

ON THE CANONICAL MODULES

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A ring will always mean a commutative noetherian ring with unit. Let R be a ring, M a finitely generated R -module and N a submodule of M . We denote by $\text{Min}_R(M)$ the set of minimal elements in $\text{Supp}_R(M)$ and put $U_M(N) = \bigcap Q$ where Q runs through all the primary components of N in M such that $\dim M/Q = \dim M/N$. Let T be an R -module and $\underline{a}$ an ideal of R . $E_R(T)$ denotes an injective envelope of T and $H_{\underline{a}}^i(T)$ is the i -th local cohomology module of T with respect to $\underline{a}$. We denote by $\hat{}$ the Jacobson radical adic completion over a semi-local ring. For a ring R , $Q(R)$ denotes the total quotient ring of R . Throughout this note A denotes a local ring of dimension d and with maximal ideal $\underline{m}$.

Definition ([7, Definition 5.6]). An A -module K is called a canonical module of A if $K \otimes_A \hat{A} \cong \text{Hom}_A(H_{\underline{m}}^d(A), E_A(A/\underline{m}))$.

For elementary properties of canonical modules, we refer the reader to [6, §6], [7, 5 Vortrag und 6 Vortrag] and [2, §1]. It is not obvious that the localization of a canonical module is a canonical module of the localization ring, which was known only

for local rings with dualizing complexes, and Ogoma [9] showed that there is a non-acceptable (hence without dualizing complex) local ring with canonical module. Our purposes are to prove that $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ for every $\underline{p}$ in $\text{Supp}_A(K)$ (A is a local ring with canonical module K) and to consider endomorphism rings of canonical modules.

Lemma 1(Corollary to [5, Theorem 1]). Let B be a faithfully flat local A -algebra with maximal ideal $\underline{n}$. Then:

- (1) If $B/\underline{m}B$ is an artinian Gorenstein ring, then $E_A(A/\underline{m}) \otimes_A B \cong E_B(B/\underline{n})$.
- (2) If T is an A -module such that $T \otimes_A B \cong E_B(B/\underline{n})$, then $T \cong E_A(A/\underline{m})$ and $B/\underline{m}B$ is an artinian Gorenstein ring.

Theorem 2([4]). Assume that A has a canonical module K and let B be a faithfully flat local A -algebra. Then the following are equivalent:

- (a) $B/\underline{m}B$ is a Gorenstein ring.
- (b) $K \otimes_A B$ is a canonical module of B and $B/\underline{m}B$ is a Cohen-Macaulay ring.

(Proof) Suppose that $B/\underline{m}B$ is a Cohen-Macaulay ring and let $y_1, \dots, y_r$ be a system of elements in $\underline{n}$, the maximal ideal of B , which is a maximal $B/\underline{m}B$ -regular sequence ($r = \dim B/\underline{m}B$). Let $R = A[X_1, \dots, X_r]_{(\underline{m}, X_1, \dots, X_r)}$ with indeterminates $X_1, \dots, X_r$ over A and let f be the natural A -algebra homomorphism from R to B such that $f(X_i) = y_i$ for $i = 1, \dots, r$. Then f is a flat local homomorphism. By [7, Korollar 5.12], $C = K \otimes_A R$ is a canonical module of R . Hence we may assume that $B/\underline{m}B$

is artinian. Furthermore we may assume that A and B are both complete. In this case it is shown that $K \otimes_A B$ is a canonical module of B if and only if $E_A(A/\underline{m}) \otimes_A B \cong E_B(B/\underline{n})$ ([2, Proof of Proposition 4.1]). Hence the assertion follows from Lemma 1. (Q.E.D.)

Suppose that A has a canonical module K . Let M be a finitely generated A -module and h_M the natural map from M to $\text{Hom}_A(\text{Hom}_A(M, K), K)$.

Proposition 3 ([2, (1.11)]). The following are equivalent:

- (a) The map h_M is an isomorphism.
- (b) $\hat{M}$ is (S_2) and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\text{Min}_A(M)$.

Corollary 4 ([1, Proposition 2]). $A \cong \text{Hom}_A(K, K)$ if and only if $\hat{A}$ is (S_2) .

Next we show some elementary properties of the endomorphism ring of a canonical module. Assume that A has a canonical module K and put $H = \text{End}_A(K)$.

