

On the construction and classification of almost complex curves in a nearly Kähler 6-sphere

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This report is based on a joint work with Prof H. Hashimoto(Meijo University), Dr. T. Taniguchi(Tohoku University) (cf. [HTU]).

It is well known that a standard 6-dimensional sphere S^6 has a nearly Kähler structure. We denote it by J . Although a submanifold whose tangent space is invariant under the action of J is 2-dimensional or 4-dimensional, there is no 4-dimensional J -invariant submanifold by the result of A. Gray([Gr]). Therefore, the only possible case of J -invariant submanifold is that of immersed surface. We express the J -invariant surface as the image of almost complex conformal immersion of some Riemann surface M . In this case, we denote it by $f : M \rightarrow S^6$. On the other hand, (totally real or CR) 3-dimensional submanifold can be often constructed as a tube of some radius over some almost complex curve(cf. Mashimo's article in this volume). For example, Ejiri immersion : $S^3(\frac{1}{16}) \rightarrow S^6$ can be realized as a tube of radius $\frac{\pi}{2}$ in the direction of second normal space over almost complex curve $S^2(\frac{1}{6}) \rightarrow S^6$. Almost complex curves of S^6 is divided into the four types of the following : (I) linearly full and superminimal in S^6 , (II) linearly full and non-superminimal in S^6 , (III) linearly fully immersed in some totally geodesic 5-dimensional sphere S^5 (which is necessarily non-superminimal), (IV) totally geodesic almost complex 2-sphere.

Since the automorphism group of the nearly Kähler structure is the exceptional Lie group G_2 , S^6 can be expressed as a homogeneous space $S^6 = G_2/SU(3)$, which is a 3-symmetric space. For Type (I), it can be lifted to a horizontal holomorphic curve in $Q^5 = G_2/U(2)$ which is the twister space over $S^6 = G_2/SU(3)$. Bryant([Br]) gave the representation formula for almost complex curve of type (I) using this twister space. For types (II) and (III), Bolton-Pedit-Woodward([BPW]) showed that f has a Toda-framing into a 6-symmetric space $\tilde{f} : M \rightarrow G_2/\mathcal{T}^2$, where $\mathcal{T}^2$ is the maximal torus of $SU(3)$. From these points of views, we may consider the following problem : "Construct and classify the cases of type (II) and (III)". For the classification of type (II) and (III), there are some pioneering works by Bolton-Vrancken-Woodward([BVW]). In this note, we present some construction and classification of almost complex 2-tori of type (III).

1. Primitive map of finite type into 6-symmetric space

Theorem 1.1([BPW]). *Any almost complex 2-torus of type (II) or (III) $f :$*

$T^2 \longrightarrow S^6$ can be lifted to a primitive map of finite type into 6-symmetric space $G_2/\mathcal{T}^2$.

This theorem, together with the results of McIntosh([Mc1], [Mc2]), says that any almost complex 2-torus can be constructed from the spectral data. However, since the spectral data describes the moduli space of primitive map of finite type from 2-tori, it is not so easy to pick up only the data for almost complex 2-tori.

In the following, we consider only the almost complex curves of type (III). We denote by V_6 the hyperplane of $\mathbf{R}^7$ whose sixth coordinate is identically zero. We then consider the following correspondence.

$$(*) \quad V_6 \ni \mathbf{x}_R = {}^t(x_1, x_2, x_3, x_4, x_5, 0, x_7) \longleftrightarrow \mathbf{x}_C = \begin{pmatrix} x_1 + ix_7 \\ x_2 + ix_4 \\ -x_3 + ix_5 \end{pmatrix} \in \mathbf{C}^3.$$

By this correspondence (*), we identify an almost complex curve of type (III) $f : M \longrightarrow S^6 \cap V_6 (= S_N^5)$ with a conformal immersion $f_C : M \longrightarrow S^5 \subset \mathbf{C}^3$. We have the following theorem.

Theorem 1.2. *Let $f_C : T^2 \longrightarrow S^5 \subset \mathbf{C}^3$ be the one corresponding to an almost complex 2-torus $f : T^2 \longrightarrow S_N^5$ by (*). Then, f_C may be lifted to a primitive map of finite type into 6-symmetric space $SU(3)/\mathcal{T}^1$. Moreover, its homogeneous projection into S^6 , $f_C : T^2 \longrightarrow S^6 = G_2/SU(3)$, is a harmonic map of finite type.*

Remark. (1) If a harmonic map into a symmetric space has a lift $\tilde{f}$ into a generalized flag manifold so that $\tilde{f}$ is a primitive map of finite type, then its homogeneous projection into some symmetric space is a harmonic map of finite type in very many cases (cf. [OU]).

