

Dual Graphs of Curves on Surfaces of Class VII_0

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§0 Introduction. A compact complex surface is in class VII_0 if the first Betti number of it is equal to one and if it is minimal. We know many examples of surfaces of class VII_0 (in short VII_0 surfaces) with curves — minimal surfaces with global spherical shells [2],[6],[7]. Some of them are characterized by the existence of certain curves [1],[4],[5]. Here we recall a table of classification of VII_0 surfaces from [4],[5].

(0.1) Table.

curves	structure of surfaces
1) (more than) 3 elliptic curves	elliptic VII_0 surfaces
2) two elliptic curves	Hopf surfaces
3) an elliptic curve, no cycles	Hopf surfaces
4) an elliptic curve and a cycle	parabolic Inoue surfaces
5) two cycles	hyperbolic Inoue surfaces
6) a cycle C with $C^2=0$	$S_{n,\beta,t}$ (see [1])
7) a cycle C with $C^2<0$	
7-1) $b_2(S)=b_2(C)$	half Inoue surfaces
7-2) $b_2(S)>b_2(C)$	examples exist
8) no elliptic curves, no cycles	?

It is necessary to study the class 7) in the table in more detail in order to complete the classification of VII_0 surfaces. The purpose of this note is to complete the LHS of 7) in (0.1) under the assumption that the VII_0 surfaces has at least (in fact, then exactly) b_2 rational curves. Here we make a brief review on curves on VII_0 surfaces. An irreducible curve on a VII_0 surface is either a nonsingular rational curve, or an elliptic curve, or a rational curve with a node. If a VII_0 surface has a cycle of rational curves, then it has positive b_2 and no meromorphic functions except constants. The number of elliptic curves and cycles of rational curves on a VII_0 surface is at most two in total if the surface has no meromorphic functions except constants. If a VII_0 surface has a cycle C of rational curves with $C^2 < 0$, then there are no elliptic curves on the surface. If moreover it has another cycle of rational curves, then (0.1) 5) asserts that it is a hyperbolic Inoue surface. Therefore in (0.1) 7), any VII_0 surface has a unique cycle C with $C^2 < 0$, but no elliptic curves.

Let S be a VII_0 surface in (0.1) 7). Assume moreover that S has b_2 rational curves. If the unique cycle C on S has no branches (see (2.1)), then S is a half Inoue surface (see (2.4)). If C has branches, then our main theorem asserts that the dual graph of all the curves on S is the same as one of those of known surfaces — minimal surfaces with global spherical shells. We also give a uniform characterization of Inoue surfaces with $b_2 > 0$ by the Delzant invariant (§3).

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Notations.

$\mathcal{O}_S$	the sheaf of germs over S of holomorphic functions
Ω_S^1	the sheaf of germs over S of holomorphic one forms
K_S	the canonical line bundle of S
$c_1(L)$	the first Chern class of a line bundle L
$b_2(E)$	#(irreducible components of a reduced divisor E)
$b_i(S)$	the i -th Betti number of S
$H^q(S, F)$	q -th cohomology group of S with F coefficients
$h^q(S, F)$	$\dim H^q(S, F)$
$\chi(S, F)$	$\sum_{q=0}^{\infty} (-1)^q h^q(S, F)$

§1 Smoothability of a cycle by deformations

(1.1) Theorem. Let S be a VII_0 surface with $b_2 > 0$. Then

$$H^2(S, \theta_S) = 0.$$

Proof. Assume the contrary to derive a contradiction. By Serre duality, $H^0(S, \Omega_S^1(K_S)) \neq 0$, K_S being the canonical line bundle of S . Let D be the maximum effective divisor of S such that $H^0(S, \Omega_S^1(K_S - D)) \neq 0$ and let ω be a nonzero element of $H^0(S, \Omega_S^1(K_S - D))$. By definition, $\text{zero}(\omega)$ is isolated. Then we have,

(1.1.1) Lemma. The following is exact;

$$0 \rightarrow O_S(K_S - D) \xrightarrow{f} \Omega_S^1 \xrightarrow{g} O_S(2K_S - D)$$

where $f(a) = a\omega$, $g(b) = b \wedge \omega$.

