

コンパクト次数 cmp に関する de Groot と Nishiura の問題

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1 Introduction

A regular space X is called *rim-compact* if there exists a base $\mathcal{B}$ for the open sets of X such that the boundary $\text{Bd } U$ is compact for each U in $\mathcal{B}$.

In 1942 de Groot (cf. [1]) proved the following:

(*) *A separable metrizable space X is rim-compact if and only if there is a metrizable compactification Y of X such that $\text{ind}(Y \setminus X) \leq 0$.*

In an attempt to generalize (*), de Groot introduced two notions, *the small inductive compactness degree* cmp and *the compactness definiency* def (we will recall the definitions in Section 2 and Section 3 respectively). It is known that the inequality $\text{cmp } X \leq \text{def } X$ holds for every separable metrizable space X . The well known conjecture of de Groot (see for example [4]) was that the two invariants coincide in the class of separable metrizable spaces. As a way either to disprove or to support the conjecture de Groot and Nishiura [4] posed the following:

Question 1.1 *Let $Z_n = [0, 1]^{n+1} \setminus (0, 1)^n \times \{0\}$. Is it true that $\text{cmp } Z_n \geq n$ for $n \geq 3$?*

In the quoted article, de Groot and Nishiura proved that $\text{def } Z_n = n$ for every $n \geq 1$, and they also stated that $\text{cmp } Z_i = i$ for $i = 1, 2$.

In [9], R. Pol constructed a space $P \subset R^4$ such that $\text{cmp } P = 1 < \text{def } P = 2$. The space P is a modification of an example given by Luxemburg [7] of a compactum with noncoinciding transfinite inductive dimensions. After that, some other counterexamples to the de Groot's conjecture were constructed by Hart (cf. [1]), Kimura [6], Levin and Segal [8]). However, Question 1.1 remained open (see also [10, Question 418] and [1, Problem 3, page 71]).

One of our main results is the following.

Theorem 1.1 *Let $n \leq 2^m - 1$ for some integer m . Then $\text{cmp } Z_n \leq m + 1$. In particular $\text{cmp } Z_n < \text{def } Z_n$ for $n \geq 5$.*

This is the answer to Question 1.1 for $n \geq 5$. Our paper is based on a construction of examples of compacta with noncoinciding transfinite inductive dimensions given in [2]. Our terminology follows [5] and [1].

2 Finite sum theorem for $\mathcal{P}$ -ind

In this part, topological spaces are assumed to be regular T_1 and all classes of topological spaces considered are assumed to be nonempty and to contain any space homeomorphic with a closed subspace of one of their members. The letter $\mathcal{P}$ is used to denote such classes.

Recall the definition of *the small inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$ -ind*. Let X be a space.

- (i) $\mathcal{P}$ -ind $X = -1$ iff $X \in \mathcal{P}$;
- (ii) $\mathcal{P}$ -ind $X \leq n$ (≥ 0) if each point in X has arbitrarily small neighbourhoods V with $\mathcal{P}$ -ind $\text{Bd } V \leq n - 1$.
- (iii) $\mathcal{P}$ -ind $X = n$ if $\mathcal{P}$ -ind $X \leq n$ and $\mathcal{P}$ -ind $X > n - 1$;
- (iv) $\mathcal{P}$ -ind $X = \infty$ if $\mathcal{P}$ -ind $X > n$ for $n = -1, 0, 1, \dots$

It is clear that if $\mathcal{P} = \{\emptyset\}$ then $\mathcal{P}$ -ind $X = \text{ind } X$. If $\mathcal{P}$ is the class of compact spaces then $\mathcal{P}$ -ind $X = \text{cmp } X$.

The following is a list of properties of $\mathcal{P}$ -ind we shall use in the paper.

