

Robustness and Global Bifurcation of Three-species Ecological Model

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§0. INTRODUCTION

We shall consider three-dimensional Lotka-Volterra equations defined by the following vector fields:

$$(LV) \quad \dot{x}_i = \gamma_i x_i (1 + \sum_{j \in \mathbf{Z}_3} a_{ij} x_j), \quad i \in \mathbf{Z}_3,$$

on the closed positive cone $\mathbf{R}_+^3 := \{x \in \mathbf{R}^3 : x \geq 0\}$, where $x = (x_1, x_2, x_3)$. We denote by $\cdot$ a differential d/dt and $\mathbf{Z}_3 := \{1, 2, 3\}$ is considered cyclic.

In the case of the competitive systems (see Hirsch [H]), the global behaviors of the three-dimensional Lotka-Volterra systems (with $a_{ij} < 0$) have been studied by Driessche and Zeeman [DZ], Chi, Hsu and Wu [CHW].

The analysis of the global behaviors of orbits in this paper is featured that the system is not always competitive. For example in the case when $\gamma_1 = \gamma_2 = \gamma_3 (> 0)$, we have Theorem 1.1. below.

Consider the following systems:

$$(LV_1) \quad \dot{x}_i = x_i (1 + \sum_{j \in \mathbf{Z}_3} a_{ij} x_j), \quad i \in \mathbf{Z}_3.$$

THEOREM 1.1.[UO] *In the vector field of (LV_1) with $\gamma_i > 0$, $a_{ii} < 0$ and $a_{i,i-1} < a_{ii} < a_{i,i+1}$ ($i \in \mathbf{Z}_3$), almost every orbit ψ in $(\mathbf{R}_+^3)^\circ$ satisfies one of the following:*

(1) *If $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) > \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$ and $\det A < 0$, then every orbit ψ in $(\mathbf{R}_+^3)^\circ$ tends toward the equilibrium point x^* .*

(2) *If $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) < \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$, then every orbit ψ in $(\mathbf{R}_+^3)^\circ \setminus \Gamma_1$ winds toward the heteroclinic cycle T .*

Here we denote by $(\cdot)^\circ$ the interior of $\cdot$, by $A = (a_{ij})$ called the *interactive matrix* of the system, by x^* the equilibrium point which is in $(\mathbf{R}_+^3)^\circ$ and by Γ_1 the half line from 0 passing through the equilibrium x^* . Three singularities $e_1 := (\frac{-1}{a_{11}}, 0, 0)$, $e_2 := (0, \frac{-1}{a_{22}}, 0)$, $e_3 := (0, 0, \frac{-1}{a_{33}})$ are connected one another by three orbits called the *heteroclinic orbits*. The union of these singularities and orbits forms a curved triangle T , called the *heteroclinic cycle*.

On the other hand, in the case when γ_i 's are not necessarily the same, there is an impressive result Theorem 1.3. below.

DEFINITION 1.2. The system (LV) is said to be *permanent*, if there exists a compact set $K \subset (\mathbf{R}_+^3)^\circ$ such that for any $\psi(0) \in (\mathbf{R}_+^3)^\circ$, we have $\psi(t) \in K$ for t sufficiently large.

Namely the system (LV) is permanent, if $\partial \mathbf{R}_+^3$ is a repeller on $\mathbf{R}_+^3$, where $\partial \mathbf{R}_+^3$ denotes the boundary of $\mathbf{R}_+^3$ including points at infinity.

THEOREM 1.3.[HS] *Consider the vector field of (LV) with $\gamma_i > 0$, $a_{ii} < 0$ and $a_{i,i-1} < a_{ii} < a_{i,i+1}$ ($i \in \mathbf{Z}_3$).*

- (1) If $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) > \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$ and $\det A < 0$, then the system (LV) is permanent.
- (2) If $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) < \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$, then the heteroclinic cycle T is an attractor.

The above results determine the behaviors of the orbits in the neighborhood of $\partial \mathbf{R}_+^3$ in $\mathbf{R}_+^3$.

Now we shall define the structural stable-like idea on a stability of the system (LV).

