

Characterization of certain spaces of C^∞ -vectors of irreducible representations of solvable Lie groups

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Abstract

Let π be an irreducible unitary representation of an exponential solvable Lie group G . Realizing π on $L^2(G/H, \chi_l)$ as an induced representation from a unitary character χ_l of a subgroup H of G , we are concerned with certain subspaces of C^∞ -vectors. We describe the subspace $\mathcal{SE}$ of vectors with a certain property of rapidly decreasing at infinity as the space of C^∞ -vectors of an irreducible unitary representation of an exponential solvable Lie group F containing G . Furthermore, the space $\mathcal{ASE}$ introduced by Ludwig in [8] is expressed by our space $\mathcal{SE}$. Here we shall announce some results in [5], and we shall give brief discussions on fundamental examples.

1 Introduction

Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, and π be an irreducible unitary representation of G . By the orbit method, which associates π with a coadjoint orbit, we realize π as an induced representation from a unitary character of a subgroup as follows: There exists a linear form $l \in \mathfrak{g}^*$ and a real polarization $\mathfrak{h}$ at l such that $\pi \simeq \text{ind}_H^G \chi_l$, where $H = \exp \mathfrak{h}$ is the connected and simply connected subgroup corresponding to $\mathfrak{h}$, and χ_l is the unitary character of H defined by $\chi_l(X) = e^{il(X)}$ ($X \in \mathfrak{h}$).

We give the standard construction of the induced representation $\pi = \pi_{l,H} = \text{ind}_H^G \chi_l$: Let $\mathcal{K}(G/H)$ be the space of continuous functions f on G with compact support modulo H such that $f(gh) = \Delta_{H,G}(h)f(g)$ for all $g \in G$, $h \in H$, where Δ_G and Δ_H are the

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modular functions of G and H , respectively, and $\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$. Then there exists a positive left invariant linear functional

$$(1.1) \quad f \mapsto \mu(f) = \oint_{G/H} f(g) d\mu_{G/H}(g)$$

uniquely up to a constant factor (see [3]). Let $C(G/H, \chi_l)$ be the space of continuous functions ϕ on G with compact support modulo H such that

$$\phi(gh) = \chi_l(h)^{-1} \Delta_{H,G}(h)^{1/2} \phi(g), \quad \forall g \in G, h \in H,$$

and let $\mathcal{H}_\pi = L^2(G/H, \chi_l)$ be the completion of the space $C(G/H, \chi_l)$ with respect to the norm

$$\|\phi\|_\pi := \left(\oint_{G/H} |\phi(g)|^2 d\mu_{G/H}(g) \right)^{1/2}.$$

Then we define the action of $g \in G$ in $\mathcal{H}_\pi$ by

$$\pi_{l,H}(g)\phi(x) = \phi(g^{-1}x), \quad \phi \in L^2(G/H, \chi_l), \quad g, x \in G.$$

Let us briefly recall some well-known facts of the case of nilpotent Lie groups. Suppose G is nilpotent, and taking a supplementary Malcev basis for $\mathfrak{h}$ in $\mathfrak{g}$, identify G/H with $\mathbb{R}^k$, where $k = \dim(G/H)$, and realize π on $L^2(\mathbb{R}^k)$. Then by results of Kirillov [6] and Corwin-Greenleaf-Penney [4], the actions of the enveloping algebra $u(\mathfrak{g})$ form the algebra of differential operators on $\mathbb{R}^k$ with polynomial coefficients. Thus the space of C^∞ vectors $\mathcal{H}_\pi^\infty$ coincides with the space of Schwartz functions $\mathcal{S}(\mathbb{R}^k)$ on $\mathbb{R}^k$ as a Fréchet space.

We next observe an example of exponential groups which are not nilpotent, where the specific descriptions of C^∞ vectors are different from those of nilpotent groups.