(2) Arbitrary non-isotropic harmonic map of 2-torus $\varphi : T^2 \longrightarrow S^6 = SO(7)/SO(6)$ can be lifted to a primitive map of finite type $\tilde{\varphi} : T^2 \longrightarrow SO(7)/\mathcal{T}^2$, where $SO(7)/\mathcal{T}^2$ is a 6-symmetric space and the above $G_2/\mathcal{T}^2$ is a 6-symmetric submanifold of $SO(7)/\mathcal{T}^2$.

We then have an diagram :

$$\begin{array}{ccccc} SO(7)/\mathcal{T}^2 & \supset & G_2/\mathcal{T}^2 & \supset & SU(3)/\mathcal{T}^1 \\ \downarrow & & \downarrow & & \downarrow \\ S^6 \cong SO(7)/SO(6) \cong G_2/SU(3) & \supset & S^5 \cong SU(3)/SU(2) & & \end{array}$$

The last inclusion is due to the correspondence (*).

2. Kähler angle of conformal immersion into S^5 and examples

Since an almost complex curve of type (III) $f_C : M \longrightarrow S^5$ is a horizontal curve with respect to the Hopf fibration $S^5 \longrightarrow \mathbf{CP}^2$, f_C can be realized as a horizontal

lift of a totally real minimal surface in $\mathbf{CP}^2$. However, to construct an almost complex curve as a horizontal lift of a totally real minimal surface in $\mathbf{CP}^2$, we need to know the Kähler angle of the lift with respect to the nearly Kähler structure J . Once we come to know the Kähler angle, we may make the lifted horizontal curve into an almost complex curve by rotating it in $\mathbf{C}^3$. The latter fact is due to [BVW]. Hence, we need to know the Kähler angle of a horizontal surface in S^5 and the following proposition gives the answer.

Proposition 2.1. *Suppose that $f : M \rightarrow S_N^5 \subset S^6$ is a conformal immersion. Let θ be the Kähler angle of f with respect to the nearly Kähler structure J on S^6 . If $f_c : M \rightarrow S^5 \subset \mathbf{C}^3$ is horizontal with respect to the Hopf fibration $S^5 \rightarrow \mathbf{CP}^2$, then*

$$(2.2) \quad \cos \theta = \operatorname{Re} \left\{ i \det \begin{pmatrix} f_c & e^{-\frac{\omega}{2}} \frac{\partial f_c}{\partial z} & e^{-\frac{\omega}{2}} \frac{\partial f_c}{\partial \bar{z}} \end{pmatrix} \right\}$$

Proposition 2.1 yields the machinery method of constructing almost complex curves of type (III) as follows.

Theorem 2.3. *Let $s_0 : M \rightarrow \mathbf{C}^3$ be a smooth map and $\omega : M \rightarrow \mathbf{R}$ a smooth function. Set $s_1 = e^{-\frac{\omega}{2}} \frac{\partial s_0}{\partial z}$, $s_2 = e^{-\frac{\omega}{2}} \frac{\partial s_0}{\partial \bar{z}}$. If $S = (s_0 \ s_1 \ s_2)$ has values in $U(3)$ and satisfies $\det S = -i$, then $f : M \rightarrow S_N^5 \subset S^6$ corresponding to $f_c : M \rightarrow S^5 \subset \mathbf{C}^3$ defined by $f_c = s_0$ is a conformal immersion and an almost complex curve with respect to J . The converse is also true.*

Using Theorem 2.3, we may construct a 1-parameter family of almost complex curve of type (III) in terms of Jacobi elliptic functions based on the known example of totally real minimal surface due to Castro-Urbano([CU]).

