)\Psi_k^*(u_2) = \chi(u_1 - u_2)\Psi_k^*(u_2)\Phi_j(u_1).$$

□

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§ A. Relation to the Quasi-Hopf Formulation $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$

§ A.1. Definition of $\mathcal{B}_{q,\lambda}(B_N^{(1)})$

Let $U_q = U_q(B_N^{(1)})$ be the Drinfeld-Jimbo affine quantum group [3, 9]. Namely, $U_q(B_N^{(1)})$ is a quasi-triangular Hopf algebra realized by the Chevalley generators and

equipped with the standard coproduct Δ_0 , counit ε , antipode S and universal R matrix $\mathcal{R}$. Our conventions on the coalgebra structure follows [12]. Let $\mathfrak{h}$ and $\bar{\mathfrak{h}}$ be the Cartan subalgebras as in Sec.2.1. We denote a basis and its dual basis of $\mathfrak{h}$ by $\{\hat{h}_l\}$ and $\{\hat{h}^l\}$, respectively. More explicitly, they are given by $\{\hat{h}_l\} = \{d, c, h_j\}$ and $\{\hat{h}^l\} = \{c, d, h^j\}$ ($1 \leq j \leq N$), where $\{h_j\}$ and $\{h^j\}$ are a basis and a dual basis of $\bar{\mathfrak{h}}$.

The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ is a quasi-Hopf deformation of $U_q(B_N^{(1)})$ by the face type twistor $F(\lambda)$ ($\lambda \in \mathfrak{h}$). The twistor $F(\lambda)$ is an invertible element in $U_q \otimes U_q$ satisfying

$$(A.1) \quad (\text{id} \otimes \varepsilon)F(\lambda) = 1 = F(\lambda)(\varepsilon \otimes \text{id}),$$

$$(A.2) \quad F^{(12)}(\lambda)(\Delta_0 \otimes \text{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\text{id} \otimes \Delta_0)F(\lambda).$$

where $\lambda = \sum_l \lambda_l \hat{h}^l$ ($\lambda_l \in \mathbb{C}$), $\lambda + h^{(1)} = \sum_l (\lambda_l + \hat{h}_l^{(1)}) \hat{h}^l$ and $\hat{h}_l^{(1)} = \hat{h}_l \otimes 1 \otimes 1$. An explicit construction of the twistor $F(\lambda)$ is given in [12]. Then we define a new coproduct by

$$(A.3) \quad \Delta_\lambda(x) = F(\lambda)\Delta_0(x)F(\lambda)^{-1} \quad \forall x \in U_q(B_N^{(1)}).$$

Δ_λ satisfies a weaker coassociativity

$$(A.4) \quad (\text{id} \otimes \Delta_\lambda)\Delta_\lambda(x) = \Phi(\lambda)(\Delta_\lambda \otimes \text{id})\Delta_\lambda(x)\Phi(\lambda)^{-1} \quad \forall x \in U_q(B_N^{(1)}),$$

$$(A.5) \quad \Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda + h^{(1)})^{-1}.$$

The universal R -matrix is also deformed to

$$(A.6) \quad \mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}F^{(12)}(\lambda)^{-1}.$$

Definition A.1. [12] The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ is a quasi-triangular quasi-Hopf algebra $(U_q(B_N^{(1)}), \Delta_\lambda, \varepsilon, S, \Phi(\lambda), \alpha, \beta, \mathcal{R}(\lambda))$, where α, β are defined by

$$(A.7) \quad \alpha = \sum_i S(k_i)l_i, \quad \beta = \sum_i m_i S(n_i).$$

Here we set $\sum_i k_i \otimes l_i = F(\lambda)^{-1}$, $\sum_i m_i \otimes n_i = F(\lambda)$.

The new universal R matrix $\mathcal{R}(\lambda)$ satisfies the dynamical Yang-Baxter equation.

$$(A.8) \quad \mathcal{R}^{(12)}(\lambda + h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}) = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda).$$

In [21], we derived vector representations of $\mathcal{R}(\lambda)$ for $\hat{\mathfrak{g}} = A_N^{(1)}, B_N^{(1)}, C_N^{(1)}, D_N^{(1)}$ and found that if we parametrize $\lambda \in \mathfrak{h}^*$ as $\lambda = \lambda(r^*, P) = (r^* + h^\vee)d + \sum_{j=1}^N (P_{\alpha'_j} + 1)\bar{\Lambda}_j$ with α'_j being the simple roots of the dual Lie algebra $\bar{\mathfrak{g}}^\vee$ of $\bar{\mathfrak{g}}$, the vector representation

of $R(\lambda)$ coincides with the corresponding face weight derived by Jimbo, Okado and Miwa[10]. In particular, for the $B_N^{(1)}$ type, if we set

$$\begin{aligned}\mathcal{R}^{+*}(z, P) &= (\pi_{V,z} \otimes \pi_{V,1}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda) \right), \\ \mathcal{R}^+(z, P + h) &= (\pi_{V,z} \otimes \pi_{V,1}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda + h) \right),\end{aligned}$$

with

$$\bar{\theta}(\lambda) = -\bar{\lambda} + \bar{\rho} - \frac{1}{2} \sum_j \bar{h}_j \bar{h}^j, \quad \bar{T} = \sum_j \bar{h}_j \otimes \bar{h}^j$$

for $\lambda = \lambda(r^*, P)$, then $\mathcal{R}^{+*}(z, P)$ and $\mathcal{R}^+(z, P + h)$ coincide with (3.2) up to a gauge transformation. Moreover we define the L operators of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$ by

$$\begin{aligned}\mathcal{L}^+(z, P) &= (\pi_{V,z} \otimes \text{id}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda) \right), \\ \mathcal{L}^-(z, P) &= (\pi_{V,z} \otimes \text{id}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} \mathcal{R}^{(21)}(\lambda)^{-1} q^{-c \otimes d - d \otimes c} \right).\end{aligned}$$