Proof. Clearly f is injective. Take $b \in \text{Ker } g$. Then $b \wedge \omega = 0$. Hence $b = h\omega$ locally for a germ h of a meromorphic function. Then $\text{pole}(h) \subset \text{zero}(\omega) = \text{isolated}$, whence h is holomorphic. Hence h is contained in $\text{Im } f$. $\square$

Let H be $\text{Coker } g$. Then the following is exact;

$$0 \rightarrow O_S(K_S - D) \rightarrow \Omega_S^1 \rightarrow O_S(2K_S - D) \rightarrow H \rightarrow 0$$

We remark that $\text{supp}(H)$ is isolated points, so that $H^1(S, H) = 0$. We also see $h^0(S, \Omega_S^1) = h^2(S, \Omega_S^1) = 0$, $h^1(S, \Omega_S^1) = b_2$ by [3], or [5, p.405]. Therefore by taking Euler-Poincare characteristics, we see

$$\begin{aligned} b_2 &= -\chi(S, \Omega_S^1) = -\chi(S, -K_S + D) - \chi(S, 2K_S - D) + h^0(S, H) \\ &= -2\chi(S, -K_S + D) + h^0(S, H) \\ &= -2K_S^2 + 3K_S D - D^2 + h^0(S, H) \\ &= 2b_2 + 3K_S D - D^2 + h^0(S, H) \quad (\text{by } -K_S^2 = b_2), \end{aligned}$$

and therefore,

$$b_2 + 3K_S D - D^2 + h^0(S, H) = 0.$$

(1.1.2) Lemma. $K_S D \geq 0$, $D^2 \leq 0$.

See [3] or [5, p. 399, (2.1.2) and (2.2)].

Consequently, $b_2 = 0$, $K_S D = D^2 = h^0(S, H) = 0$. This contradicts the assumption $b_2 > 0$. q.e.d.

(1.2) Theorem. Let S be a VII_0 surface with C a cycle of rational curves. Let E be a reduced divisor containing C . Then $H^2(S, \theta_S(-\log E)) = 0$.

Proof. Assume the contrary. Then as in (1.1), we see

$$4 \geq b_2(E) + (K_S E + E^2)/2.$$

Let $E = C + H$, $H = \sum H_j$, H_j irreducible. Then

$$\begin{aligned} b_2(E) + (K_S E + E^2)/2 &= b_2(C) + CH + b_2(H) + (K_S H + H^2)/2 \\ &= b_2(C) + CH + \sum_{i < j} H_i H_j \end{aligned}$$

whence $4 \geq b_2(C)$. Take a fivefold unramified covering $\pi: S^* \rightarrow S$. Then since $H^2(S^*, \theta_{S^*}(-\log E^*)) \neq 0$ for $E^* = \pi^* E$, we see $4 > b_2(C^*)$, C^* being $\pi^* C$. However $b_2(C^*) = 5b_2(C)$ since π is five-sheeted. This is a contradiction. q.e.d.

A geometric consequence of this theorem is

(1.3) Theorem. Let S be a VII_0 surface with C a cycle of rational curves, H a reduced divisor with no common components with C , $E = C + H$. Then there is a smooth proper family $\pi: S \rightarrow \Delta$ with π -flat divisors C and H of S such that

$$(1.3.1) \quad (S_0, C_0, H_0) \cong (S, C, H),$$

$$(1.3.2) \quad H_t = H \text{ (together with embeddings) for any } t \in \Delta$$

(1.3.3) $\omega := \pi|_C: C \rightarrow \Delta$ is a versal deformation of C with natural isomorphism $\text{Ext}^1(\Omega_C^1, \mathcal{O}_C) \cong T_0(\Delta)$.