Consider the system (LV_ϵ) which is obtained by perturbing the system (LV) as follows:

$$(LV_\epsilon) \quad \dot{x}_i = \gamma_i x_i (1 + \sum_{j \in \mathbf{Z}_3} a_{ij} x_j + \epsilon_i \phi_i(x)), \quad i \in \mathbf{Z}_3,$$

where the ϕ_i are affine linear, the $|\epsilon_i|$ are sufficiently small and $\epsilon := (\epsilon_1, \epsilon_2, \epsilon_3)$.

DEFINITION 1.4. We say a property of the system (LV) is *robust*, if it holds in the system (LV_ϵ) .

The following theorem is our first result.

THEOREM 1.5. Consider a vector field of (LV_1) as in Theorem 1.1.

- (1) The global property that x^* is a global attractor on $(\mathbf{R}_+^3)^\circ$ are robust.
- (2) The global property that T is a global attractor on $(\mathbf{R}_+^3)^\circ \setminus \Gamma_\epsilon$ are robust.

Here Γ_ϵ is the one-dimensional stable manifold of the x^* . This result has the following corollary.

COROLLARY 1.6. Consider the system (LV) as in Theorem 1.3.[HS]. Then there exists sufficient small $\epsilon > 0$, such that for $|\gamma_i - \gamma_j| < \epsilon$ ($i, j \in \mathbf{Z}_3$), every orbit ψ in $(\mathbf{R}_+^3)^\circ \setminus \Gamma_\epsilon$ satisfies one of the following:

- (1) If $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) > \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$ and $\det A < 0$, then ψ tends toward the equilibrium point x^* .
- (2) If $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) < \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i,i-1})$ then ψ winds toward the heteroclinic cycle T .

In the case (1) of Theorem 1.3.[HS], the system is permanent but the x^* need not be locally stable. Infact it may be locally unstable with a certain condition. And with some additional conditions, the system may have a limit cycle which is robust in competitive systems. We shall show the above case in Proposition 2.2. and Corollary 2.3. as our second result.

PROPOSITION 2.2. Consider the system (LV) with $a_{ii} = -1$, $\det A < 0$. Suppose that $\gamma_i x_i^*$ are constant k , $a_{i,i-1}$ ($i \in \mathbf{Z}_3$) have negative values (and not all the same) and satisfy the following conditions (C1) and (C2):

$$(C1) \quad 1 + \prod_{i \in \mathbf{Z}_3} (1 + a_{i,i-1}) > 0.$$

$$(C2) \quad 8 + \prod_{i \in \mathbf{Z}_3} a_{i,i-1} < 0.$$

If $a_{i,i+1}$ ($i \in \mathbf{Z}_3$) close enough to 0, then there exists a non-trivial ω -limit set in $(\mathbf{R}_+^3)^\circ$. In particular, when $a_{i,i+1}$ ($i \in \mathbf{Z}_3$) close enough to 0 from below, the ω -limit set is a limit cycle.

For our interests where the limit sets exist, we conclude by showing the existence of a positively invariant set which includes them in Theorem 3.1. and Corollary 3.2.

THEOREM 3.1. *Given system (LV) with $a_{ij} + a_{ji} < 0$ ($i, j \in \mathbf{Z}_3$). If the set I satisfies (G1), (G2) and (G3), then the set I is positively invariant and the every orbit from $(\mathbf{R}_+^3)^\circ$ has an ω -limit in the set I .*

For details see §3.

§1. PROOF OF THEOREM 1.5.

We denote by $(LV_{1\epsilon})$ the system (LV_ϵ) with $\gamma_1 = \gamma_2 = \gamma_3 (> 0)$. The assertion of Theorem 1.5. means that, in the system $(LV_{1\epsilon})$, every orbit ψ from $(\mathbf{R}_+^3)^\circ$ tends toward x^* or every orbit ψ from $(\mathbf{R}_+^3)^\circ \setminus \Gamma_\epsilon$ winds toward T if $\det A < 0$ and $\prod_{i \in \mathbf{Z}_3} (a_{i,i+1} - a_{ii}) +$

$$\prod_{i \in \mathbf{Z}_3} (a_{i,i-1} - a_{ii}) \neq 0.$$

LEMMA 1.7. *There exists a open set $K_\epsilon \subset (\mathbf{R}_+^3)^\circ$, a neighborhood $n(\Gamma_1)$ of Γ_1 and a smooth scalar function $G(x)$ on $(\mathbf{R}_+^3)^\circ \setminus \Gamma_1$ such that $(LV_{1\epsilon})$ is transverse to $C_\theta := \{x \in (\mathbf{R}_+^3)^\circ : G(x) = \theta\}$ for any θ in $(0, \pi)$, and $\dot{G}$ has a constant sign on $K_\epsilon \setminus n(\Gamma_1)$.*