Note that $\mathcal{L}^+(z, P)$ and $\mathcal{L}^-(z, P)$ are not independent: we have

$$(A.9) \quad \mathcal{L}^-(z, P) = \mathcal{L}^+(zp^*q^c, P).$$

In addition, if we define

$$\begin{aligned}\mathcal{R}^{-*}(z, P) &= (\pi_{V,z_1} \otimes \pi_{V,z_2}) \mathcal{R}^{(21)}(\lambda)^{-1} q^{-c \otimes d - d \otimes c}, \\ \mathcal{R}^-(z, P + h) &= (\pi_{V,z_1} \otimes \pi_{V,z_2}) \mathcal{R}^{(21)}(\lambda + h)^{-1} q^{-c \otimes d - d \otimes c},\end{aligned}$$

Then we have $\mathcal{R}^{-*}(z, P) = \mathcal{R}^{+*}(zp^*q^c, P)$, $\mathcal{R}^-(z, P + h) = \mathcal{R}^+(zpq^{-c}, P + h)$. Combining these formulas we obtain from (A.8) the following dynamical RLL relations [13]

$$(A.10) \quad \begin{aligned}\mathcal{R}^\pm(z, P + h) \mathcal{L}^\pm(z_1, P) \mathcal{L}^\mp(z_2, P + h^{(1)}) &= \mathcal{L}^\pm(z_2, P) \mathcal{L}^\pm(z_1, P + h^{(2)}) \mathcal{R}^{\pm*}(z, P), \\ \mathcal{R}^\pm(zq^{\pm c}, P + h) \mathcal{L}^\pm(z_1, P) \mathcal{L}^\mp(z_2, P + h^{(1)}) &= \mathcal{L}^\mp(z_2, P) \mathcal{L}^\pm(z_1, P + h^{(2)}) \mathcal{R}^{\pm*}(zq^{\mp c}, P).\end{aligned}$$

Furthermore define

$$\widehat{\mathcal{L}}^\pm(z) = \mathcal{L}^\pm(z, P) e^{-\sum_j \pi_V(h_{\varepsilon_j}) \otimes Q_{\varepsilon_j}} \in \mathcal{B}_{q,\lambda(r^*, P)}(\widehat{\mathfrak{g}})[[z, z^{-1}]] [z^{\pm \frac{1}{r^*}}, z^{\pm \frac{1}{r}}] \# \mathbb{C}[\mathcal{R}_Q],$$

where $\pi_V(h_{\varepsilon_j}) = E_{jj} - E_{-j-j}$ for the case $B_N^{(1)}$. Then one can verify that $\widehat{\mathcal{L}}^\pm(z)$ satisfy the RLL relations

$$(A.11) \quad \mathcal{R}^\pm(z, P + h) \widehat{\mathcal{L}}^\pm(z_1) \widehat{\mathcal{L}}^\mp(z_2) = \widehat{\mathcal{L}}^\pm(z_2) \widehat{\mathcal{L}}^\pm(z_1) \mathcal{R}^{\pm*}(z, P),$$

$$(A.12) \quad \mathcal{R}^\pm(zq^{\pm c}, P + h) \widehat{\mathcal{L}}^\pm(z_1) \widehat{\mathcal{L}}^\mp(z_2) = \widehat{\mathcal{L}}^\mp(z_2) \widehat{\mathcal{L}}^\pm(z_1) \mathcal{R}^{\pm*}(zq^{\mp c}, P).$$

These RLL -relations coincide with (3.19) and (3.20).

§ B. Integral Expressions for the Half Currents

For $X = E, F$, let us denote by $[X_{j_1}(v_1) \cdots X_{j_m}(v_m)]$ the product of the elliptic currents $X_{j_1}(v_1), \dots, X_{j_m}(v_m)$ where all the zero-modes, $w_j^{-\frac{P\alpha_j-1}{r^*}}$ of $E_j(v_j)$ ($j = 1, \dots, N-1$), $w_N^{-\frac{P\alpha_N-1/2}{r^*}}$ of $E_N(v_N)$, $w_j^{\frac{(P+h)\alpha_j-1}{r}}$ of $F_j(v_j)$ ($j = 1, \dots, N-1$) and $w_N^{\frac{(P+h)\alpha_N-1/2}{r}}$ of $F_N(v_N)$, are normally ordered, i.e. they are moved to the right of all $e_j(w_j)$ and $f_j(w_j)$ by using (2.5) and (2.6).

i) The $j \prec k \prec N$ case:

$$\begin{aligned}
 (B.1) \quad & [E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_k(v_k) \cdots E_N(v_N)E_N(v'_N) \cdots E_k(v'_k)] \\
 & = e_j(v_j)e_{j+1}(v_{j+1}) \cdots e_k(v_k) \cdots e_N(v_N)e_N(v'_N) \cdots e_k(v'_k) \prod_{m=j}^{k-2} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_{k-1}^{-\frac{P\alpha_{k-1}+1}{r^*}} \\
 & \quad \times w_k^{-\frac{P\alpha_k-1}{r^*}} \prod_{m=k+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_N^{-\frac{P\alpha_N-1/2}{r^*}} w'_N^{-\frac{P\alpha_N+1/2}{r^*}} \prod_{m=k+1}^{N-1} w'_m^{-\frac{P\alpha_m}{r^*}} \cdot w'_k^{-\frac{P\alpha_k-1}{r^*}},
 \end{aligned}$$