(1.3.4) S_t is either a blown-up Hopf surface or a blown-up parabolic Inoue surface if C_t is an elliptic curve.

We note that (1.3.4) is a consequence of (0.1). See [5, (12.1) and (12.2)]. Since any parabolic Inoue surface is deformed into a blown-up Hopf surface, we may assume that S_t in (1.3.4) is a blown-up primary Hopf surface for generic t .

(1.4) Corollary. Let S, C and H_j be the same as in (1.3). Assume moreover that S is not a half Inoue surface, and $C^2 < 0$.

Then there are line bundles $L_j \in H^1(S, O_S^*)$, $E_j = c_1(L_j) \in H^2(S, \mathbb{Z})$, ($1 \leq j \leq n$), $n = b_2(S)$ such that

(1.4.1) E_j ($1 \leq j \leq n$) is a $\mathbb{Z}$ basis of $H^2(S, \mathbb{Z})$,

(1.4.2) $K_S L_j = -1$, $L_j L_k = -\delta_{jk}$,

(1.4.3) $C = -(L_{r+1} + \dots + L_n)$, $K_S = L_1 + \dots + L_n$ for some $1 \leq r \leq n-1$.

(1.4.4) $c_1(H_j) = E_{j,1} - (E_{j,2} + \dots + E_{j,k_j})$, $E_{j,k}$ being one of E_i ($1 \leq i \leq n$), all distinct for j fixed.

(1.5) Remark. If $C^2 = 0$, $c_1(C) = 0$, but $C \neq 0$ in $H^1(S, O_S^*)$.

If S is a half Inoue surface, then $C^2 < 0$, $c_1(C) = -c_1(K_S) = -(E_1 + \dots + E_n)$, $r=0$, but $C \neq -K_S$. In fact, $K_S + C$ is a line bundle of order two. See [5, §9].

§2 Dual graphs of curves

(2.1) Definition. A VII_0 surface S with $b_2 > 0$ is said to be special if S has at least b_2 rational curves.

It is still unknown whether there are un-special VII_0 surfaces with $b_2 > 0$.

We notice that the maximum number of (possibly singular) rational curves on a VII_0 surface is $b_2 (= b_2(S))$, the second Betti number of S (Ma. Kato, see [5, (3.5)] for the proof.)

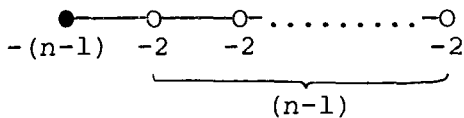
Any special VII_0 surface has a cycle of rational curves.

Any (minimal) surface with a global spherical shell is special.

Let C be a cycle of rational curves. A branch D of C is by definition a reduced connected divisor with $CD = 1$ having no common irreducible components with C .

As a simplest case, we shall prove the following

(2.2) Theorem. If a special VII_0 surface S has a rational curve C with a node having branches, then the dual graph of b_2 rational curves is



where $n = b_2(S)$, $\bullet$ (resp. $\circ$) denotes C (resp. a nonsingular $\mathbb{P}^1$).

Proof. By (1.4), $C = -(L_{r+1} + \dots + L_n)$, $K_S = L_1 + \dots + L_n$, $1 \leq r \leq n-1$.

By the assumption, there is a branch of C . So we take a nonsingular rational curve D with $CD=1$. We assume $c_1(D) = a_1 E_1 + \dots + a_n E_n$. Since $2 + K_S D + D^2 = 0$, we have $\sum_i (a_i^2 + a_i) = 2$. Hence there is a unique i such that $a_i^2 + a_i = 2$ and except i , $a_j^2 + a_j = 0$.