- (1) If A is closed in X then $\mathcal{P}$ -ind $A \leq \mathcal{P}$ -ind X .
- (2) If $\mathcal{P}$ -ind $X \leq n \geq 0$ and U is open in X then $\mathcal{P}$ -ind $U \leq n$.
- (3) If $X = O_1 \cup O_2$, where O_i is open in $X, i = 1, 2$, and $\max\{\mathcal{P}$ -ind $O_1, \mathcal{P}$ -ind $O_2\} \leq n \geq 0$. Then $\mathcal{P}$ -ind $X \leq n$.
- (4) $\mathcal{P}$ -ind $X \leq n \geq 0$ iff for each point p and for each closed set G of X with $p \notin G$ there is a partition S between p and G such that $\mathcal{P}$ -ind $S \leq n - 1$.

The following statement is contained implicitly in the proofs of [2, Theorem 3.9] and [3, Theorem 2].

Lemma 2.1 . *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed in X , and A, B be two closed disjoint subsets of X such that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose a partition C_1 in X_1 between the sets $A \cap X_1$ and $B \cap X_1$ such that $X_1 \setminus C_1 = U_1 \cup V_1$, where U_1, V_1 are open in X_1 and disjoint, and $A \cap X_1 \in U_1, B \cap X_1 \subset V_1$. Choose also a partition C_2 in X_2 between the the sets $A \cap X_2$ and $((C_1 \cup V_1) \cup B) \cap X_2$ such that $X_2 \setminus C_2 = U_2 \cup V_2$, where U_2, V_2 are open in X_2 and disjoint, and $A \cap X_2 \in U_2, (C_1 \cup V_1) \cup B \cap X_2 \subset V_2$. Then the set $C = X \setminus (((U_1 \setminus X_2) \cup U_2) \cup (V_1 \cup (V_2 \setminus X_1)))$ is a partition in X between the sets A and B such that $C \subset C_1 \cup C_2 \cup (X_1 \cap X_2)$.*

Moreover, if X is a regular T_1 -space then the same statement is valid for a pair of closed subsets of X , where one of the sets is a point.

The following theorem and corollary are generalizations of [3, Theorem 2] and [2, Corollary 3.10 (a)] respectively.

Theorem 2.1 *Let X be a space such that $X = X_1 \cup X_2$, where X_i is closed in X and $\mathcal{P}\text{-ind } X_i \leq n \geq 0$ for every $i = 1, 2$. Then $\mathcal{P}\text{-ind } X \leq n + 1$.*

Moreover, if the space X is normal then for any closed subsets A and B of X there exists a partition C between A and B such that $\mathcal{P}\text{-ind } C \leq n$.

Corollary 2.1 *Let X be a space and q be an integer. If $X = \bigcup_{k=1}^{n+1} X_k$, where each X_k is closed in X , $0 \leq n \leq 2^m - 1$ for some integer m and $\max\{\mathcal{P}\text{-ind } X_k\} \leq q \geq 0$ then $\mathcal{P}\text{-ind } X \leq q + m$.*

For every normal space X one assigns the large inductive compactness degree $\text{Cmp } X$ as follows (cf. [1]).

- (i) For $n = -1$ or 0 , $\text{Cmp } X = n$ iff $\text{cmp } X = n$.
- (ii) $\text{Cmp } X \leq n \geq 1$ if each pair of disjoint closed subsets A and B of X there exists a partition C such that $\text{Cmp } C \leq n - 1$.
- (iii) $\text{Cmp } X = n$ if $\text{Cmp } X \leq n$ and $\text{Cmp } X > n - 1$.
- (iv) $\text{Cmp } X = \infty$ if $\text{Cmp } X > n$ for every natural number n .

It is clear that the following properties of Cmp are valid.

1. If A is closed in X then $\text{Cmp } A \leq \text{Cmp } X$.
2. If X is a sum of closed subsets $X_i, i = 1, 2$, then $\text{Cmp } X = \max\{\text{Cmp } X_1, \text{Cmp } X_2\}$.