$$\begin{aligned}
 (B.2) \quad & [E_k(v'_k)E_{k+1}(v'_{k+1}) \cdots E_N(v'_N)E_N(v_N) \cdots E_k(v_k) \cdots E_k(v_j)] \\
 & = e_k(v'_k)e_{k+1}(v'_{k+1}) \cdots e_N(v'_N)e_N(v_N) \cdots e_k(v_k) \cdots e_k(v_j) \prod_{m=k}^{N-1} w'_m^{-\frac{P\alpha_m}{r^*}} \\
 & \quad \times w'_N^{-\frac{P\alpha_N-1/2}{r^*}} w_N^{-\frac{P\alpha_N+1/2}{r^*}} \prod_{m=j+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_j^{-\frac{P\alpha_j-1}{r^*}}.
 \end{aligned}$$

ii) The $j = k \prec N$ case:

$$\begin{aligned}
 (B.3) \quad & [E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_N(v_N)E_N(v'_N) \cdots E_j(v'_j)] \\
 & = e_j(v_j)e_{j+1}(v_{j+1}) \cdots e_N(v_N)e_N(v'_N) \cdots e_j(v'_j)w_j^{-\frac{P\alpha_j-1}{r^*}} \\
 & \quad \times \prod_{m=j+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_N^{-\frac{P\alpha_N-1/2}{r^*}} w'_N^{-\frac{P\alpha_N+1/2}{r^*}} \prod_{m=j+1}^{N-1} w'_m^{-\frac{P\alpha_m}{r^*}} \cdot w'_j^{-\frac{P\alpha_j-1}{r^*}},
 \end{aligned}$$

$$\begin{aligned}
 (B.4) \quad & [E_j(v'_j)E_{j+1}(v'_{j+1}) \cdots E_N(v_N)E_N(v_N) \cdots E_j(v_j)] \\
 & = e_j(v'_j)e_{j+1}(v'_{j+1}) \cdots e_N(v_N)e_N(v_N) \cdots e_j(v_j)w'_j^{-\frac{P\alpha_j-1}{r^*}} \\
 & \quad \times \prod_{m=j+1}^{N-1} w'_m^{-\frac{P\alpha_m}{r^*}} \cdot w'_N^{-\frac{P\alpha_N-1/2}{r^*}} w_N^{-\frac{P\alpha_N+1/2}{r^*}} \prod_{m=j+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_j^{-\frac{P\alpha_j-1}{r^*}}.
 \end{aligned}$$

iii) The $k \prec j \prec N$ case:

$$(B.5) \quad [E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_N(v_N)E_N(v'_N) \cdots E_j(v'_j) \cdots E_k(v'_k)] \\ = e_j(v_j)e_{j+1}(v_{j+1}) \cdots e_N(v_N)e_N(v'_N) \cdots e_j(v'_j) \cdots e_k(v'_k) \\ \times \prod_{m=j}^{N-1} w_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w_N^{-\frac{P_{\alpha_N-1/2}}{r^*}} w'_N^{-\frac{P_{\alpha_N+1/2}}{r^*}} \cdot \prod_{m=k+1}^{N-1} w'_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w'_k^{-\frac{P_{\alpha_k-1}}{r^*}},$$

$$(B.6) \quad [E_j(v'_k)E_{k+1}(v'_{k+1}) \cdots E_j(v'_j) \cdots E_N(v'_N)E_N(v_N) \cdots E_k(v_j)] \\ = e_j(v'_k)e_{k+1}(v'_{k+1}) \cdots e_j(v'_j) \cdots e_N(v'_N)e_N(v_N) \cdots e_k(v_j) \prod_{m=k}^{j-2} w'_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w'_{j-1}^{-\frac{P_{\alpha_{j-1}+1}}{r^*}} \\ \times w'_j^{-\frac{P_{\alpha_j-1}}{r^*}} \cdot \prod_{m=j+1}^{N-1} w'_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w'_N^{-\frac{P_{\alpha_N-1/2}}{r^*}} w_N^{-\frac{P_{\alpha_N+1/2}}{r^*}} \cdot \prod_{m=j+1}^{N-1} w_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w_j^{-\frac{P_{\alpha_j-1}}{r^*}}.$$

iv) The $k \prec j = N$ case:

$$(B.7) \quad [E_N(v_N)E_N(v'_N) \cdots E_k(v'_k)] \\ = e_N(v_N)e_N(v'_N) \cdots e_j(v'_j) \cdots e_k(v'_k) w_N^{-\frac{P_{\alpha_N-1/2}}{r^*}} w'_N^{-\frac{P_{\alpha_N+1/2}}{r^*}} \prod_{m=k+1}^{N-1} w'_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w'_k^{-\frac{P_{\alpha_k-1}}{r^*}},$$

$$(B.8) \quad [E_k(v'_k) \cdots E_N(v'_N)E_N(v_N)] \\ = e_k(v'_k) \cdots e_N(v'_N)e_N(v_N) \prod_{m=k}^{N-2} w'_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w'_{N-1}^{-\frac{P_{\alpha_{N-1}+1}}{r^*}} \cdot w'_N^{-\frac{P_{\alpha_k-3/2}}{r^*}} w_N^{-\frac{P_{\alpha_N-1/2}}{r^*}}.$$

v) The $j \prec k = N$ case:

$$(B.9) \quad [E_j(v_j) \cdots E_N(v_N)E_N(v'_N)] \\ = e_k(v_j) \cdots e_N(v_N)e_N(v'_N) \prod_{m=j}^{N-2} w_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w_{N-1}^{-\frac{P_{\alpha_{N-1}+1}}{r^*}} \cdot w_N^{-\frac{P_{\alpha_k-3/2}}{r^*}} w'_N^{-\frac{P_{\alpha_N-1/2}}{r^*}},$$

$$(B.10) \quad [E_N(v'_N)E_N(v_N) \cdots E_k(v_j)] \\ = e_N(v'_N)e_N(v_N) \cdots e_k(v_j) w'_N^{-\frac{P_{\alpha_N-1/2}}{r^*}} w_N^{-\frac{P_{\alpha_N+1/2}}{r^*}} \prod_{m=j+1}^{N-1} w_m^{-\frac{P_{\alpha_m}}{r^*}} \cdot w_j^{-\frac{P_{\alpha_j-1}}{r^*}}.$$

vi) The $j = k = N$ case:

$$(B.11) \quad [E_N(v_N)E_N(v'_N)] = e_N(v_N)e_N(v'_N) w_N^{-\frac{P_{\alpha_k-3/2}}{r^*}} w'_N^{-\frac{P_{\alpha_N-1/2}}{r^*}},$$

$$(B.12) \quad [E_N(v'_N)E_N(v_N)] = e_N(v'_N)e_N(v_N) w'_N^{-\frac{P_{\alpha_k-3/2}}{r^*}} w_N^{-\frac{P_{\alpha_N-1/2}}{r^*}}.$$