Theorem 5 ([2, Theorem 3.2]). The following statements hold for H :

- (1) H is a semi-local ring which is a finitely generated A -module and $A/U \subseteq H \subseteq Q(A/U)$ where $U = U_A(0) = \text{ann}_A(K)$.
- (2) Every maximal chain of prime ideals in H is of length d .
- (3) $\hat{H}$ is (S_2) .
- (4) For every maximal ideal $\underline{n}$ of H , $K_{\underline{n}}$ is a canonical module of $H_{\underline{n}}$. (K is an H -module by the usual way.)
- (5) $\dim_A \text{Coker}(A \rightarrow H) \leq d - 2$.

(Proof) We may assume that $\text{ann}_A(K) = U_A(0) = 0$.

(1) Let $\underline{p}$ be a prime ideal of A with $\dim A/\underline{p} = d$ and $\underline{q}$ a minimal prime ideal of $\hat{A}$. Then $\dim \hat{A}/\underline{q} = d$ and $\hat{K}_{\underline{q}}$ is a canonical module of $\hat{A}_{\underline{q}}$. Since $\dim \hat{A}_{\underline{q}} = 0$, $\hat{K}_{\underline{q}} \cong E_{\hat{A}}(\hat{A}/\underline{q})$. Since $K_{\underline{p}} \otimes_{A_{\underline{p}}} \hat{A}_{\underline{q}} \cong \hat{K}_{\underline{q}}$, $K_{\underline{p}} \cong E_A(A/\underline{p})$ by Lemma 1(2). Let $\text{Ass}(A) = \{\underline{p}_1, \dots, \underline{p}_t\}$ and $S = A \setminus \bigcup_{i=1}^t \underline{p}_i$, the set of non-zero-divisors of A . Since K is torsion free, so is H and the natural map $H \rightarrow S^{-1}H$ is injective. Since $S^{-1}K \cong \bigoplus_{i=1}^t K_{\underline{p}_i} \cong \bigoplus_{i=1}^t E_A(A/\underline{p}_i)$, $S^{-1}H \cong \text{Hom}_A(S^{-1}K, S^{-1}K) \cong \bigoplus_{i=1}^t A_{\underline{p}_i} \cong Q(A)$.

(2) Because A is unmixed.

(3) Because $\hat{K}$ is (S_2) .

(4) The map $h_K : K \rightarrow \text{Hom}_A(H, K)$ is an isomorphism by Proposition 3. Hence the assertion follows from [7, Satz 5.12] and (3).

(5) We may assume that A is complete. Let $\underline{p}$ be a prime ideal such that $\text{height } \underline{p} \leq 1$. Then $A_{\underline{p}}$ is Cohen-Macaulay and $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ because A is complete and $U_A(0) = 0$. Hence $A_{\underline{p}} = H_{\underline{p}}$, that is, $\text{Coker}(A \rightarrow H)_{\underline{p}} = 0$, which means $\dim_A \text{Coker}(A \rightarrow H) \leq d - 2$. (Q.E.D.)

Theorem 6 ([2, Theorem 4.2]). Let $(A, \underline{m}) \rightarrow (B, \underline{n})$ be a flat local homomorphism and M an A -module. If $M \otimes_A B$ is a canonical module of B , then M is a canonical module of A .

Corollary 7 ([2, Corollary 4.3]). Assume that A has a canonical module K and let $\underline{p}$ be an element of $\text{Supp}_A(K)$. Then $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ and $\hat{A}_{\underline{q}}/\underline{p}\hat{A}_{\underline{q}}$ is a Gorenstein ring for every minimal prime ideal $\underline{q}$ of $\hat{A}$.

Before proving Theorem 6, we show two lemmas.

Lemma 8. Assume that A is complete. Let T be a finitely generated (S_2) A -module such that $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\text{Min}_A(T)$ and $H_{\underline{m}}^d(T) \cong E_A(A/\underline{m})$. Then T is a canonical module of A . In this case A is (S_2) .

(Proof) By Proposition 3, the map h_T is an isomorphism. Since $\text{Hom}_A(T, K) \cong \text{Hom}_A(H_{\underline{m}}^d(T), E_A(A/\underline{m})) \cong \text{Hom}_A(E_A(A/\underline{m}), E_A(A/\underline{m})) \cong A$, $T \cong \text{Hom}_A(A, K) \cong K$, a canonical module of A . (Q.E.D.)