[Example] Define $f : \mathbf{R}^2 \rightarrow S^6$ by

$$(2.4) \quad f = \left(\sqrt{\frac{r_2}{r_1 + r_2}} \cos \left(r_1 y + \frac{\pi}{6} \right) \operatorname{dn}(rx, p), \sqrt{\frac{r_1}{r_1 + r_2}} \cos \left(\frac{\pi}{6} - r_2 y \right) \operatorname{cn}(rx, p), \right. \\ \left. - \sqrt{\frac{r_2 p^2 + r_1}{r_1 + r_2}} \cos \left(\frac{\pi}{6} - r_3 y \right) \operatorname{sn}(rx, p), \sqrt{\frac{r_1}{r_1 + r_2}} \sin \left(\frac{\pi}{6} - r_2 y \right) \operatorname{cn}(rx, p), \right. \\ \left. \sqrt{\frac{r_2 p^2 + r_1}{r_1 + r_2}} \sin \left(\frac{\pi}{6} - r_3 y \right) \operatorname{sn}(rx, p), 0, \sqrt{\frac{r_2}{r_1 + r_2}} \sin \left(r_1 y + \frac{\pi}{6} \right) \operatorname{dn}(rx, p) \right),$$

where

$$(2.5) \quad \begin{cases} r^2 = \frac{\alpha^3 - 2 + 2\sqrt{\alpha^3 + 1}}{\alpha^2} & , & p^2 = \frac{\alpha^3 - 2 - 2\sqrt{\alpha^3 + 1}}{\alpha^3 - 2 + 2\sqrt{\alpha^3 + 1}}, \\ r_1 = \frac{\sqrt{\alpha^3 + 1} + 1}{\alpha} & , & r_2 = \frac{\sqrt{\alpha^3 + 1} - 1}{\alpha} & , & r_3 = \frac{2}{\alpha}, \end{cases}$$

and $\alpha(\geq 2)$ is a real number. Then, it follows from the correspondence (*) and Theorem 2.3 that f gives a 1-parameter family of almost complex curves of type (III).

3. Spectral data and representation formula in term of Prym-theta function

The spectral data for constructing almost complex 2-torus of type (III), $f_C : T^2 \rightarrow S^5$ is given by the triplet $(\hat{C}, \hat{\mathcal{L}}, \pi)$ which satisfies the following four conditions : For $d \equiv 1 \pmod{6}$

- (1) $\hat{C}$: compact Riemann surface of genus $\hat{g} = 2d$ admitting an anti-holomorphic involution ρ and a holomorphic involution σ , which satisfy $\rho\sigma = \sigma\rho$.
- (2) $\pi : \hat{C} \rightarrow \mathbf{CP}^1$ is a three-fold holomorphic covering map with $\pi \circ \rho = \bar{\pi}^{-1}$. Moreover, the divisor (π) and the ramification divisor R of π are given by

$$(\pi) = 3P_0 - 3P_\infty, \quad R = 2P_0 + 2P_\infty + D_0 + \rho D_0,$$

where two points P_0 and P_∞ satisfy $\sigma(P_0) = P_0, \sigma(P_\infty) = P_\infty, \rho(P_0) = P_\infty$ and D_0 is defined by

$$\begin{cases} D_0 = \{P_1, \dots, P_g, P_{g+1}, \dots, P_{\hat{g}}\}, \\ P_{g+j} = \sigma\rho(P_j) \quad \text{for } j = 1, \dots, g, \end{cases}$$

and arbitrary two points of D_0 are distinct each other.

- (3) A complex line bundle $\hat{\mathcal{L}} = \mathcal{O}_{\hat{C}}(2P_0 + D_0)$ of degree $(\hat{g} + 2)$ over $\hat{C}$.
- (4) π has no branch points over $S_\lambda^1 = \{\lambda \in P^1(\mathbf{C}) : |\lambda| = 1\}$ and ρ fixes each point of $\pi^{-1}(S_\lambda^1)$.

If we define a complex line bundle $\hat{\mathcal{L}}_0$ over $\hat{C}$ by $\hat{\mathcal{L}}_0 = \mathcal{O}_{\hat{C}}(2P_0 - 2P_\infty + D_0)$, then we have $\deg(\hat{\mathcal{L}}_0) = \hat{g}$. It is known ([Mc1]) that $\hat{\mathcal{L}}_0$ is non-special, hence $H^0(\hat{C}, \hat{\mathcal{L}}_0)$ is of 1-dimension. Therefore, if we express a non-trivial section ψ of $\hat{\mathcal{L}}_0$ explicitly, then we can write down $f_C : T^2 \rightarrow S^5$ explicitly. The section ψ is given in terms of Prym-theta function η (cf. [Fa], [MaMa], [HTU]) as follows :