The $F_j(v_j)$'s counterpart of the product $[E_{j_1}(v_1) \cdots E_{j_m}(v_m)]$ is obtained by replacing $e_{j_k}(v_k)$ with $f_{j_k}(v_k)$, w_j with w_j^{-1} and r^* with r .

Conjecture B.1. For $1 \preceq j \prec l \preceq 0$,

$$\begin{aligned}
F_{j,l}^+(v) &= a_{j,l} \oint_{C_{j,l}^+} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j)] f_{j,l}^+(v, v_j \cdots, v_{l-1}, P+h) \\
&\quad + a_{j,l} \oint_{C_{j,l}^-} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_j(v_j) F_{j+1}(v_{j+1}) \cdots F_{l-1}(v_{l-1})] f_{j,l}^-(v, v_j \cdots, v_{l-1}, P+h), \\
F_{-l,-j}^+(v) &= a_{-l,-j} \oint_{C_{-l,-j}^+} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_j(v_j) F_{j+1}(v_{j+1}) \cdots F_{l-1}(v_{l-1})] f_{-l,-j}^+(v, v_j, \cdots, v_{l-1}, P+h) \\
&\quad + a_{-l,-j} \oint_{C_{-l,-j}^-} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j)] f_{-l,-j}^-(v, v_j \cdots, v_{l-1}, P+h), \\
E_{l,j}^+(v) &= a_{l,j}^* \oint_{C_{l,j}^{*+}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1})] g_{l,j}^+(v, v_j \cdots, v_{l-1}, P) \\
&\quad + a_{l,j}^* \oint_{C_{l,j}^{*-}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_{l-1}(v_{l-1}) E_{l-2}(v_{l-2}) \cdots E_j(v_j)] g_{l,j}^-(v, v_j \cdots, v_{l-1}, P), \\
E_{-j,-l}^+(v) &= a_{-j,-l}^* \oint_{C_{-j,-l}^{*+}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_{l-1}(v_{l-1}) E_{l-2}(v_{l-2}) \cdots E_j(v_j)] g_{-j,-l}^+(v, v_j, \cdots, v_{l-1}, P) \\
&\quad + a_{-j,-l}^* \oint_{C_{-j,-l}^{*-}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1})] g_{-j,-l}^-(v, v_j \cdots, v_{l-1}, P), \\
f_{j,l}^\pm(v, v_j \cdots, v_{l-1}, P+h) &= \frac{[v - v_{l-1} + (P+h)_{j,l} + \frac{l-1}{2} - 1][1]}{[v - v_{l-1} + \frac{l-1}{2}][(P+h)_{j,l} - 1]} \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m + (P+h)_{j,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m \pm \frac{1}{2}][(P+h)_{j,m+1}]}, \\
f_{-l,-j}^\pm(v, v_j \cdots, v_{l-1}, P+h) &= \frac{[v - v_j + (P+h)_{-l,-j} - \frac{j}{2} - \eta - 1 + \delta_{l,0}][1]}{[v - v_j - \frac{j}{2} - \eta][(P+h)_{-l,-j} - 1 + \delta_{l,0}]} \\
&\quad \times \prod_{m=j}^{l-2} \frac{[v_m - v_{m+1} + (P+h)_{-l,-(m+1)} - \frac{1}{2} + \delta_{l,0}][1]}{[v_m - v_{m+1} \pm \frac{1}{2}][(P+h)_{-l,-(m+1)} + \delta_{l,0}]}, \\
g_{l,j}^\pm(v, v_j \cdots, v_{l-1}, P) &= \frac{[v - v_{l-1} + \frac{l-1-c}{2} + 1P_{j,l}]^*[1]^*}{[v - v_{l-1} + \frac{l-1-c}{2}]^*[P_{j,l} - 1]^*} \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^*[1]^*}{[v_{m+1} - v_m \pm \frac{1}{2}]^*[P_{j,m+1} - 1]^*},
\end{aligned}$$

$$g_{-j,-l}^{\pm}(v, v_j, \dots, v_{l-1}, P) = \frac{[v-v_j-\frac{j+c}{2}-\eta+1-P_{-l,-j}-\delta_{l,0}]^*[1]^*}{[v-v_j-\frac{j+c}{2}-\eta]^*[P_{-l,-j}-1+\delta_{l,0}]^*} \prod_{m=j}^{l-2} \frac{[v_m-v_{m+1}-P_{-l,-(m+1)}+\frac{1}{2}-\delta_{l,0}]^*[1]^*}{[v_m-v_{m+1} \pm \frac{1}{2}]^*[P_{-l,-(m+1)}-1+\delta_{l,0}]^*}.$$

$$\begin{aligned} C_{j,l}^{\pm} &: |pq^l w| < |w_{l-1}| < |q^l w|, \quad |pq^{\pm 1} w_{m+1}| < |w_m| < |q^{\pm 1} w_{m+1}|, \\ C_{-l,-j}^{\pm} &: |pq^{-j+1} \xi w| < |w_j| < |q^{-j+1} \xi w|, \quad |pq^{\pm 1} w_m| < |w_{m+1}| < |q^{\pm 1} w_m|, \\ C_{l,j}^{*\pm} &: |q^{l+c} w| < |w_{l-1}| < |p^{*-1} q^{l+c} w|, \quad |q^{\pm 1} w_{m+1}| < |w_m| < |p^{*-1} q^{\pm 1} w_{m+1}|, \\ C_{-j,-l}^{*\pm} &: |q^{-j+1+c} \xi w| < |w_j| < |p^{*-1} q^{-j+1+c} \xi w|, \quad |q^{\pm 1} w_m| < |w_{m+1}| < |p^{*-1} q^{\pm 1} w_m| \end{aligned}$$

for $j \leq m \leq l-2$. Here $N+1 \equiv 0$. The case $l \neq 0$ can be proved in the same way as for $U_{q,p}(A_N^{(1)})$ [18].