Corollary 2.2 *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed in X and $\text{Cmp } X_i \leq 0$ for every i . Then $\text{Cmp } X \leq 1$. Moreover, if $\text{Cmp } (X_1 \cap X_2) = -1$ then $\text{Cmp } X \leq 0$; if $\text{Cmp } X_1 = -1$ then $\text{Cmp } X = \text{Cmp } X_2$.*

Now we are ready to prove the following theorem.

Theorem 2.2 *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed for $i = 1, 2$. Then $\text{Cmp } X \leq \max\{\text{Cmp } X_1, \text{Cmp } X_2\} + \text{Cmp } (X_1 \cap X_2) + 1 \leq \text{Cmp } X_1 + \text{Cmp } X_2 + 1$.*

Proof. Put $\text{Cmp } (X_1 \cap X_2) = k$ and $\max\{\text{Cmp } X_1, \text{Cmp } X_2\} = m$. Observe that $k \leq m$. Let $k = -1$. First we will prove the theorem for any $m \geq -1$ ($k = -1$). By Corollary 2.2 the statement is valid for $m = -1$ and $m = 0$. Assume that our theorem is valid for $m < p \geq 1$. Put $m = p$. Consider two disjoint closed subsets A and B of X . We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose partitions $C_i, i = 1, 2$, as we

did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p-1$. Denote $Y_1 = C_1 \cup C_2$ (recall that C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\text{Cmp } (Y_1 \cap Y_2) = -1$, $\text{Cmp } Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p-1$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \leq p-1$. By inductive assumption, $\text{Cmp } Y \leq \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \leq -1 + (p-1) + 1 = p-1$. By Lemma 2.1 there is a partition C between A and B in X such that $C \subset Y$. Hence, $\text{Cmp } X \leq p = k + m + 1$.

Assume that our theorem is valid for any pair $(k, m) : k < q \geq 0$ and $k \leq m$.

Put $k = q$. Consider the case $m = k \geq 0$. If $k = m = 0$ then $\text{Cmp } X_i \leq 0$ for every $i = 1, 2$, and by Corollary 2.2, $\text{Cmp } X \leq 1 = k + m + 1$. Let $k = m = q \geq 1$. Consider two disjoint closed subsets A and B of X . We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose partitions $C_i, i = 1, 2$, as we did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq q-1$. Denote $Y_1 = C_1 \cup C_2$ (C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\text{Cmp } Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq q-1$, $\text{Cmp } (Y_1 \cap Y_2) \leq \min\{q, q-1\} = q-1 < q$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \leq q$. By inductive assumption, $\text{Cmp } Y \leq \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \leq q + (q-1) + 1 = 2q$. By Lemma 2.1 there is a partition C between A and B in X such that $C \subset Y$. Hence, $\text{Cmp } X \leq 2q + 1 = k + m + 1$.

Assume that our theorem is valid for any $m : k \leq m < p \geq 1$ ($k=q$). Put $m = p$. Consider two disjoint closed subsets A and B of X . We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose partitions $C_i, i = 1, 2$, as we did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p-1$. Denote $Y_1 = C_1 \cup C_2$ (C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\text{Cmp } Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p-1$, $\text{Cmp } (Y_1 \cap Y_2) \leq \min\{q, p-1\} = q$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \leq p-1$. By inductive assumption, $\text{Cmp } Y \leq \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \leq q + (p-1) + 1 = q + p$. By Lemma 2.1 there is a partition C between A and B in X such that $C \subset Y$. Hence, $\text{Cmp } X \leq q + p + 1 = k + m + 1$.

