

Asymptotic behavior of solutions for p -Laplace parabolic equations

佐賀大学・理工学部 梶木屋龍治 (Ryuji Kajikiya)

Faculty of Science and Engineering, Saga University

This lecture is based on the joint work with Professor Goro Akagi. We study the asymptotic behavior of solutions for the one-dimensional p -Laplace parabolic equation

$$\begin{aligned} u_t &= \Delta_p u := (|u_x|^{p-2} u_x)_x \quad \text{in } (0, 1) \times (0, \infty), \\ u(0, t) &= u(1, t) = 0 \quad \text{in } (0, \infty), \\ u(x, 0) &= u_0(x) \quad \text{in } (0, 1), \end{aligned} \tag{1}$$

where $p > 2$ and $u_0 \in W_0^{1,p}(0, 1) \setminus \{0\}$.

Definition 1. We call $u(x, t)$ a solution of (1) if $u \in C([0, \infty), W_0^{1,p}(0, 1)) \cap W_{loc}^{1,2}(0, \infty; L^2(0, 1))$, $\Delta_p u \in L_{loc}^2(0, \infty; L^2(0, 1))$, $u(x, 0) = u_0(x)$ and $u(x, t)$ satisfies the first equation of (1) a.e. $t \in (0, \infty)$.

We denote the $L^q(0, 1)$ and $W_0^{1,q}(0, 1)$ norms by

$$\begin{aligned} \|u\|_q &:= \left(\int_0^1 |u(x)|^q dx \right)^{1/q} \quad \text{for } u \in L^q(0, 1), \\ \|u\|_{1,q} &:= \left(\int_0^1 |u'(x)|^q dx \right)^{1/q} \quad \text{for } u \in W_0^{1,q}(0, 1). \end{aligned}$$

The next theorem can be proved by using Theorem 3.6 of [1].

Theorem A. *Problem (1) has a unique solution.*

The next theorem is proved in [4, 5].

Theorem B. *Any nontrivial solution $u(x, t)$ of (1) decays as $t \rightarrow \infty$, more precisely, there exist constants $C_i > 0$ such that*

$$C_1(t+1)^{-1/(p-2)} \leq \|u(t)\|_2 \leq C_2 \|u(t)\|_{1,p} \leq C_3(t+1)^{-1/(p-2)}$$

for $t \in [0, \infty)$.

We investigate the asymptotic behavior of solutions as $t \rightarrow \infty$. To this end, we use a change of variable

$$v(x, s) = (t + 1)^{1/(p-2)} u(x, t), \quad s = \log(t + 1).$$

Then (1) is reduced to

$$\begin{aligned} v_s &= \Delta_p v + \alpha v && \text{in } (0, 1) \times (0, \infty), \\ v(0, t) &= v(1, t) = 0 && \text{in } (0, \infty), \\ v(x, 0) &= u_0(x) && \text{in } (0, 1), \end{aligned} \tag{2}$$

where $\alpha := 1/(p - 2)$. The stationary problem for (2) is written in the following form:

$$\begin{aligned} -(|\phi'(x)|^{p-2} \phi'(x))' &= \alpha \phi(x), && x \in (0, 1), \\ \phi(0) &= \phi(1) = 0. \end{aligned} \tag{3}$$

The next theorem implies that each stationary solution is characterized by its nodal number.

Theorem C. *For each $k \in \mathbb{N}$, there exists a unique solution ϕ_k of (3) which has exactly $k - 1$ zeros in $(0, 1)$ and $\phi'_k(0) > 0$. Moreover, the set of all nontrivial solutions of (3) consists of $\pm \phi_k$ with $k \in \mathbb{N}$.*