The following is a conjectural expression for the half currents of the second type, which we obtained by requiring that the integrand should be single-valued and the vector representation of the L -operator should reproduce the R -matrix.

Conjecture B.2. *i) For $j \prec k \preceq N$,*

$$\begin{aligned} E_{-k,j}^+(v) &= a_{-k,j}^* \oint_{C_{-k,j}^{*+}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} \\ &\times [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_k(v_k) \cdots E_N(v_N) E_N(v'_N) \cdots E_k(v'_k)] g_{-k,j}^+(v, v_j, \dots, v'_k, P) \\ &+ a_{-k,j}^* \frac{[1]^*}{[\frac{1}{2}]^*} \oint_{C_{-k,j}^{*-}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} \\ &\times [E_k(v'_k) E_{k+1}(v'_{k+1}) \cdots E_N(v'_N) E_N(v_N) \cdots E_k(v_k) \cdots E_j(v_j)] g_{-k,j}^-(v, v_j, \dots, v'_k, P), \end{aligned}$$

$$\begin{aligned} g_{-k,j}^+(v, v_j, \dots, v'_k, P) &= \prod_{m=j+1}^{k-1} \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^*[1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^*[P_{j,m} - 1]^*} \\ &\times \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^*[1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^*[P_{j,-k} - 1]^*} \prod_{m=k+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^*[1]^*}{[v'_{m-1} - v'_m + \frac{1}{2}]^*[P_{j,-m} - 1]^*} \\ &\times \frac{[v'_N - v_N - P_{j,0} + \frac{1}{2}]^*[1]^*}{[v'_N - v_N - \frac{1}{2}]^*[P_{j,0}]^*} \prod_{m=k+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^*[1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^*[P_{j,m} - 1]^*} \frac{[v_k - v_{k-1} - P_{j,k} - \frac{1}{2}]^*[1]^*}{[v_k - v_{k-1} + \frac{1}{2}]^*[P_{j,k}]^*}. \end{aligned}$$

$$\begin{aligned} g_{-k,j}^-(v, v_j, \dots, v'_k, P) &= \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^*[1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^*[P_{j,-k} - 1]^*} \prod_{m=k+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^*[1]^*}{[v'_{m-1} - v'_m - \frac{1}{2}]^*[P_{j,-m} - 1]^*} \\ &\times \frac{[v'_N - v_N]^*}{[v'_N - v_N + \frac{1}{2}]^*} \frac{[v'_N - v_N - P_{j,0}]^*[1]^*}{[v'_N - v_N - \frac{3}{2}]^*[P_{j,0} - \frac{3}{2}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^*[1]^*}{[v_m - v_{m-1} - \frac{1}{2}]^*[P_{j,m} - 1]^*}. \end{aligned}$$

$$\begin{aligned}
C_{-k,j}^{*\pm} : & |q^{-k+1+c}\xi w| < |w'_k| < |p^{*-1}q^{-k+1+c}\xi w|, \\
& |q^{\pm 1}w'_{m-1}| < |w'_m| < |p^{*-1}q^{\pm 1}w'_{m-1}| \quad (k+1 \leq m \leq N), \\
& |q^{\mp 1}w'_N| < |w_N| < |p^{*-1}q^{\mp 1}w'_N|, \quad |q^{\pm 1}w_n| < |w_{n-1}| < |p^{*-1}q^{\pm 1}w_n| \quad (j+1 \leq n \leq N)
\end{aligned}$$

ii) For $j = k \prec N$, we obtain from (B.3) and (B.11),

$$\begin{aligned}
E_{-j,j}^+(v) &= a_{-j,j}^* \oint_{C_{-j,j}^{*+}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_N(v_N)E_N(v'_N) \cdots E_k(v'_j)] g_{-j,j}^+(v, v_j, \dots, v'_j, P) \\
&+ a_{-j,j}^* \frac{[1]^*}{[\frac{1}{2}]^*} \oint_{C_{-j,j}^{*-}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [E_j(v'_j)E_{j+1}(v'_{j+1}) \cdots E_N(v'_N)E_N(v_N) \cdots E_k(v_j)] g_{-j,j}^-(v, v_j, \dots, v'_j, P), \\
g_{-j,j}^+(v, v_j, \dots, v'_j, P) &= \frac{[v - v'_j - \frac{j+c}{2} - \eta - 2P_j + 1]^*[1]^*}{[v - v'_j - \frac{j+c}{2} - \eta]^*[2P_j - 1]^*} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^*[1]^*}{[v'_{m-1} - v'_m + \frac{1}{2}]^*[P_{j,-m} - 1]^*} \\
&\times \frac{[v'_N - v_N - P_{j,0} + \frac{1}{2}]^*[1]^*}{[v'_N - v_N - \frac{1}{2}]^*[P_{j,0}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^*[1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^*[P_{j,m} - 1]^*} \frac{[v - v_j + \frac{-j+1+c}{2} - \eta + 1]^*}{[v - v_j + \frac{-j+1+c}{2} - \eta]^*}, \\
g_{-j,j}^-(v, v_j, \dots, v'_j, P) &= \frac{[v - v'_j - \frac{j+c}{2} - \eta - 2P_j + 2]^*[1]^*}{[v - v'_j - \frac{j+c}{2} - \eta]^*[2P_j - 1]^*} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^*[1]^*}{[v'_{m-1} - v'_m - \frac{1}{2}]^*[P_{j,-m} - 1]^*} \\
&\times \frac{[v'_N - v_N]^*}{[v'_N - v_N + \frac{1}{2}]^*} \frac{[v'_N - v_N - P_{j,0}]^*[1]^*}{[v'_N - v_N - \frac{3}{2}]^*[P_{j,0} - \frac{3}{2}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^*[1]^*}{[v_m - v_{m-1} - \frac{1}{2}]^*[P_{j,m} - 1]^*}.
\end{aligned}$$