Proof of Lemma 1.7. We consider the projected vector field $(\hat{x})$ of (LV_1) on $S_+^2 := \{x \in \mathbf{R}_+^3 : |x| = 1\}$ as follows:

$$(\hat{x}) \quad \dot{x} = F - |x|^{-2}(x \cdot F)x,$$

where $F = (F_1, F_2, F_3)$ is the vector field of the system (LV_1) . We denote by $\gamma_x(t)$ the orbit of $(\hat{x})$ through the point x at $t = 0$.

We define the map $\hat{\psi} : \mathbf{R} \times S_+^2 \rightarrow S_+^2$ by $\hat{\psi}(t, x) = \gamma_x(t)$.

Then we have

$$\hat{\psi}(0, x) = x \quad x \in S_+^2$$

and

$$\hat{\psi}(t_1 + t_0, x) = \hat{\psi}(t_1, \hat{\psi}(t_0, x)) \quad t_1, t_0 \in \mathbf{R}.$$

For each $t \in \mathbf{R}$ we have a map

$$\widehat{\psi}_t : S_+^2 \rightarrow S_+^2$$

defined by

$$\widehat{\psi}_t(x) = \hat{\psi}(t, x) \quad (t, x) \in \mathbf{R} \times S_+^2.$$

We shall consider the system (LV_1) in the case (1). Because x^* is locally asymptotically stable, in the vector field $(\hat{x})$ there exists a positive real number d such that $(\hat{x})$ is transverse to the closed curve $\widehat{C}_d$ inward, where $\widehat{C}_d := \{x \in (S_+^2)^\circ : |x - x^*| = d\}$.

For each $t < 0$ the closed curve $\widehat{\psi}_t(\widehat{C}_d) = \{\widehat{\psi}_t(x) \in (S_+^2)^\circ : x \in \widehat{C}_d\}$ is smooth because $\widehat{\psi}_t$ is a diffeomorphism. For each $t < 0$, let

$$C_\theta := \bigcup_{s>0} s\widehat{\psi}_t(\widehat{C}_d) \subset (\mathbf{R}_+^3)^\circ,$$

where $\theta = \cot^{-1} t$.

Now we consider the system $(LV_{1\epsilon})$ in the case (1). We define the function $G : (\mathbf{R}_+^3)^\circ \rightarrow \mathbf{R}$ as follows:

$$G(x) = \begin{cases} \theta & (x \in (\mathbf{R}_+^3)^\circ \setminus \Gamma_1), \\ 0 & (x \in \Gamma_1). \end{cases}$$

We consider

$$\dot{G} := \nabla G \cdot f = \sum_{i \in \mathbf{Z}_3} \frac{\partial G}{\partial x_i} \dot{x}_i,$$

where $\nabla := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ and $f := (f_1, f_2, f_3)$ is the vector field of the system $(LV_{1\epsilon})$.

Hence for the sufficiently small ϵ , there exists a open set $K_\epsilon \subset (\mathbf{R}_+^3)^\circ$ and the neighborhood $n(\Gamma_1)$ of Γ_1 such that

$$\dot{G} < 0 \quad \text{on } K_\epsilon \setminus n(\Gamma_1).$$

Similarly in the case (2), we have

$$\dot{G} > 0 \quad \text{on } K_\epsilon \setminus n(\Gamma_1).$$

□

Proof of Theorem 1.5. First we consider sail-like surfaces $D_r := \{x \in (\mathbf{R}_+^3)^\circ : D(x) = r\}$ and a family of cone-like annuluses C_θ , where $D(x) := |x|^2$ on $(\mathbf{R}_+^3)^\circ$ and $G(x)$ are smooth scalar functions defined in Lemma 1.7.