Example 1.1. ($ax+b$ group) Let $\mathfrak{g}$ be a two-dimensional Lie algebra with basis $\{X, Y\}$ whose bracket relation is $[X, Y] = Y$, and let $G = \exp \mathfrak{g}$. Then with the dual basis $\{X^*, Y^*\}$ in $\mathfrak{g}^*$, the coadjoint orbits of G are described as follows:

$$\mathcal{O}_+ := \{l \in \mathfrak{g}^*; l(Y) > 0\}, \quad \mathcal{O}_- := \{l \in \mathfrak{g}^*; l(Y) < 0\}, \\ \{\xi X^*\}, \quad \xi \in \mathbb{R}.$$

Let $l_\varepsilon := \varepsilon Y^*$ ($\varepsilon = \pm 1$) and $\mathfrak{h} := \mathbb{R}Y$, $H := \exp \mathfrak{h}$. Then $\mathfrak{h}$ is a polarization at l_ε and $\pi_\varepsilon := \text{ind}_H^G \chi_{l_\varepsilon}$ is an irreducible representation of G . We realize π_ε on $L^2(\mathbb{R})$ identifying $\mathbb{R}$ with G/H by $\mathbb{R} \ni x \mapsto \exp(xX)H$, as follows:

$$\pi(\exp aX)\phi(x) = \phi(x - a) \\ \pi(\exp bY)\phi(x) = e^{i\varepsilon b e^{-x}} \phi(x), \quad \phi \in L^2(\mathbb{R}), \quad a, b \in \mathbb{R}.$$

Then the actions of $\mathfrak{g}$ are described by

$$d\pi(X)\phi(x) = -\frac{d\phi}{dx} \\ d\pi(Y)\phi(x) = i\varepsilon e^{-x} \phi(x).$$

It shows that if ϕ is a C^∞ vector, then ϕ decreases rapidly at $x \rightarrow -\infty$, but it does not necessarily decrease so rapidly at $x \rightarrow +\infty$ as at $x \rightarrow -\infty$.

Here we shall announce some results in [5], giving brief discussions on fundamental examples. For an exponential solvable group G and an irreducible unitary representation π of G , we construct π in $L^2(G/H, \chi_l)$ by taking $l \in \mathfrak{g}^*$ and a suitable polarization $\mathfrak{h}$. Then we shall define a subspace $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ of vectors with some property of rapidly decreasing at infinity and show that it can be identified with the space of C^∞ vectors of an irreducible representation of an exponential solvable group F containing G . Next, using our $\mathcal{SE}$ space, we shall describe the space $\mathcal{ASE}$ introduced by Ludwig in [8].

2 The space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$

In the sequel, let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. For example, we can take the nilradical of $\mathfrak{g}$ as $\mathfrak{n}$, or we can also take $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. Let $l \in \mathfrak{g}^*$ as above, and let

$$\mathfrak{n}^l := \{X \in \mathfrak{g}; l([X, \mathfrak{n}]) = \{0\}\}.$$

Definition 2.1. (See [9].) We say that a polarization $\mathfrak{h}$ at l is adapted to $\mathfrak{n}$ if it satisfies (1) and (2).

- (1) The subalgebra $\mathfrak{h} \cap \mathfrak{n}$ is a polarization at $l|_{\mathfrak{n}}$ in $\mathfrak{n}$.
- (2) $[\mathfrak{n}^l, \mathfrak{h} \cap \mathfrak{n}] \subset \mathfrak{h} \cap \mathfrak{n}$.

Remark 2.2. (1) Suppose that a polarization $\mathfrak{h}$ at l is adapted to $\mathfrak{n}$. Then there exists a polarization $\mathfrak{h}_0 \subset \mathfrak{n}^l$ at $l|_{\mathfrak{n}^l}$ such that $\mathfrak{h} = \mathfrak{h}_0 + (\mathfrak{h} \cap \mathfrak{n})$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{n}^l$.