$$\begin{aligned}
C_{-j,j}^{*\pm} : & |q^{-j+1+c}\xi w| < |w'_j| < |p^{*-1}q^{-j+1+c}\xi w|, \quad |q^{\pm 1}w'_{m-1}| < |w'_m| < |p^{*-1}q^{\pm 1}w'_{m-1}|, \\
& |q^{\mp 1}w'_N| < |w_N| < |p^{*-1}q^{\mp 1}w'_N|, \quad |q^{\pm 1}w_m| < |w_{m-1}| < |p^{*-1}q^{\pm 1}w_m|
\end{aligned}$$

for $j+1 \leq m \leq N$ with

$$\begin{aligned}
& |q^{-j+1+c}\xi w| < |w_j| < |p^{*-1}q^{-j+1+c}\xi w| \quad \text{for } C_{-j,j}^{*+}, \\
& |q^{-3}w'_N| < |w_N| < |p^{*-1}q^{-3}w'_N| \quad \text{for } C_{-j,j}^{*-}.
\end{aligned}$$

iii) For $k \prec j \preceq N$, we obtain from (B.5)

$$\begin{aligned}
E_{-k,j}^+(v) &= a_{-k,j}^* \oint_{C_{-k,j}^{*+}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_N(v_N) E_N(v'_N) \cdots E_j(v'_j) \cdots E_k(v'_k)] g_{-k,j}^+(v, v_j, \dots, v'_k, P) \\
&+ a_{-k,j}^* \frac{[1]^*}{[\frac{1}{2}]^*} \oint_{C_{-k,j}^{*-}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [E_k(v'_k) E_{k+1}(v'_{k+1}) \cdots E_j(v'_j) \cdots E_N(v'_N) E_N(v_N) \cdots E_k(v_j)] g_{-k,j}^-(v, v_j, \dots, v'_k, P), \\
g_{-k,j}^+(v, v_j, \dots, v'_k, P) &= \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^* [1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^* [P_{j,-k} - 1]^*} \prod_{m=k+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m + \frac{1}{2}]^* [P_{j,-m} - 1]^*} \\
&\quad \times \frac{[v'_N - v_N - P_{j,0} + \frac{1}{2}]^* [1]^*}{[v'_N - v_N - \frac{1}{2}]^* [P_{j,0}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^* [P_{j,m} - 1]^*}, \\
g_{-k,j}^-(v, v_j, \dots, v'_k, P) &= \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^* [1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^* [P_{j,-k} - 1]^*} \\
&\quad \times \prod_{\substack{m=k+1 \\ m \neq j}}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m - \frac{1}{2}]^* [P_{j,-m} - 1]^*} \frac{[v'_{j-1} - v'_j - 2P_j + \frac{3}{2}]^* [1]^*}{[v'_{j-1} - v'_j - \frac{1}{2}]^* [2P_j - 1]^*} \\
&\quad \times \frac{[v'_N - v_N]^*}{[v'_N - v_N + \frac{1}{2}]^*} \frac{[v'_N - v_N - P_{j,0}]^* [1]^*}{[v'_N - v_N - \frac{3}{2}]^* [P_{j,0} - \frac{3}{2}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} - \frac{1}{2}]^* [P_{j,m} - 1]^*}.
\end{aligned}$$

$$\begin{aligned}
C_{-k,j}^{*\pm} : |q^{-k+1+c}\xi w| &< |w'_k| < |p^{*-1}q^{-k+1+c}\xi w|, \\
|q^{\pm 1}w'_{m-1}| &< |w'_m| < |p^{*-1}q^{\pm 1}w'_{m-1}| \quad (k+1 \leq m \leq N), \\
|q^{\mp 1}w'_N| &< |w_N| < |p^{*-1}q^{\mp 1}w'_N|, \quad |q^{\pm 1}w_n| < |w_{n-1}| < |p^{*-1}q^{\pm 1}w_n| \quad (j+1 \leq n \leq N),
\end{aligned}$$

in addition, for $C_{-k,j}^{*-}$

$$|q^{-3}w'_N| < |w_N| < |p^{*-1}q^{-3}w'_N|.$$

vi) For $j = k = N$, we obtain from (B.11) and (B.12),

$$\begin{aligned}
E_{-N,N}^+(v) &= a_{-N,N}^* \oint_{C_{-N,N}^{*+}} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [E_N(v_N) E_N(v'_N)] g_{-N,N}^+(v, v_N, v'_N, P) \\
&+ a_{-N,N}^* \oint_{C_{-N,N}^{*-}} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [E_N(v'_N) E_N(v_N)] g_{-N,N}^-(v, v_N, v'_N, P),
\end{aligned}$$