Lemma 9. Let R be a finite over-ring of A such that $\dim_A R/A \leq d-2$ and $\dim R_{\underline{p}} = d$ for every maximal ideal $\underline{p}$ of R . If T is a finitely generated R -module such that $T_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ for every maximal ideal $\underline{p}$ of R , then T , as an A -module, is a canonical module of A .

(Proof) We may assume that A is complete. For every maximal ideal $\underline{p}$ of R , $\text{Hom}_A(R, K)_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ by [7, Satz 5.12] (K is a canonical module of A). Hence $T_{\underline{p}} \cong \text{Hom}_A(R, K)_{\underline{p}}$ for every maximal ideal $\underline{p}$ of R and therefore $T \cong \text{Hom}_A(R, K)$. Since $\dim_A R/A \leq d-2$, we have $\text{Hom}_A(R/A, K) = 0$ and $\text{Ext}_A^1(R/A, K) = 0$ (cf. [2, (1.10)]). Hence, from the exact sequence $0 \rightarrow A \rightarrow R \rightarrow R/A \rightarrow 0$, we have $\text{Hom}_A(R, K) \cong \text{Hom}_A(A, K) \cong K$, a canonical module of A . (Q.E.D.)

(Proof of Theorem 6) We may assume that A and B are both complete and $\underline{m}B$ is $\underline{n}$ -primary. Let K (resp. L) be a canonical module of A (resp. B).

(I) The case that B is (S_2) : Since B is (S_2) , $B \cong \text{Hom}_B(L, L)$, i.e., $H_{\underline{n}}^d(L) \cong E_B(B/\underline{n})$. Since $H_{\underline{m}}^d(M) \otimes_A B \cong H_{\underline{n}}^d(M \otimes_A B) \cong H_{\underline{n}}^d(L) \cong E_B(B/\underline{n})$, $H_{\underline{m}}^d(M) \cong E_A(A/\underline{m})$ by Lemma 1(2). Since L is (S_2) ,

so is M . Since $\text{Ass}_B(L) = \{ \underline{q} \in \text{Spec}(B) \mid \dim B/\underline{q} = d \}$, $\text{Ass}_A(M) = \{ \underline{p} \in \text{Spec}(A) \mid \dim A/\underline{p} = d \}$. Hence we have $M \cong K$ by Lemma 8.

(II) The general case: Since $\text{Ass}_A(M) = \{ \underline{p} \in \text{Spec}(A) \mid \dim A/\underline{p} = d \}$ and $M_{\underline{p}} \cong E_A(A/\underline{p})$ for every $\underline{p}$ in $\text{Ass}_A(M)$ (cf. Proof of Theorem 5(1)), we have $\text{ann}_A(M) = U_A(0)$. Hence we may assume that $U_A(0) = 0$ and $U_B(0) = 0$. Put $R = \text{End}_A(M)$ and $S = \text{End}_B(L)$. Since $R \otimes_A B \cong S$ is a finite over-ring of B , R is a finite over-ring of A . For every maximal ideal $\underline{p}$ of R , $\dim R_{\underline{p}} = d$ because A is unmixed. We have $\dim_A R/A \leq d-2$ because $\dim_B S/B \leq d-2$. Let $\underline{p}$ be a maximal ideal of R and $\underline{q}$ a maximal ideal of S lying over $\underline{p}$. Since $M_{\underline{p}} \otimes_{R_{\underline{p}}} S_{\underline{q}} \cong L_{\underline{q}}$ is a canonical module of $S_{\underline{q}}$ by Theorem 5(4) and $S_{\underline{q}}$ is (S_2) by Theorem 5(3), $M_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ by the case (I). Hence we have that M is a canonical module of A by Lemma 9. (Q.E.D.)

Remark. Goto (Nihon University) proved the following lemma and gave another proof of Theorem 6. ([3, Appendix])

Lemma. Let $(A, \underline{m}) \rightarrow (B, \underline{n})$ be a flat local homomorphism such that $\underline{m}B$ is $\underline{n}$ -primary. If there is a finitely generated A -module T such that $T \otimes_A B$ is a canonical module of B , then $B/\underline{m}B$ is a Gorenstein ring.

By virtue of Corollary 7, we can prove the following proposition by induction on $\dim A$ (cf. [1, Proof of Proposition 2]). Assume that A has a canonical module K . For a finitely generated A -module M , h_M denotes the natural map from M to

$\text{Hom}_A(\text{Hom}_A(M, K), K)$.