Proof. This theorem is a known result, but for the reader's convenience we give a sketch of proof. Observe that if ϕ satisfies the first equation of (3), so is $\lambda^{-p/(p-2)} \phi(\lambda x)$ for any $\lambda > 0$. We consider the first equation of (3) with the initial condition,

$$\phi(0) = 0, \quad \phi'(0) = 1.$$

This problem has a unique solution, which is denoted by $\phi_0(x)$. Moreover, $\phi_0(x)$ is a periodic solution and it has the first zero $T > 0$. Thus kT with $k \in \mathbb{Z}$ are all the zeros of $\phi_0(x)$. Then we put

$$\phi_k(x) := (kT)^{-p/(p-2)} \phi_0(kTx) \quad \text{with } k \in \mathbb{N}, \tag{4}$$

which is the desired solution. Furthermore, it is easy to check that the set of all nontrivial solutions of (3) consists of $\pm \phi_k$ with $k \in \mathbb{N}$. $\square$

By using Theorem C with the same way as in Berryman and Holland [2], we can prove the next theorem.

Theorem D. *For any nontrivial solution $v(s)$ of (2), there exists a unique nontrivial stationary solution ϕ (i.e., $\phi = \phi_k$ or $-\phi_k$ with a certain $k \in \mathbb{N}$) such that*

$$\lim_{s \rightarrow \infty} \|v(s) - \phi\|_{1,p} = 0.$$

We give a definition of the stability of stationary solutions.

Definition 2. Let ϕ be a nontrivial solution of (3).

(i) ϕ is called stable if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{0 \leq s < \infty} \|v(s) - \phi\|_{1,p} < \varepsilon \quad \text{when } \|v(0) - \phi\|_{1,p} < \delta.$$

(ii) ϕ is called asymptotically stable if it is stable and moreover there exists a $\delta_0 > 0$ such that

$$\lim_{s \rightarrow \infty} \|v(s) - \phi\|_{1,p} = 0 \quad \text{when } \|v(0) - \phi\|_{1,p} < \delta_0.$$

We state our main result.

Theorem 1. *The positive solution ϕ_1 and the negative solution $-\phi_1$ of (3) are asymptotically stable and $\pm\phi_k$ with $k \geq 2$ are unstable.*

To prove Theorem 1, we define the energy

$$J(v) := \int_0^1 \left(\frac{1}{p} |v'(x)|^p - \frac{\alpha}{2} v(x)^2 \right) dx \quad \text{for } v \in W_0^{1,p}(0, 1).$$

Then J becomes a Lyapunov functional for (2). Indeed, multiplying (2) by v_s and integrating it over $(0, 1)$, we have

$$-\|v_s\|_2^2 = (|v_x|^{p-2} v_x, (v_x)_s) - \alpha(v, v_s).$$

Here (u, v) denotes the duality pairing of u and v . The above expression is rewritten as

$$\frac{d}{ds} J(v(s)) = -\|v_s\|_2^2 \leq 0.$$

Thus, if $v(s)$ is a solution of (2), then $J(v(s))$ is decreasing. Consequently J becomes a Lyapunov functional.

Lemma 1. *Each stationary solution is isolated from each other. Moreover, we have*

$$J(\pm\phi_1) < J(\pm\phi_2) < J(\pm\phi_3) < \cdots \nearrow 0. \quad (5)$$

Proof. Multiplying the first equation of (3) by $\phi(x)$ and integrating it over $(0, 1)$, we have

$$\int_0^1 |\phi'|^p dx = \alpha \int_0^1 \phi^2 dx.$$

Using this relation with $\alpha = 1/(p-2)$, we get

$$J(\phi) = -\frac{1}{2p} \int_0^1 \phi^2 dx,$$

provided that ϕ is a solution of (3). Substituting (4) into the relation above, we obtain

$$\begin{aligned} J(\phi_k) &= -\frac{1}{2p} (kT)^{-2p/(p-2)} \int_0^1 \phi_0(kTx)^2 dx \\ &= -\frac{1}{2p} (kT)^{-2p/(p-2)} T^{-1} \int_0^T \phi_0(x)^2 dx. \end{aligned}$$

This expression assures (5), which implies that each stationary solution is isolated from each other. $\square$