Furthermore, it satisfies the Pukanszky condition

$$\text{Ad}^*(H)l = \mathfrak{h}^\perp + l,$$

where $\mathfrak{h}^\perp := \{f \in \mathfrak{g}^*; f(\mathfrak{h}) = \{0\}\}$, and thus we obtain a realization of the irreducible representation corresponding to the orbit $\text{Ad}^*(G)l$ by $\text{ind}_H^G \chi_l$.

- (2) For any l and $\mathfrak{n}$, there exists a polarization $\mathfrak{h}$ at l adapted to $\mathfrak{n}$. For example, a Vergne polarization associated with a refinement of the sequence of ideals $\{0\} \subset \mathfrak{n} \subset \mathfrak{g}$ satisfies the condition (1) and (2) of Definition 2.1 above.

Starting from $\mathfrak{n}$, l and a polarization $\mathfrak{h}$ at l adapted to $\mathfrak{n}$, we realize the irreducible representation $\pi = \pi_{l,H} = \text{ind}_H^G \chi_l$ in $L^2(\mathbb{R}^n)$ ($n = \dim(G/H)$) as follows.

Let $\{T_1, \dots, T_m, R_1, \dots, R_v\}$ be a coexponential basis for $\mathfrak{h}$ in $\mathfrak{g}$ such that

$$\begin{aligned} G &= \exp \mathbb{R}T_1 \cdots \exp \mathbb{R}T_m \cdot NH, \\ NH &= \exp \mathbb{R}R_1 \cdots \exp \mathbb{R}R_v \cdot H, \end{aligned}$$

and identify

$$\mathbb{R}^m \times \mathbb{R}^v \simeq (G/NH) \times (NH/H) \simeq G/H$$

by

$$\begin{aligned} \mathbb{R}^m \times \mathbb{R}^v \ni (t, r) &= (t_1, \dots, t_m, r_1, \dots, r_v) \\ &\mapsto E(t, r) := \exp t_1 T_1 \cdots \exp t_m T_m \exp r_1 R_1 \cdots \exp r_v R_v \pmod{H}. \end{aligned}$$

Then the left invariant functional (1.1) is described by

$$\mu(f) = \int_{\mathbb{R}^{m+v}} f(E(t, r)) dt dr, \quad f \in \mathcal{K}(G/H)$$

(see [7]), and we have $L^2(G/H, \chi_l) \simeq L^2(\mathbb{R}^{m+v})$.

Denoting by $\mathcal{D}_{t,r}$ the algebra of differential operators on $\mathbb{R}^{m+v}$ with polynomial coefficients, we define the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ as follows:

Definition 2.3. Let $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of vectors $\phi \in \mathcal{H}_{\pi_l, H} = L^2(G/H, \chi_l)$ such that

(1) ϕ is a C^∞ function.

(2)

$$\|\phi\|_{a,D}^2 := \int_{\mathbb{R}^{m+v}} e^{a\|t\|} |D(\phi \circ E)(t, r)|^2 dt dr < \infty, \quad \forall a \in \mathbb{R}_+, \forall D \in \mathcal{D}_{t,r},$$

where $\|t\|$ denotes a norm on $\mathbb{R}^m$.

Let us remark that the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ is independent of the choice of coexponential basis.

In [5], we obtained the following result. There exists an exponential solvable Lie group F containing G , and an irreducible representation π_0 of F such that $\pi_0|_G \simeq \pi$ and the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ is naturally identified with the space of C^∞ vectors of π_0 . More specifically, we can construct an exponential solvable Lie algebra $\mathfrak{f}$ which has the properties (i), (ii) and (iii):

(i) $\mathfrak{f}$ is described as $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal of $\mathfrak{f}$ satisfying

$$[\mathfrak{a}, \mathfrak{n} + \mathfrak{h}] = \{0\}.$$

(ii) $\dim \mathfrak{a} = 2 \dim(\mathfrak{g}/(\mathfrak{n} + \mathfrak{h})) = 2m$, and there exist a coexponential basis $\{X_1, \dots, X_m\}$ for $\mathfrak{n} + \mathfrak{h}$ in $\mathfrak{g}$ and a basis $\{A_1, \dots, A_m, B_1, \dots, B_m\}$ of $\mathfrak{a}$ such that

$$[X_j, A_k] = \delta_{jk} A_k, \quad [X_j, B_k] = -\delta_{jk} B_k, \quad 1 \leq j, k \leq m.$$