Proposition 10([2, Proposition 4.4]). The following are equivalent:

- (a) The map h_M is an isomorphism.
- (b) $\hat{M}$ is (S_2) and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\text{Min}_A(M)$.
- (c) M is (S_2) and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\text{Min}_A(M)$.

Corollary 11([9, Proposition 4.2] and [4]). The following are equivalent:

- (a) $A \cong \text{Hom}_A(K, K)$.
- (b) $\hat{A}$ is (S_2) .
- (c) A is (S_2) .

Remark. The implication (c) $\Rightarrow$ (a) was first proved by Ogoma (Kochi University), not by induction. (See [9, §4]. cf. [3, (二)])

Corollary 12([4]). Assume that A has a canonical module and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\text{Min}(A)$. Then the (S_2) -locus of A is open in $\text{Spec}(A)$.

Corollary 13([4]). Assume that A has a canonical module. Let $(A, \underline{m}) \rightarrow (B, \underline{n})$ be a flat local homomorphism such that $B/\underline{n}B$ is a Gorenstein ring.

- (1) Let M be a finitely generated (S_2) A -module such that $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\text{Min}_A(M)$. Then $M \otimes_A B$ is (S_2) and $\dim B/\underline{q} = \dim B$ for every $\underline{q}$ in $\text{Min}_B(M \otimes_A B)$.
- (2) If A is (S_2) , then B is also (S_2) .

Next we show that the endomorphism ring of a canonical module is characterized by the properties described in Theorem 5.

Theorem 14([4]). Assume that A has a canonical module K .

Let R be a ring satisfying the following conditions:

- (i) R is a finite (S_2) over-ring of $A/U_A(0)$,
- (ii) For every maximal ideal $\underline{n}$ of R , $\dim R_{\underline{n}} = d$, and
- (iii) $\dim_A \text{Coker}(A \rightarrow R) \leq d - 2$.

Then $R \cong \text{End}_A(K)$ as A -algebras.

(Proof) We may assume that $U_A(0) = 0$. Put $L = \text{Hom}_A(R, K)$. Then $L_{\underline{n}}$ is a canonical module of $R_{\underline{n}}$ for every maximal ideal $\underline{n}$ of R . By Lemma 9, we have $L \cong K$. From this isomorphism, we have an A -algebra isomorphism $\text{End}_A(K) \xrightarrow{\sim} \text{End}_A(L)$. Since $\text{End}_A(K)$ is commutative, so is $\text{End}_A(L)$ and $\text{End}_A(L) = \text{End}_R(L)$. Since R is (S_2) , $R \cong \text{End}_R(L)$. Hence we have $R \cong \text{End}_A(K)$ as A -algebras. (Q.E.D.)

In the following we assume that A has a canonical module K , $d \geq 2$ and $U_A(0) = 0$. Put $H = \text{End}_A(K)$ and $\underline{c} = A :_A H$, the conductor. Let T be the $\underline{c}$ -transform of A , i.e., $T = \{ x \in Q(A) \mid \underline{c}^t x \subseteq A \text{ for some } t \}$. Let $\underline{q}$ be a prime ideal of $\hat{A}$ containing $\underline{c}\hat{A}$ and $\underline{p}$ an associated prime ideal of $\hat{A}_{\underline{q}}$. Since $U_{\hat{A}}(0) = U_A(0)\hat{A} = 0$ and $\text{height } \underline{c} \geq 2$, we have $\dim \hat{A}_{\underline{q}}/\underline{p} \geq 2$. Hence by [8, Proposition(2.7)] we have:

(15.1) T is a finitely generated A -module.

The following two assertions are obvious:

(15.2) $\dim_A T/A \leq d - 2$.

(15.3) T is (S_2) .

Hence, from Theorem 14, we obtain the following

Proposition 16([4]). $T \cong H$ as A -algebras.

We denote by A^g the global transform of A , i.e., $A^g = \{ x \in Q(A) \mid \underline{m}^t x \subseteq A \text{ for some } t \}$. Since $U_A(0) = 0$ and $d \geq 2$, A^g is a finitely generated A -module by [8, Proposition (2.3)].

Corollary 17([4]). $A^g \cong H$ as A -algebras if and only if $\text{depth } A_{\underline{p}} \geq \min \{ 2, \dim A_{\underline{p}} \}$ for every non-maximal prime ideal $\underline{p}$ of A . In particular, if $H_{\underline{m}}^i(A)$ is of finite length for $i \neq d$, $A^g \cong H$ as A -algebras.

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