$$\begin{aligned}
g_{-N,N}^+(v, v_N, v'_N, P) &= \frac{[v - v'_N - \frac{N+c}{2} - \eta - 2P_N + \frac{3}{2}]^*[1]^*}{[v - v'_N - \frac{N+c}{2} - \eta]^*[2P_N - 1]^*} \\
&\quad \times \frac{[v'_N - v_N - P_{N,0} + 1]^*[1]^*}{[v'_N - v_N - \frac{1}{2}]^*[P_{N,0} - \frac{3}{2}]^*} \frac{[v - v_N - \frac{N+c}{2} - \eta + \frac{1}{2}]^*}{[v - v_N - \frac{N+c}{2} - \eta]^*}, \\
g_{-N,N}^-(v, v_N, v'_N, P) &= \frac{[v - v'_N - \frac{N+c}{2} - \eta - 2P_N + \frac{3}{2}]^*[1]^*}{[v - v'_N - \frac{N+c}{2} - \eta]^*[2P_N - 1]^*} \\
&\quad \times \frac{[v'_N - v_N - P_{N,0} + 1]^*[1]^*}{[v'_N - v_N + \frac{1}{2}]^*[P_{N,0} - \frac{3}{2}]^*} \frac{[v - v_N - \frac{N+c}{2} - \eta + \frac{1}{2}]^*}{[v - v_N - \frac{N+c}{2} - \eta]^*}.
\end{aligned}$$

$$C_{-N,N}^{*\pm} : |q^{-N+1+c}\xi w| < |w_N|, |w'_N| < |p^{*-1}q^{-N+1+c}\xi w|, |q^{\mp 1}w'_N| < |w_N| < |p^{*-1}q^{\mp 1}w'_N|.$$

Conjecture B.3. *i) For $j \prec k \preceq N$,*

$$\begin{aligned}
F_{k,-j}^+(v) &= a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k,-j}^+} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [F_k(v_k)F_{k+1}(v_{k+1}) \cdots F_N(v_N)F_N(v'_N) \cdots F_k(v'_k) \cdots F_j(v'_j)] f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) \\
&+ a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k,-j}^-} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [F_j(v'_j)F_{j+1}(v'_{j+1}) \cdots F_k(v'_k) \cdots F_N(v'_N)F_N(v_N) \cdots F_k(v_k)] f_{k,-j}^-(v, v_k, \dots, v'_j, P+h), \\
f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) &= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m - \frac{1}{2}][(P+h)_{k,-m}]} \\
&\quad \times \frac{[v'_N - v_N]}{[v'_N - v_N + \frac{1}{2}]} \frac{[v'_N - v_N + (P+h)_{k,0} - 2][1]}{[v'_N - v_N - \frac{3}{2}][(P+h)_{k,0} - \frac{1}{2}]} \prod_{m=k+1}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} - \frac{1}{2}][(P+h)_{k,m}]}, \\
f_{k,-j}^-(v, v_k, \dots, v'_j, P+h) &= \prod_{m=k+1}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} + \frac{1}{2}][(P+h)_{k,m}]} \\
&\quad \times \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{\substack{m=j+1 \\ \neq k}}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m + \frac{1}{2}][(P+h)_{k,-m}]} \\
&\quad \times \frac{[v'_{k-1} - v'_k + 2(P+h)_k - \frac{3}{2}][1]}{[v'_{k-1} - v'_k + \frac{1}{2}][2(P+h)_k - 1]} \frac{[v'_N - v_N + (P+h)_{k,0} - \frac{3}{2}][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{k,0} - 2]}.
\end{aligned}$$

$$\begin{aligned}
C_{k,-j}^{\pm} : & |pq^{\pm 1}w'_N| < |w_N| < |q^{\pm 1}w'_N|, \quad |pq^{\mp 1}w_n| < |w_{n-1}| < |q^{\mp 1}w_n| \quad (k+1 \leq n \leq N), \\
& |pq^{-j+1}\xi w| < |w'_j| < |q^{-j+1}\xi w|, \quad |pq^{\mp 1}w'_{m-1}| < |w'_m| < |q^{\mp 1}w'_{m-1}| \quad (j+1 \leq m \leq N).
\end{aligned}$$

in addition, for $C_{k,-j}^+$

$$|pq^{-3}w'_N| < |w_N| < |q^{-3}w'_N|.$$

ii) For $j = k \prec N$, we obtain from (B.3) and (B.11),

$$\begin{aligned}
F_{j,-j}^+(v) &= a_{j,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{j,-j}^+} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [F_j(v_j) F_{j+1}(v_{j+1}) \cdots F_N(v_N) F_N(v'_N) \cdots F_j(v'_j)] f_{j,-j}^+(v, v_j, \dots, v'_j, P+h) \\
&\quad + a_{j,-j} \oint_{C_{j,-j}^-} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [F_j(v'_j) F_{j+1}(v'_{j+1}) \cdots F_N(v'_N) F_N(v_N) \cdots F_j(v_j)] f_{j,-j}^-(v, v_j, \dots, v'_j, P+h), \\
f_{j,-j}^+(v, v_j, \dots, v'_j, P+h) &= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + 2(P+h)_j - 1][1]}{[v - v'_j - \frac{j}{2} - \eta][2(P+h)_j - 3]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{j,-m} - \frac{3}{2}][1]}{[v'_{m-1} - v'_m - \frac{1}{2}][(P+h)_{j,-m} - 1]} \\
&\quad \times \frac{[v'_N - v_N]}{[v'_N - v_N + \frac{1}{2}]} \frac{[v'_N - v_N + (P+h)_{j,0} - 3][1]}{[v'_N - v_N - \frac{3}{2}][(P+h)_{j,0} - \frac{3}{2}]} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} + (P+h)_{j,m} - \frac{3}{2}][1]}{[v_m - v_{m-1} - \frac{1}{2}][(P+h)_{j,m} - 1]}, \\
f_{j,-j}^-(v, v_j, \dots, v'_j, P+h) &= \frac{[v - v'_j - \frac{3j-2}{2} - 2\eta]}{[v - v'_j - \frac{3j-2}{2} - 2\eta - 1]} \\
&\quad \times \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + 2(P+h)_j - 2][1]}{[v - v'_j - \frac{j}{2} - \eta][2(P+h)_j - 3]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{j,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m + \frac{1}{2}][(P+h)_{j,-m}]} \\
&\quad \times \frac{[v'_N - v_N + (P+h)_{j,0} - \frac{3}{2}][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{j,0} - 2]} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} + (P+h)_{j,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} + \frac{1}{2}][(P+h)_{j,m}]}
\end{aligned}$$