Lemma 2. *J has a global minimizer and it is equal to either ϕ_1 or $-\phi_1$.*

Proof. We use the Sobolev imbedding to get a constant $C > 0$ such that

$$J(v) = \frac{1}{p} \|v'\|_p^p - \frac{\alpha}{2} \|v\|_2^2 \geq \frac{1}{p} \|v'\|_p^p - C \|v'\|_p^2, \quad (6)$$

which shows the lower boundedness of J because $p > 2$. In the standard way, we can prove that J satisfies the Palais-Smale condition. Then J has a global minimizer (for the proof, refer to [3, Theorem 2.7]). If ϕ is a global minimizer, so is $|\phi|$, which becomes a critical point of J . Hence $|\phi|$ is a solution of (3). By the strong maximum principle, $|\phi| > 0$ in $(0, 1)$. Thus ϕ is a positive or negative solution. Since a positive solution is unique by Theorem C, ϕ is equal to either ϕ_1 or $-\phi_1$. $\square$

Lemma 3. *For any $\varepsilon > 0$, there exists an $a > J(\phi_1)$ such that*

$$\{v \in W_0^{1,p}(0,1) : J(v) \leq a\} \subset B(\phi_1, \varepsilon) \cup B(-\phi_1, \varepsilon), \quad (7)$$

where

$$B(\phi_1, \varepsilon) := \{v \in W_0^{1,p}(0,1) : \|v - \phi_1\|_{1,p} < \varepsilon\}.$$

Proof. Recall that $\pm\phi_1$ are minimizers of J and the $W_0^{1,p}(0,1)$ -norm is defined by $\|v\|_{1,p} = \|v'\|_p$. We use contradiction. Suppose that there exist $\varepsilon > 0$ and a sequence $v_n \in W_0^{1,p}(0,1)$ such that $J(v_n)$ converges to $J(\phi_1)$ but

$$\|v_n - \phi_1\|_{1,p} \geq \varepsilon, \quad \|v_n + \phi_1\|_{1,p} \geq \varepsilon. \quad (8)$$

By (6), $\|v'_n\|_p$ is bounded. Hence a subsequence (denoted by v_n again) of v_n converges to v_∞ weakly in $W_0^{1,p}(0,1)$ and strongly in $L^2(0,1)$. Since $J(v_n)$ converges to $J(\phi_1)$, we have

$$\frac{1}{p} \|v'_n\|_p^p = J(v_n) + \frac{\alpha}{2} \|v_n\|_2^2 \rightarrow J(\phi_1) + \frac{\alpha}{2} \|v_\infty\|_2^2$$

Since ϕ_1 is a global minimizer of J , we get

$$J(\phi_1) \leq J(v_\infty) = \frac{1}{p} \|v'_\infty\|_p^p - \frac{\alpha}{2} \|v_\infty\|_2^2.$$

From two inequalities above, it follows that $\limsup_{n \rightarrow \infty} \|v'_n\|_p \leq \|v'_\infty\|_p$. Moreover, $\liminf_{n \rightarrow \infty} \|v'_n\|_p \geq \|v'_\infty\|_p$ because v_n weakly converges in $W_0^{1,p}(0,1)$. Since $W_0^{1,p}(0,1)$ is uniformly convex, v_n converges strongly in $W_0^{1,p}(0,1)$. Letting $n \rightarrow \infty$ in (8), we have

$$\|v_\infty - \phi_1\|_{1,p} \geq \varepsilon, \quad \|v_\infty + \phi_1\|_{1,p} \geq \varepsilon.$$

On the other hand, since $J(v_\infty) = J(\phi_1)$, v_∞ is equal to ϕ_1 or $-\phi_1$ by Lemma 2. This is a contradiction. Thus the proof is complete. $\square$

To prove instability of ϕ_k with $k \geq 2$, we use

Lemma 4. *Let $k \geq 2$. Then for any $\varepsilon > 0$ there exists a $v_0 \in W_0^{1,p}(0,1)$ such that*

$$\|v_0 - \phi_k\|_{1,p} < \varepsilon \quad \text{and} \quad J(v_0) < J(\phi_k).$$

In other words, there is a point v_0 sufficiently close to ϕ_k whose energy is less than that of ϕ_k .