(iii) For all extension $l_1 \in \mathfrak{f}^*$ of $l \in \mathfrak{g}^*$, we have that

$$\dim(\mathfrak{f}(l_1)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a}),$$

where $\mathfrak{f}(l_1) := \{X \in \mathfrak{f}; l_1([X, \mathfrak{f}]) = \{0\}\}$, $\mathfrak{g}(l) := \{X \in \mathfrak{g}; l([X, \mathfrak{g}]) = \{0\}\}$. Thus the subalgebra $\mathfrak{p} := \mathfrak{h} + \mathfrak{a}$ is a polarization at l_1 adapted to the nilpotent ideal $\mathfrak{n} + \mathfrak{a}$ of $\mathfrak{f}$.

Let $F = \exp \mathfrak{f}$, $P = \exp \mathfrak{p}$, and χ_{l_1} be the unitary character of F defined by $\chi_{l_1}(\exp X) = e^{i l_1(X)}$ for $X \in \mathfrak{p}$. Then by (iii) above, the induced representation $\pi_{l_1, P} := \text{ind}_P^G \chi_{l_1}$ is irreducible and $\pi_{l_1, P}|_G \simeq \pi$. In fact, the intertwining operator

$$\mathcal{R}_{l_1} : \mathcal{H}_{\pi_{l_1, P}} = L^2(F/P, \chi_{l_1}) \rightarrow \mathcal{H}_{\pi, H} = L^2(G/H, \chi_l)$$

is defined by

$$\mathcal{R}_{l_1} \psi = \psi|_G, \quad \psi \in L^2(F/P, \chi_{l_1}),$$

and the inverse $\mathcal{S}_{l_1} := \mathcal{R}_{l_1}^{-1}$ is

$$\mathcal{S}_{l_1} \phi(g \exp Y) := e^{-i l_1(Y)} \phi(g), \quad \phi \in L^2(G/H, \chi_l), \quad g \in G, Y \in \mathfrak{a}.$$

It can be seen easily that

$$\mathcal{S}_{l_1}(\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})) \subset \mathcal{H}_{\pi_{l_1, P}}^\infty.$$

Now we define another family of seminorms $\{\|\cdot\|_{l_1, U}\}$ on $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$:

$$\|\phi\|_{l_1, U} := \|d\pi_{l_1, P}(U) \mathcal{S}_{l_1} \phi\|_{\pi_{l_1, P}}, \quad U \in \mathfrak{u}(\mathfrak{f}).$$

Theorem 2.4. ([5]) *Let $G, \mathfrak{n}, l, \mathfrak{h}$ be as above. Then there exists an exponential solvable Lie algebra $\mathfrak{f}$ having the property (i), (ii), (iii) above and satisfying the following:*

(iv) *There exists an extension $l_0 \in \mathfrak{f}^*$ of l such that the family of seminorms $\{\|\cdot\|_{l_0, U}, U \in \mathfrak{u}(\mathfrak{f})\}$ is equivalent to the family of seminorms $\{\|\cdot\|_{a, D}, a \in \mathbb{R}_+, D \in \mathcal{D}_{t, r}\}$; and thus we have*

$$\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}_{\pi_{l_0, P}}^\infty).$$