$$\begin{aligned}
C_{j,-j}^\pm : |pq^{-j+1}\xi w| &< |w'_j| < |q^{-j+1}\xi w|, \quad |pq^{\mp 1}w'_{m-1}| < |w'_m| < |q^{\mp 1}w'_{m-1}|, \\
|pq^{\pm 1}w'_N| &< |w_N| < |q^{\pm 1}w'_N|, \quad |q^{\mp 1}w_m| < |w_{m-1}| < |q^{\mp 1}w_m|
\end{aligned}$$

$$\text{for } j+1 \leq m \leq N \text{ with } \begin{cases} |pq^{-3}w'_N| < |w_N| < |q^{-3}w'_N| & \text{for } C_{j,-j}^+, \\ |pq^{-3j-5}\xi^2 w| < |w'_j| < |q^{-3j-5}\xi^2 w| & \text{for } C_{j,-j}^-. \end{cases}$$

iii) $k \prec j \preceq N$

$$\begin{aligned}
F_{k,-j}^+(v) &= a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k,-j}^+} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [F_k(v_k) F_{k+1}(v_{k+1}) \cdots F_j(v_j) \cdots F_N(v_N) F_N(v'_N) \cdots F_j(v'_j)] f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) \\
&\quad + a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k,-j}^-} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} \\
&\quad \times [F_j(v'_j) F_{j+1}(v'_{j+1}) \cdots F_N(v'_N) F_N(v_N) \cdots F_j(v_j) \cdots F_k(v_k)] f_{k,-j}^-(v, v_k, \dots, v'_j, P+h),
\end{aligned}$$

$$\begin{aligned}
& f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) \\
&= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m - \frac{1}{2}][(P+h)_{k,-m}]} \\
&\times \frac{[v'_N - v_N]}{[v'_N - v_N + \frac{1}{2}]} \frac{[v'_N - v_N + (P+h)_{k,0} - 2][1]}{[v'_N - v_N - \frac{3}{2}][(P+h)_{k,0} - \frac{1}{2}]} \prod_{\substack{m=k+1 \\ \neq j}}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} - \frac{1}{2}][(P+h)_{k,m}]} \\
&\times \frac{[v_j - v_{j-1} + (P+h)_{k,j} + \frac{1}{2}][1]}{[v_j - v_{j-1} - \frac{1}{2}][(P+h)_{k,j}]}, \\
& f_{k,-j}^-(v, v_k, \dots, v'_j, P+h) \\
&= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m + \frac{1}{2}][(P+h)_{k,-m}]} \\
&\times \frac{[v'_N - v_N + (P+h)_{k,0} - \frac{3}{2}][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{k,0} - 2]} \prod_{m=k+1}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} + \frac{1}{2}][(P+h)_{k,m}]}.
\end{aligned}$$

$$\begin{aligned}
C_{k,-j}^\pm : & |pq^{\pm 1}w'_N| < |w_N| < |q^{\pm 1}w'_N|, \quad |pq^{\mp 1}w_n| < |w_{n-1}| < |q^{\mp 1}w_n| \quad (k+1 \leq n \leq N), \\
& |pq^{-j+1}\xi w| < |w'_j| < |q^{-j+1}\xi w|, \quad |pq^{\mp 1}w'_{m-1}| < |w'_m| < |q^{\mp 1}w'_{m-1}| \quad (j+1 \leq m \leq N),
\end{aligned}$$

in addition, for $C_{k,-j}^+$

$$|pq^{-3}w'_N| < |w_N| < |q^{-3}w'_N|.$$

vi) $j = k = N$

$$\begin{aligned}
F_{N,-N}^+(v) &= a_{N,-N} \oint_{C_{N,-N}^+} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [F_N(v'_N)F_N(v_N)] f_{N,-N}^+(v, v_N, v'_N, P+h) \\
&+ a_{N,-N} \oint_{C_{N,-N}^-} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [F_N(v_N)F_N(v'_N)] f_{N,-N}^-(v, v_N, v'_N, P+h),
\end{aligned}$$

$$\begin{aligned}
f_{N,-N}^+(v, v_N, v'_N, P+h) &= \frac{[v - v'_N - \frac{N}{2} - \eta + 2(P+h)_N - \frac{5}{2}][1]}{[v - v'_N - \frac{N}{2} - \eta][2(P+h)_N - 3]} \\
&\times \frac{[v'_N - v_N + (P+h)_{N,0} - 2][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{N,0} - \frac{3}{2}]} \frac{[v - v_N - \frac{N-1}{2} - \eta + \frac{1}{2}]}{[v - v_N - \frac{N-1}{2} - \eta]},
\end{aligned}$$

$$\begin{aligned}
f_{N,-N}^-(v, v_N, v'_N, P+h) &= \frac{[v - v'_N - \frac{N}{2} - \eta + 2(P+h)_N - \frac{5}{2}][1]}{[v - v'_N - \frac{N}{2} - \eta][2(P+h)_N - 3]} \\
&\times \frac{[v'_N - v_N + (P+h)_{N,0} - 2][1]}{[v'_N - v_N + \frac{1}{2}][(P+h)_{N,0} - \frac{3}{2}]} \frac{[v - v_N - \frac{N-1}{2} - \eta + \frac{1}{2}]}{[v - v_N - \frac{N-1}{2} - \eta]}.
\end{aligned}$$

$$C_{N,-N}^\pm : |pq^{-N+1}\xi w| < |w_N|, |w'_N| < |q^{-N+1}\xi w|, \quad |pq^{\mp 1}w'_N| < |w_N| < |q^{\mp 1}w'_N|.$$

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