Corollary 2.3 *Let X be a normal space with $\text{Cmp } X = n \geq 1$. Then*

- (a) *X cannot be represented as a union of n many closed subsets $P_1, P_2, \dots, P_n$ with $\text{Cmp } P_i \leq 0$ for each i .*

Furthermore, we suppose now that $X = \bigcup_{i=1}^{n+1} Z_i$, where each Z_i is closed and $\text{Cmp } Z_i \leq 0$ for every $i = 1, \dots, n+1$, then we have

- (b) *$\text{Cmp } (Z_1 \cup \dots \cup Z_{k+1}) = k$ for any k with $0 \leq k \leq n$;*
- (c) *$\text{Cmp } ((Z_1 \cup \dots \cup Z_{1+i}) \cap (Z_{i+2} \cup \dots \cup Z_{i+j+2})) = \min \{i, j\}$ for any nonnegative integers i, j such that $i + j + 1 \leq n$.*

Remark. The estimations from Corollary 2.2 and Theorem 2.2 can not be improved (see Corollary 3.3).

3 Spaces with $\text{cmp} \neq \text{def}$ ($\text{cmp} \neq \text{Cmp}$).

The deficiency def is defined in the following way: For a separable metrizable space X ,

$$\text{def } X = \min\{\text{ind}(Y \setminus X) : Y \text{ is a metrizable compactification of } X\}.$$

In this section, the concept of B -special decomposition introduced in [2] essentially works. A decomposition $X = F \cup \bigcup_{i=1}^{\infty} E_i$ of a metric space X into disjoint sets is called B -special if E_i is clopen in X and $\lim_{i \rightarrow \infty} \delta(E_i) = 0$, where $\delta(A)$ is the diameter of A .

The following proposition is easily obtained by use of [2, Lemma 2.3].

Proposition 3.1 *Let $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be a B -special decomposition of a metric space X and $n \geq 0$ be an integer. If $\max\{\mathcal{P}\text{-ind } F, \mathcal{P}\text{-ind } E_i\} \leq n$ then $\mathcal{P}\text{-ind } X \leq n$.*

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all i and $\lim_{i \rightarrow \infty} x_i = 0$. Put $C^n = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [x_{2i}, x_{2i-1}]) \subset I^{n+1}$.

Theorem 3.1 (a) *There are closed subsets $X_1, X_2, \dots, X_{n+1}$ of C^n such that $C^n = \bigcup_{k=1}^{n+1} X_k$ and $\text{cmp } X_k = 0$ for each $k = 1, 2, \dots, n+1$.*

(b) *The equalities $\text{def } C^n = \text{Cmp } C^n = n$ ($= \text{Comp } C^n$) hold (see [1] for the definition of Comp).*

(c) *Let m be an integer such that $0 \leq n \leq 2^m - 1$. Then we have $\text{cmp } C^n \leq m$. In particular $\text{cmp } C^n < \text{Cmp } C^n = \text{def } C^n$ for $n \geq 3$.*

Proof. (a) For every i choose finite systems $B_k^i, k = 1, \dots, n+1$, consisting of disjoint compact subsets of I^n with diameter $< \frac{1}{i}$ such that $I^n = \bigcup_{k=1}^{n+1} (\bigcup B_k^i)$. We put $X_k = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} ((\bigcup B_k^i) \times [x_{2i}, x_{2i-1}])$ for every $k = 1, \dots, n+1$. Observe that the space X_k admits a B -special decomposition into compact subsets and, by Proposition 3.1, $\text{cmp } X_k = 0$ for every $k = 1, \dots, n+1$.

(b) It is enough to prove that $\text{Comp } C^n \geq n$ i.e. there exist n pairs $(F_1, G_1), \dots, (F_n, G_n)$ of disjoint compact subsets of C^n such that for any partitions S_i between F_i and G_i in $X, i = 1, \dots, n$, the intersection $S_1 \cap \dots \cap S_n$ is not compact. (Recall that for every separable metrizable space W we have $\text{Comp } W \leq \text{Cmp } W \leq \text{def } W$ (cf. [1]) and evidently $\text{def } C^n \leq n$.) For example such pairs are $((\{0\} \times I^n) \cap C^n, (\{1\} \times I^n) \cap C^n), \dots, ((I^{n-1} \times \{0\} \times [0, 1]) \cap C^n, (I^{n-1} \times \{1\} \times [0, 1]) \cap C^n)$.