Example 2.5. ($ax + b$ group) Let $\mathfrak{g} = \mathbb{R}X + \mathbb{R}Y$ and $\mathfrak{h} = \mathbb{R}Y$ be as in Example 1.1, and let $l := Y^*$ and $\mathfrak{n} = \mathbb{R}Y$, which is the nilradical of $\mathfrak{g}$. Then the polarization $\mathfrak{h}$ is obviously adapted to $\mathfrak{n}$. We construct π_l in $L^2(\mathbb{R})$ as in Example 1.1. Then a square integrable smooth function ϕ belongs to $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ if and only if

$$\int_{\mathbb{R}} e^{a|x|} |D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad \forall D \in \mathcal{D}_x,$$

where $\mathcal{D}_x$ is the algebra of differential operators on $\mathbb{R}$ with polynomial coefficients. Applying Theorem 2.4 above, we have

$$\begin{aligned} \mathfrak{f} &= \mathfrak{g} \ltimes \mathfrak{a}, \quad \mathfrak{a} = \mathbb{R}A + \mathbb{R}B \\ [X, A] &= A, \quad [X, B] = -B, \quad [Y, A] = [Y, B] = 0. \end{aligned}$$

Let $l_0 \in \mathfrak{f}^*$ be an extension of l such that $l_0(B) \neq 0$. Then we have

$$(2.2) \quad \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}_{\pi_{l_0, P}}^\infty).$$

In fact, realizing $\pi_{l_0, P} = \text{ind}_P^F \chi_{l_0}$ in $L^2(F/P, \chi_{l_0}) = \mathcal{S}_{l_0}(L^2(G/P, \chi_l)) \simeq L^2(\mathbb{R})$, we have that $\mathfrak{f}$ acts by

$$(2.3) \quad d\pi_{l_0, P}(X)\phi(x) = -\frac{d\phi}{dx},$$

$$(2.4) \quad d\pi_{l_0, P}(Y)\phi(x) = ie^{-x}\phi(x),$$

$$(2.5) \quad d\pi_{l_0, P}(A)\phi(x) = il_0(A)e^{-x}\phi(x),$$

$$(2.6) \quad d\pi_{l_0, P}(B)\phi(x) = il_0(B)e^x\phi(x).$$

Thus we can directly verify the equality (2.2).

Remark 2.6. In Example 2.5, replacing F with a subgroup F' of F , we can also identify the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ with the space of C^∞ vectors of an extension of π_l . Let $\mathfrak{a}' := \mathbb{R}B$, $\mathfrak{f}' := \mathfrak{g} \ltimes \mathfrak{a}'$, $\mathfrak{p}' := \mathfrak{h} + \mathfrak{a}'$, $F' := \exp \mathfrak{f}'$ and $P' := \exp \mathfrak{p}'$. Then $\mathfrak{p}'$ is a polarization at any extension $l'_0 \in \mathfrak{f}'^*$ of l and $\pi_{l'_0, P'}|_G \simeq \pi_l$, where $\pi_{l'_0, P'} = \text{ind}_{P'}^{F'} \chi_{l'_0}$. We denote the intertwining operator by $\mathcal{R}_{l'_0} : L^2(F'/P', \chi_{l'_0}) \rightarrow L^2(G/H, \chi_l)$ as above. Suppose that an extension $l'_0 \in \mathfrak{f}'^*$ of l satisfies $l'_0(B) \neq 0$. Then we also obtain that

$$(2.7) \quad \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l'_0}(\mathcal{H}_{\pi_{l'_0, P'}}^\infty).$$

In fact, letting $l_0 \in \mathfrak{f}^*$ be any extension of l'_0 , we have $\pi_{l_0, P}|_{F'} \simeq \pi_{l'_0, P'}$, and using the descriptions (2.3), (2.4) and (2.6), we obtain the equality (2.7).

Example 2.7. (Heisenberg group) Taking $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ instead of the nilradical, we observe an example of nilpotent groups. Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X, Y, Z\}$ be the 3-dimensional Lie algebra whose non-trivial bracket relation is $[X, Y] = Z$. Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{R}Z$, $l = Z^*$ and $\mathfrak{h} = \mathbb{R}Y + \mathbb{R}Z$. Then $\mathfrak{h}$ is a polarization adapted to $\mathfrak{n}$. We realize the representation $\pi_{l, H}$ in $L^2(\mathbb{R})$ by the coexponential basis $\{X\}$ for $\mathfrak{h}$ in $\mathfrak{g}$. Then we have

$$(2.8) \quad d\pi_{l, H}(X)\phi(x) = -\frac{d\phi}{dx}, \quad d\pi_{l, H}(Y)\phi(x) = -ix\phi(x), \quad d\pi_{l, H}(Z) = i.$$

