

On Kato's inequality for the relativistic Schrödinger operators with magnetic fields *

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This lecture deals with whether *Kato's inequality* holds for the *magnetic* relativistic Schrödinger operator H_A with vector potential $A(x)$ and mass $m \geq 0$ associated with the classical relativistic Hamiltonian symbol $\sqrt{(\xi - A(x))^2 + m^2}$ such as

$$\operatorname{Re}[(\operatorname{sgn} u)H_A u] \geq \sqrt{-\Delta + m^2} |u|, \quad (1)$$

in the distribution sense, for u is in $L^2(\mathbf{R}^d)$ with $H_A u$ in $L^1_{\text{loc}}(\mathbf{R}^d)$.

In the literature there are three magnetic relativistic Schrödinger operators associated with the classical symbol (1) (e.g. [I12], [I13]). The first two $H_A^{(1)}$ and $H_A^{(2)}$ are to be defined as pseudo-differential operators: for $f \in C_0^\infty(\mathbf{R}^d)$,

$$(H_A^{(1)} f)(x) := \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) dy d\xi, \quad (2)$$

$$(H_A^{(2)} f)(x) := \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\left(\xi - \int_0^1 A((1-\theta)x + \theta y) d\theta\right)^2 + m^2} f(y) dy d\xi. \quad (3)$$

The third $H_A^{(3)}$ is defined as the square root of the nonnegative selfadjoint (nonrelativistic Schrödinger) operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbf{R}^d)$:

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2}. \quad (4)$$

$H_A^{(1)}$ is the so-called Weyl pseudo-differential operator ([ITa 86], [I89]). $H_A^{(2)}$ is a modification of $H_A^{(1)}$ given in [IfMP 07], and $H_A^{(3)}$ used in [LSei 10] to discuss *relativistic stability of matter*.

All these three operators are nonlocal operators, and, under suitable condition on $A(x)$, become selfadjoint. For $A = 0$ we put $H_0 = \sqrt{-\Delta + m^2}$, where $-\Delta$ is the *minus-signed* Laplacian in $\mathbf{R}^d$. $H_A^{(2)}$ and $H_A^{(3)}$ are gauge-covariant, but not $H_A^{(1)}$.

Inequality (1) for $H_A^{(1)}$ has been shown in [I89], [ITs 76], and similarly will be for $H_A^{(2)}$.

For $H_A^{(3)}$, we assume that $d \geq 2$, as in case $d = 1$ any magnetic vector potential can be removed by a gauge transformation. We want to show

Theorem 1 (Kato's inequality). *Let $m \geq 0$ and assume that $A \in [L^2_{\text{loc}}(\mathbf{R}^d)]^d$. Then if u is in $L^2(\mathbf{R}^d)$ with $H_A^{(3)} u$ in $L^1_{\text{loc}}(\mathbf{R}^d)$, then the distributional inequality holds:*

$$\operatorname{Re}[(\operatorname{sgn} u)H_A^{(3)} u] \geq \sqrt{-\Delta + m^2} |u|, \quad (5)$$

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or

$$\operatorname{Re}[(\operatorname{sgn} u)H_A^{(3)}u] \geq [\sqrt{-\Delta + m^2} - m]|u|. \quad (6)$$

Here $(\operatorname{sgn} u)(x) := \overline{u(x)}/|u(x)|$, if $u(x) \neq 0$; $= 0$, if $u(x) = 0$.

From Theorem 1 follows the following corollary.

Corollary (Diamagnetic inequality) (cf. [FLSei 08], [HILo 12, 13]) *Let $m \geq 0$ and assume that $A \in [L_{\text{loc}}^2(\mathbf{R}^d)]^d$. Then $f, g \in L^2(\mathbf{R}^d)$*

$$|(f, e^{-t[H_A^{(3)} - m]}g)| \leq (|f|, e^{-t[H_0 - m]}|g|). \quad (7)$$

Once Theorem 1 is established, we can apply it to show the following theorem on essential selfadjointness of the relativistic Schrödinger operator with both vector and scalar potentials $A(x)$ and $V(x)$:

$$H := H_A^{(3)} + V. \quad (8)$$

Theorem 2. *Let $m \geq 0$ and assume that $A \in [L_{\text{loc}}^2(\mathbf{R}^d)]^d$. If $V(x)$ is in $L_{\text{loc}}^2(\mathbf{R}^d)$ with $V(x) \geq 0$ a.e., then $H = H_A^{(3)} + V$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ and its unique selfadjoint extension is bounded below by m .*

The characteristic feature is that, unlike $H_A^{(1)}$ and $H_A^{(2)}$, $H_A^{(3)}$ is, since being defined as an operator square root (4), neither an integral operator nor a pseudo-differential operator associated with a certain tractable symbol. $H_A^{(3)}$ is, under the condition of the theorem, essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ so that $H_A^{(3)}$ has domain

$$D[H_A^{(3)}] = \{u \in L^2(\mathbf{R}^d); (i\nabla + A(x))u \in L^2(\mathbf{R}^d)\},$$

which contains $C_0^\infty(\mathbf{R}^d)$ as an operator core. Although we can know the domain of $H_A^{(3)}$ is determined, the point which becomes crucial is in how to derive regularity of the weak solution $u \in L^2(\mathbf{R}^d)$ of equation

$$H_A^{(3)}u \equiv \sqrt{(-i\nabla - A(x))^2 + m^2}u = f, \quad \text{for given } f \in L_{\text{loc}}^1(\mathbf{R}^d).$$

We shall show inequality (5)/(6), modifying the method used in the case ([I 89], [ITs 92]) for the Weyl pseudo-differential operator $H_A^{(1)}$, basically along the idea of Kato's original proof for the magnetic nonrelativistic Schrödinger operator $\frac{1}{2}(-i\nabla - A(x))^2$ in [K 72]. However, the present case seems to be not so simple as to need much further modification within "operator theory *plus alpha*".

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