(3.1)

$$\psi(P) = \frac{\eta(\mathcal{B}(P) + iUz + iV\bar{z} - \mathbf{e})}{\eta(\mathcal{B}(P) - \mathbf{e})} \nu^{-2} \exp \left(\int_{P_0}^P iz\Omega_\infty + \int_{P_\infty}^P i\bar{z}\Omega_0 \right),$$

where Ω_0 and Ω_∞ are normalized(= “zero $\mathcal{A}$ -periods”) Abelian differentials of second kind. They satisfy the relations $\rho^*\Omega_\infty = \overline{\Omega_0}, \sigma^*\Omega_0 = -\Omega_0, \sigma^*\Omega_\infty = -\Omega_\infty$ and have the asymptotic behaviors as follows :

$$\begin{cases} \int_{P_\infty}^P \Omega_0 = c\nu^{-1} + o(\nu^{-2}) \\ \int_{P_0}^P \Omega_\infty = \nu + o(\nu^{-2}) \end{cases} \quad \text{around } P_\infty, \quad \begin{cases} \int_{P_\infty}^P \Omega_0 = \nu^{-1} + o(\nu^2) \\ \int_{P_0}^P \Omega_\infty = c\nu + o(\nu^2) \end{cases} \quad \text{around } P_0,$$

where $c \in \mathbf{R}$ and $U = (U_1, \dots, U_g), V = (V_1, \dots, V_g)$ are $\mathcal{B}$ -periods given by $U_j = \int_{B_j} \Omega_\infty, V_j = \int_{B_j} \Omega_0$. Let e^ω be the one define by

$$(3.2) \quad e^\omega = 2\partial_z \partial_{\bar{z}} \log \eta(iUz + iV\bar{z} - \mathbf{e}) + c.$$

This gives a finite gap solution for Tzitzéica equation, which is the integrability condition for almost complex curve of type (III). The induced metric on T^2 may be expressed as $2e^\omega dz d\bar{z}$. That e^ω is $\mathbf{R}$ -valued follows from the relation $\psi(\sigma(P)) = \nu^{-4} \overline{\psi(\rho(P))}$. If we define $\hat{\psi}_0, \hat{\psi}_1$ and $\hat{\psi}_2$ by

$$(3.3) \quad \hat{\psi}_0 = \psi, \quad \hat{\psi}_1 = \partial_z \psi, \quad \hat{\psi}_2 = i\lambda e^{-\omega} \partial_{\bar{z}} \psi$$

then the expression (3.1) of ψ follows from the integrability condition of the system of differential equations formed by $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$.

Now, we obtain the following theorem.

Theorem 3.4. *Given the spectral data $(\hat{C}, \hat{\mathcal{L}}, \pi)$, we define $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$ by (3.3). Moreover, define $\hat{S}$ by*

$$(3.5) \quad \hat{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} \hat{\psi}_0(Q_1) & e^{-\frac{\omega}{2}} \hat{\psi}_1(Q_1) & e^{\frac{\omega}{2}} \hat{\psi}_2(Q_1) \\ \hat{\psi}_0(Q_2) & e^{-\frac{\omega}{2}} \hat{\psi}_1(Q_2) & e^{\frac{\omega}{2}} \hat{\psi}_2(Q_2) \\ \hat{\psi}_0(Q_3) & e^{-\frac{\omega}{2}} \hat{\psi}_1(Q_3) & e^{\frac{\omega}{2}} \hat{\psi}_2(Q_3) \end{pmatrix}.$$

Then, $S = \exp(\frac{\pi}{2}i + \frac{2n}{3}\pi i) (\det(\hat{S}))^{-\frac{1}{3}} \hat{S}$, ($n = 0, 1, 2$), gives a Toda-framing for almost complex curve of type (III). Thus, the first column vector of S gives an almost complex curve of type (III), $f : \mathbf{R}^2 \rightarrow S_N^5$. Moreover, the necessary condition for f be doubly-periodic is that there are two complex numbers c_1, c_2 satisfying $c_1 \bar{c}_2 \neq \bar{c}_1 c_2$ such that

$$\operatorname{Re}(c_k U_j), \operatorname{Re}\left(c_k \int_{P_0}^{Q_l} \Omega_\infty\right) \in \pi \mathbf{Z} \quad \text{for } j = 1, \dots, g; k = 1, 2; l = 1, 2, 3.$$

hold.

Remark. There are some overlaps in fundamental concepts of the works between [Mc3] and [HTU]. Although the paper [Mc3] has been already published, our work had been announced in [U].

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