We have that a smooth function $\phi(x) \in L^2(\mathbb{R})$ belongs to $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ if and only if

$$\int_{\mathbb{R}} e^{a|x|} |D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad \forall D \in \mathfrak{D}_x,$$

Applying Theorem 2.4, we have

$$\begin{aligned} \mathfrak{f} &= \mathfrak{g} \ltimes \mathfrak{a}, \quad \mathfrak{a} = \mathbb{R}A + \mathbb{R}B, \\ [X, A] &= A, \quad [X, B] = -B, \quad [Y, A] = [Y, B] = [Z, A] = [Z, B] = 0. \end{aligned}$$

Let $l_0 \in \mathfrak{f}^*$ be an extension of l such that $l_0(A) \neq 0$ and $l_0(B) \neq 0$. Then we have

$$\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}_{\pi_{l,P}}^\infty).$$

In fact, we have

$$(2.9) \quad d\pi_{l_0,P}(A)\phi(x) = il_0(A)e^{-x}\phi(x),$$

$$(2.10) \quad d\pi_{l_0,P}(B)\phi(x) = il_0(B)e^x\phi(x).$$

By the actions (2.8), (2.9) and (2.10), we can obtain the conclusion.

3 The space $\mathcal{ASE}$ and the space $\mathcal{SE}^\infty$

Let $G, \mathfrak{n}, l, \mathfrak{h}$ be as above. As we mentioned in Remark 2.2, we have that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{n}^l) + (\mathfrak{h} \cap \mathfrak{n})$, so we have $\mathfrak{h} \subset \mathfrak{n} + \mathfrak{n}^l$. We choose a coexponential basis $\{T_1, \dots, T_\nu, S_1, \dots, S_u\}$ for $\mathfrak{h} + \mathfrak{n}$ in $\mathfrak{g}$ along with the sequence $\mathfrak{g} \supset \mathfrak{n} + \mathfrak{n}^l \supset \mathfrak{n} + \mathfrak{h}$, where $\nu + u = m$, so that

$$\begin{aligned} G &= \exp \mathbb{R}T_1 \cdots \exp \mathbb{R}T_\nu \cdot N^l N \\ N^l N &= \exp \mathbb{R}S_1 \cdots \exp \mathbb{R}S_u \cdot NH. \end{aligned}$$

(Here we write $N^l := \exp \mathfrak{n}^l$.) In the sequel, we identify

$$\mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^\nu \simeq (G/N^l N) \times (N^l N/NH) \times (NH/H) \simeq G/H$$

by

$$\begin{aligned} \mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^\nu \ni (t, s, r) &= (t_1, \dots, t_\nu, s_1, \dots, s_u, r_1, \dots, r_\nu) \\ &\mapsto E(t, s, r) = E(t)E(s)E(r) \text{ (modulo } H), \end{aligned}$$

where

$$\begin{aligned} E(t) &= \exp t_1 T_1 \cdots \exp t_\nu T_\nu, \quad E(s) = \exp s_1 S_1 \cdots \exp s_u S_u, \\ E(r) &= \exp r_1 R_1 \cdots \exp r_\nu R_\nu, \quad (t, s, r) \in \mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^\nu. \end{aligned}$$

For $\phi \in \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$, let $\hat{\phi}_s(t, s, r)$ be the partial Fourier transform of ϕ in s :

$$\hat{\phi}_s(t, s, r) := \int_{\mathbb{R}^u} \phi(E(t, x, r)) e^{i\langle x, s \rangle} dx,$$

where $\langle x, s \rangle$ is the standard inner product of $\mathbb{R}^u$. Denoting by $\mathfrak{D}_{t,s,r}$ the algebra of differential operators on $\mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^\nu$ with polynomial coefficients, we define the space $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ introduced in [8], where this space has been denoted by $\mathcal{ES}$.