For $r_1, r_2 (0 < r_1 < r_2 < \infty)$ and $\theta_1, \theta_2 (0 < \theta_1 < \theta_2 < \pi)$ we suppose the domain:

$$D_{(r_1, r_2)} := \{x : r_1 < D(x) < r_2\}, \quad C_{(\theta_1, \theta_2)} := \{x : \theta_1 < G(x) < \theta_2\}.$$

Clearly we have

$$D_{(0, p)} \subseteq D_{(0, q)}, \quad C_{(0, p)} \subseteq C_{(0, q)},$$

for $p < q$. And we have

$$\lim_{\theta \rightarrow 0} C_\theta = \Gamma_1, \quad \lim_{\theta \rightarrow \pi} C_\theta = \partial \mathbf{R}_+^3 \setminus \{\infty\} \quad \text{and} \quad \lim_{r \rightarrow 0} D_r = \{0\}.$$

We shall consider the system (LV_1) . If $\det A < 0$ and $\prod_{i \in \mathbf{Z}_3} (a_{i, i+1} - a_{ii}) + \prod_{i \in \mathbf{Z}_3} (a_{i, i-1} - a_{ii}) > 0$, then the system (LV_1) is permanent by Theorem 1.3. Therefore there is some bounded domain $I(q) \subset (\mathbf{R}_+^3)^\circ$ such that for any orbit ψ from any point in $I(q)$ and sufficiently large t , $\psi(t) \in (\mathbf{R}_+^3)^\circ \setminus I(q)$.

Hence there is some $\underline{\delta}, \bar{\delta} (0 < \underline{\delta} < \bar{\delta} < \infty)$ such that for any orbit ψ from any point in $(\mathbf{R}_+^3)^\circ$, $\liminf_{t \rightarrow \infty} \psi^i(t) > \underline{\delta}$ and $\limsup_{t \rightarrow \infty} \psi^i(t) < \bar{\delta}$ ($i \in \mathbf{Z}_3$), where the $\psi^i(t)$ are the components of $\psi(t)$.

When we consider the system (LV_ϵ) , there exists some domain $I(q)_\epsilon$, the construction of which is similar to that of $I(q)$ in the (LV_1) , because the permanence of the system is robust in this case (see [HS]). In addition to existence of $I(q)_\epsilon$, from the Appendix 1 if ϵ is enough close to 0, then $I(q)_\epsilon$ is close to $I(q)$ enough.

Now we consider the system $(LV_{1\epsilon})$ in the case (1). For sufficiently large l we have

$$\dot{D} \geq D(1 - l(x_1 + x_2 + x_3)),$$

where $\dot{D} := \nabla D \cdot f$.

Therefore for sufficiently small m we have

$$\dot{D}|_{D_{(0, m)}} > 0.$$

On the other hand for sufficiently large L and T we have

$$\psi(T) \in D_{(0, L)} \quad \text{if} \quad \psi(0) \in D_{(L, \infty)},$$

where $\psi(t)$ is the orbit of the system $(LV_{1\epsilon})$.

In the system (LV_1) , if necessary we shall exchange once, there exists a family $\{C_\theta\}$, such that $\dot{G} < 0$ on $\{x \in C_\theta : 0 < \theta < \infty\}$. Therefore in the system $(LV_{1\epsilon})$, for any $\underline{r}, \bar{r}$ ($0 < \underline{r} < \bar{r} < \infty$) and any $\underline{\theta}, \bar{\theta}$ ($0 < \underline{\theta} < \bar{\theta} < \pi$), if necessary by resetting ϵ_i enough small, then there is r_1, r_2 ($0 < r_1 < \underline{r} < \bar{r} < r_2 < \infty$) and θ_1, θ_2 ($0 < \theta_1 < \underline{\theta} < \bar{\theta} < \theta_2 < \pi$) such that

$$\dot{G} < 0 \quad \text{on} \quad D_{(r_1, r_2)} \cap C_{(\theta_1, \theta_2)}, \quad (1)$$