Proof. We denote by $\psi(x, (a, b))$ the unique positive solution of

$$\begin{aligned} -(|\psi'(x)|^{p-2}\psi'(x))' &= \alpha\psi(x), \quad \psi(x) > 0, \quad x \in (a, b), \\ \psi(a) &= \psi(b) = 0, \end{aligned}$$

Recall that $\phi_1(x)$ is a positive solution of (3). Hence it holds that $\psi(x, (0, 1)) = \phi_1(x)$ and moreover we have the relation

$$\psi(x, (a, b)) = c^{-p/(p-2)}\phi_1(c(x-a)), \quad c := 1/(b-a). \quad (9)$$

For $\lambda \in (0, 2)$, we define

$$\Psi_\lambda(x) := \begin{cases} \psi(x, (0, \lambda/k)) & \text{if } x \in [0, \lambda/k], \\ -\psi(x, (\lambda/k, 2/k)) & \text{if } x \in [\lambda/k, 2/k], \\ \phi_k(x) & \text{if } x \in [2/k, 1]. \end{cases}$$

By using (9) with (4), we can prove that $\Psi_\lambda \rightarrow \phi_k$ as $\lambda \rightarrow 1$ and $J(\Psi_\lambda) < J(\phi_k)$ if $\lambda \neq 1$. When $\lambda \neq 1$ is sufficiently close to 1, $v_0 = \Psi_\lambda$ satisfies the assertion of Lemma 4. $\square$

Proof of Theorem 1. We prove that ϕ_1 is asymptotically stable. Let $\varepsilon > 0$. We can assume that ε satisfies

$$B(\phi_1, \varepsilon) \cap B(-\phi_1, \varepsilon) = \emptyset, \quad \pm\phi_k \notin \overline{B(\phi_1, \varepsilon)} \quad (k \geq 2).$$

Then by Lemma 3, we determine $a(> J(\phi_1))$ which satisfies (7). If $\|v(0) - \phi_1\|_{1,p}$ is small enough, then $J(v(0))$ is sufficiently close to $J(\phi_1)$ and hence $J(v(0)) < a$. Thus $J(v(s)) \leq J(v(0)) \leq a$. By Lemma 3, we get

$$v(s) \in B(\phi_1, \varepsilon) \cup B(-\phi_1, \varepsilon) \quad \text{for all } s \geq 0.$$

Since $B(\phi_1, \varepsilon) \cap B(-\phi_1, \varepsilon) = \emptyset$, $v(s)$ belongs to $B(\phi_1, \varepsilon)$ for $s \geq 0$. Therefore ϕ_1 is stable. By Theorem D, $v(s)$ has a limit as $s \rightarrow \infty$. Since $\pm\phi_k \notin \overline{B(\phi_1, \varepsilon)}$ for $k \geq 2$, $v(s)$ must converge to ϕ_1 . Therefore ϕ_1 is asymptotically stable. In the same way as above, we can show the asymptotic stability of $-\phi_1$.

Let $k \geq 2$. We show the instability of ϕ_k . Let $\varepsilon > 0$. Then we choose v_0 by Lemma 4. Let $v(s)$ be the solution starting from $v(0) = v_0$. Then $v(s)$ converges to a certain stationary point v_∞ . But $v_\infty \neq \phi_k$ because

$$J(v_\infty) = \lim_{s \rightarrow \infty} J(v(s)) \leq J(v_0) < J(\phi_k).$$

Since each stationary point is isolated from each other, we define

$$d := \inf \{ \|u - \phi_k\|_{1,p} : u \text{ is any stationary solution except for } \phi_k \}.$$

Then $\|v_\infty - \phi_k\|_{1,p} \geq d > 0$. The initial data v_0 is sufficiently close to ϕ_k but the solution $v(s)$ is away from ϕ_k with at least distance $d/2$ for s large enough. Therefore ϕ_k ($k \geq 2$) is unstable. $\square$

References

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