Definition 3.1. (See [8].) Let $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of functions $\phi \in L^2(G/H, \chi_l)$ such that

$$(1) \quad \phi \in \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}),$$

$$(2)$$

$$\|\hat{\phi}_s(t, s, r)\|_{a,b,D}^2 := \int_{\mathbb{R}^{\nu+u+v}} e^{a\|t\|} e^{b\|s\|} |D\hat{\phi}_s(t, s, r)|^2 dt ds dr < \infty, \\ \forall (a, b) \in \mathbb{R}_+^2, \quad \forall D \in \mathcal{D}_{t,s,r}.$$

Remark 3.2. The space $\mathcal{ASE}$ is independent of the choice of coexponential bases. We write the letter $\mathcal{A}$ in front to indicate that the functions $\phi \in \mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ are analytic in the direction s . It has been shown in [8] and [1] that for ϕ and ψ in $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ there exists a function $f \in L^1(G)$, more precisely in the subalgebra $\mathcal{ES}(G)$ (see [8]) such that

$$\pi_{l,H} f(\xi) = \langle \xi, \psi \rangle \phi, \quad \xi \in \mathcal{H}_{\pi_{l,H}}.$$

Let $\mathcal{P}(\mathfrak{h})$ be the set of polarizations $\tilde{\mathfrak{h}}$ at l adapted to $\mathfrak{n}$ and satisfying $\tilde{\mathfrak{h}} \cap \mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}$. For $\tilde{\mathfrak{h}} \in \mathcal{P}(\mathfrak{h})$, we have $\text{ind}_{\tilde{H}}^G \chi_l \simeq \text{ind}_H^G \chi_l$, where $\tilde{H} = \exp \tilde{\mathfrak{h}}$. We denote the intertwining operator by $I_{\tilde{\mathfrak{h}}, \mathfrak{h}} : L^2(G/\tilde{H}, \chi_l) \rightarrow L^2(G/H, \chi_l)$ (see [2].)

Definition 3.3. We define

$$\mathcal{SE}^\infty(G, \mathfrak{n}, l, \mathfrak{h}) := \cap_{\tilde{\mathfrak{h}} \in \mathcal{P}(\mathfrak{h})} I_{\tilde{\mathfrak{h}}, \mathfrak{h}}(\mathcal{SE}(G, \mathfrak{n}, l, \tilde{\mathfrak{h}})).$$

Then we have the following result:

Theorem 3.4. ([5])

$$\mathcal{SE}^\infty(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h}).$$

Example 3.5. Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X, Y, Z\}$, $\mathfrak{n}, l, \mathfrak{h}$ be as in Example 2.7. Concerning Theorem 3.4, we have $\mathfrak{n}^l = \mathfrak{g}$ and a smooth function $\phi \in L^2(\mathbb{R})$ belongs to $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ if and only if

$$(3.11) \quad \int_{\mathbb{R}} e^{a|x|} |D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad D \in \mathcal{D}_x,$$

$$(3.12) \quad \int_{\mathbb{R}} e^{a|x|} |D\hat{\phi}(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad D \in \mathcal{D}_x,$$

where

$$\hat{\phi}(x) = \int_{\mathbb{R}} e^{ixs} \phi(s) ds.$$

Since $\mathfrak{n} = \mathbb{R}Z$ is the center of $\mathfrak{g}$, any polarization $\tilde{\mathfrak{h}}$ at l belongs to $\mathcal{P}(\mathfrak{h})$. Thus by Theorem 3.4, we have that the intersection of $I_{\tilde{\mathfrak{h}}, \mathfrak{h}}(\mathcal{SE}(G, \mathfrak{n}, l, \tilde{\mathfrak{h}}))$ for all polarizations $\tilde{\mathfrak{h}}$ at l consists of analytic functions ϕ satisfying (3.11) and (3.12).

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