Since $c_1(D) = E_{r+1} - (E_1 + \dots)$, we have $c_1(D) = E_2 - E_1$. We also have $C = -(L_2 + \dots + L_n)$. Let D' and D'' be the other rational curves on S . Then both D' and D'' are nonsingular rational curves as was explained in the introduction. By (1.4.4), we have

$$c_1(D') = E_j - (E_{j_1} + \dots + E_{j_s}),$$

$$c_1(D'') = E_k - (E_{k_1} + \dots + E_{k_t})$$

for some j, j_h, k, k_h . By $CD' \geq 0, DD' \geq 0$, we have $j \neq 1, 2$ and $s=1$ or $s=2$. Similarly $k \neq 1, 2$, $t=1$ or 2 and moreover $j \neq k$. By the assumption that S is special, there are $(n-2)$ rational curves D_j ($3 \leq j \leq n$) on S . By the above observation, we can index

$$c_1(D_j) = E_j - E_{I_j}$$

for a subset I_j of $\{1, \dots, n\} \setminus \{j\}$. By $D_j^2 \leq -2$, I_j is nonempty.

(2.2.4) Lemma. $\# I_j = 1$.

We omit the proof. See [8].

Assuming (2.2.4), it is not difficult to see that $c_1(D_j) = E_j - E_{j-1}$ ($2 \leq j \leq n$), where $D_2 = D$. The proof is done simply by $D_j D_k \geq 0, CD_j \geq 0$ for $j \neq k$. (For instance, if $n=3$, $c_1(D_3) \neq E_3 - E_1$ because $D_2 D_3 = (E_2 - E_1)(E_3 - E_1) = -1$ is absurd. Hence $c_1(D_3) = E_3 - E_2$.) Thus we complete the proof of (2.2). q.e.d.

After preparing various lemmas, we are able to prove

(2.3) Theorem. Let S be a special VII_0 surface with C a unique cycle of rational curves having branches. Then the dual graph of b_2 rational curves is the same as one of those of minimal surfaces with global spherical shells.

Any graph in (2.3) is given explicitly. In (2.3), C^2 is

negative because $c_1(C)=0$, hence C has no branches, if $C^2=0$.

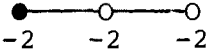
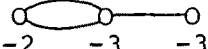
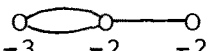
(2.4) Theorem. Let S be a special VII_0 surface with C a unique cycle of rational curves. Assume that $C^2 < 0$ and that C has no branches. Then S is isomorphic to a half Inoue surface.

(2.5) Corollary. Let S be a special VII_0 surface with C a cycle of rational curves. Then either $C^2 = b_2(C) - b_2(S)$, or $C^2 = -b_2(C) = -b_2(S)$ and S is a half Inoue surface.

In (2.5), if S has two cycles of rational curves, the assertion is a part of the strange duality.

(2.6) Problem. Give a direct proof of (2.5).

(2.7) Example. Let S be a special VII_0 surface with $b_2=3$, and a unique cycle with branches. Then there are three cases;

dual graph	homological expressions of curves (with numbering from the left)
	$C_1 = -E_2 - E_3, \quad D_2 = E_2 - E_1, \quad D_3 = E_3 - E_2$
	$C_1 = E_1 - E_2, \quad C_2 = E_2 - E_1 - E_3, \quad D_3 = E_3 - E_1 - E_2$
	$C_1 = E_1 - E_2 - E_3, \quad C_2 = E_2 - E_1, \quad D_3 = E_3 - E_2,$

where $\bullet$ (resp. $\circ$) stands for a rational curve with a node (resp. a nonsingular rational curve).

§3 A theorem.

(3.1) Definition. Let S be a VII_0 surface with $b_2 > 0$. The Dloussky invariant $Dl(S)$ of S is

$$Dl(S) = - \sum D^2 + \#(\text{rational curves with a node})$$

where D ranges over all the irreducible curves on S .

(3.2) Theorem. Let S be a VII_0 surface with $b_2 > 0$. Then

$$Dl(S) \leq 3b_2(S),$$

equality holding if and only if S is an Inoue surface with $b_2 > 0$.

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