Moreover, for any partition C between $(\{0\} \times I^n) \cap C^n$ and $(\{1\} \times I^n) \cap C^n$ in C^n , $\text{Comp } C \geq n - 1$.

(c) One can show (c) by applying Corollary 2.1 for cmp and the statement (a).

Now we are ready to show Theorem 1.1.

Proof of Theorem 1.1. Decompose the space $Z_n, n \geq 3$, into the union of two closed subsets Z_n^1 and Z_n^2 (each of them is homeomorph to C^n), where $Z_n^1 = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [1/(2i+1), 1/(2i)])$, $Z_n^2 = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [1/(2i), 1/(2i-1)])$.

Let m be the integer such that $0 \leq n \leq 2^m - 1$. It follows from Theorem 3.1 (c) that $\text{cmp } Z_n^i \leq m$ for $i = 1, 2$. Thus, by Corollary 2.1, we have $\text{cmp } Z_n \leq m + 1$.

Corollary 3.1 (a) For the space C^2 we have $\text{cmp } C^2 = \text{cmp } (C^2 \times [0, 1]) = 2$.

(b) $\text{cmp } C^3 = 2$.

The following question is discussed in [1, Problem 6, page 71].

Question 3.1 For any k and m with $0 < k < m$, does there exist a separable metrizable space X such that $\text{cmp } X = k$ and $\text{def } X = m$?

We shall partially answer the question as follows.

Corollary 3.2 Let m be an integer and $l(m) = [\log_2(m)] + 1$. Then for every k with $m \geq k \geq l(m)$ there exists a separable metrizable space X such that $\text{cmp } X = k$ and $\text{def } X = m$.

Let C^n be the space defined above and $X_1, X_2, \dots, X_{n+1}$ be closed subsets of C^n described in Theorem 3.1. It follows from Theorem 3.1 (a) and Corollary 2.3 that $\text{Cmp } (X_1 \cup \dots \cup X_{k+1}) = k$ for each k with $0 \leq k \leq n$. However, we do not know the value of the deficiency of $X_1 \cup \dots \cup X_{k+1}$. So we can ask the following.

Question 3.2 Is it true that $\text{def } (X_1 \cup \dots \cup X_{k+1}) = k$ for $1 \leq k < n$?

The question might be interesting when we consider a problem posed by Aarts and Nishiura [1, Problem 6, page 71]: Exhibit a separable metrizable space X such that $\text{cmp } X < \text{Cmp } X < \text{def } X$. If the Question 3.1 would be answered negatively for example for the case of $n = 4$ and $k = 3$, then we have $\text{def } (X_1 \cup X_2 \cup X_3 \cup X_4) = 4$. We put $Y = X_1 \cup X_2 \cup X_3 \cup X_4$. Then, by the argument above, we have $\text{Cmp } Y = 3$. On the other hand, by Theorem 3.1 (a) and Corollary 2.1, it follows that $\text{cmp } Y \leq 2$. Hence $\text{cmp } Y < \text{Cmp } Y < \text{def } Y$. Even if the Question 3.1 would be answered positively, then one gets an interesting counterpart of Corollary 3.3 (see below) for def .

Now we will obtain a complement to Theorem 2.2 showing the exactness of the theorem's estimations.

Corollary 3.3 For any integer $n \geq 1$ there exists a compact space $X_n (= C^n)$ with $\text{Cmp } X_n = n$ such that for any nonnegative integers p, q with $p + q = n - 1$ there exist its closed subsets $X_n^{(p)}$ and $X_n^{(q)}$ such that $X_n = X_n^{(p)} \cup X_n^{(q)}$, $\text{Cmp } X_n^{(p)} = p$, $\text{Cmp } X_n^{(q)} = q$ and $\text{Cmp } (X_n^{(p)} \cap X_n^{(q)}) = \min \{p, q\}$.

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