and

$$D_{(0, r_1)}, D_{(r_2, \infty)}, C_{(\theta_2, \pi)} \subset I(q). \quad (2)$$

On the other hand, there exists a tubular neighborhood $n(\Gamma_\epsilon)$ of one dimensional manifold Γ_ϵ which is tangent to the eigenspace spanned by an eigenvector with real eigen value of the Jacobian matrix $D_{x_\epsilon^*} f$ of f at x_ϵ^* such that for any orbit ψ from any point in $n(\Gamma_\epsilon)$ we have $\omega(\psi) = x_\epsilon^*$, where $\omega(\cdot)$ is ω -limit set of $\cdot$ and x_ϵ^* is an interior equilibrium point of the system (LV_ϵ) .

Now we define a domain J_ϵ :

$$J_\epsilon := \max_{r'_2, \theta'} \min_{r'_1} \{ x \in (\mathbf{R}_+^3)^\circ : x \in D_{(r'_1, r'_2)} \cap C_{(0, \theta')} \subset n(\Gamma_\epsilon) \}.$$

If necessary by resetting ϵ_i more close to 0, r'_1 can be close to 0 enough, r'_2 can be large enough and θ' can be close to π enough, namely for any r_1, r_2 (> 0) and $\theta_1 \in (0, \pi)$, there are ϵ_i ($0 < |\epsilon_i| \ll 1$), such that $0 < r'_1 < r_1 < r_2 < r'_2 < \infty$ and $0 < \theta_1 < \theta' < \pi$. Hence,

$$C_{(0, \theta_1)} \subset \{I(q)_\epsilon \cup J_\epsilon\} \subset \{I(q)_\epsilon \cup n(\Gamma_\epsilon)\}. \quad (3)$$

Therefore by (1), (2) and (3), for any orbit ψ from any point in $(\mathbf{R}_+^3)^\circ$, we have $\omega(\psi) = x_\epsilon^*$.

In the other case that $\prod_{i \in \mathbf{Z}_3} (a_{i, i+1} - a_{ii}) + \prod_{i \in \mathbf{Z}_3} (a_{i, i-1} - a_{ii}) < 0$, the proof is done similarly.

□

This theorem has the following corollary.

COROLLARY 1.6. *Consider the system (LV) as in Theorem 1.3.[HS]. Then there exists sufficient small $\epsilon > 0$, such that for $|\gamma_i - \gamma_j| < \epsilon$ ($i, j \in \mathbf{Z}_3$), every orbit ψ in $(\mathbf{R}_+^3)^\circ \setminus \Gamma_\epsilon$ satisfies one of the following:*

- (1) *If $\prod_{i \in \mathbf{Z}_3} (a_{i, i+1} - a_{ii}) > \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i, i-1})$ and $\det A < 0$, then ψ tends toward the equilibrium point x^* .*
- (2) *If $\prod_{i \in \mathbf{Z}_3} (a_{i, i+1} - a_{ii}) < \prod_{i \in \mathbf{Z}_3} (a_{ii} - a_{i, i-1})$ then ψ winds toward the heteroclinic cycle T .*

§2. EXISTENCE OF A LIMIT CYCLE

In another case that the system is not so near the system (LV_1) , an ω -limit set which is another type one in Theorem 1.1. is happened. In some case, it is a limit cycle. We shall show these cases in Proposition 2.2.

PROPOSITION 2.2. *Consider the system (LV) with $a_{ii} = -1$, $\det A < 0$. Suppose that $\gamma_i x_i^*$ are constant k , $a_{i, i-1}$ ($i \in \mathbf{Z}_3$) have negative values (and not all the same) and satisfy the following conditions (C1) and (C2):*

$$(C1) \quad 1 + \prod_{i \in \mathbf{Z}_3} (1 + a_{i, i-1}) > 0.$$

$$(C2) \quad 8 + \prod_{i \in \mathbf{Z}_3} a_{i,i-1} < 0.$$

If $a_{i,i+1}$ ($i \in \mathbf{Z}_3$) close enough to 0, then there exists a non-trivial ω -limit set in $(\mathbf{R}_+^3)^\circ$. In particular, when $a_{i,i+1}$ ($i \in \mathbf{Z}_3$) close enough to 0 from below, the ω -limit set is a limit cycle.

The above proposition has the following corollary.

COROLLARY 2.3. Consider the system (LV) with $a_{ii} = -1$, $\det A < 0$, $\gamma_i > 0$ and $x^* \in (\mathbf{R}_+^3)^\circ$ ($i \in \mathbf{Z}_3$). Suppose that $a_{i,i-1}$ are negative and not all the same, and the parameters of the system holds the following condition (D1) and (D2):

$$(D1) \quad \prod_{i \in \mathbf{Z}_3} (1 + a_{i,i+1}) + \prod_i (1 + a_{i,i-1}) > 0.$$

$$(D2) \quad \left(\sum_{i \in \mathbf{Z}_3} \gamma_i x_i^* \right) \left\{ \sum_{i \neq j \in \mathbf{Z}_3} \gamma_i \gamma_j x_i^* x_j^* (1 - a_{ij} a_{ji}) \right\} + \left(\prod_{i \in \mathbf{Z}_3} \gamma_i x_i^* \right) \det A < 0.$$

Then there exists a non-trivial ω -limit set in $(\mathbf{R}_+^3)^\circ$. In particular, if $a_{i,i+1} < 0$ ($i \in \mathbf{Z}_3$), then the ω -limit set is a limit cycle.

§3. STATEMENT AND PROOF OF THEOREM 3.1.

For our interests where the limit set exists, we shall consider the following set I and conditions (G1), (G2) and (G3).

$$I := \{x \in \mathbf{R}_+^3 : C_{\min} \leq \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3} \leq C_{\max}\}$$

where

$$C_{\min} := \min(C'_{\min}, C_{\bar{x}}) \quad \text{and} \quad C_{\max} := \max(C'_{\max}, C_{\bar{x}}).$$

Here

$$C'_{\min} := \inf\{c > 0 : (G1) \cap (G2)\}, \quad C'_{\max} := \sup\{c > 0 : (G1) \cap (G2)\}$$

and $C_{\bar{x}}$ is defined in (G3).

(G1) : For some $i, j \in \mathbf{Z}_3$ ($i \neq j$),

$$(1 - c\gamma_i)(1 - c\gamma_j) < 0. \tag{6}$$

(G2) : For some $i, j \in \mathbf{Z}_3$ ($i \neq j$) if

$$\gamma_i^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j + \gamma_j^2 > 0,$$

$$c(2\gamma_i^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j) > \gamma_i - \gamma_j \quad \text{and} \quad c(2\gamma_j^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j) > \gamma_j - \gamma_i$$

holds, then

$$h_c(1 - c\gamma_i) < 0 \quad \text{or} \quad h_c(1 - c\gamma_j) < 0, \tag{7}$$

where

$$h_c := \frac{\gamma_i^2 \gamma_j^2 (a_{ij} + a_{ji} + 2)(a_{ij} + a_{ji} - 2)c^2 + 2\gamma_i \gamma_j (a_{ij} + a_{ji} + 2)(\gamma_i \gamma_j)c + (\gamma_i - \gamma_j)^2}{\gamma_i^2 + (a_{ij} + a_{ji})\gamma_i\gamma_j + \gamma_j^2}.$$

(G3) : We put ${}_1a_1$, ${}_{\gamma}a_{\gamma}$, s and $\tilde{x}$ as follows.

$${}_1a_1 := (1, 1, 1) (A + {}^tA)^{-1} {}^t(1, 1, 1),$$

$${}_{\gamma}a_{\gamma} := \left(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}\right) (A + {}^tA)^{-1} {}^t\left(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}\right).$$

Thus

$${}_1a_{\gamma} := (1, 1, 1) (A + {}^tA)^{-1} {}^t\left(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}\right) = {}_{\gamma}a_1.$$

And

$$s := -({}_1a_1 / {}_{\gamma}a_{\gamma})^{\frac{1}{2}},$$

$$\tilde{x} := \left\{s\left(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}\right) - (1, 1, 1)\right\} (A + {}^tA)^{-1}.$$

If $\tilde{x} \in (\mathbf{R}_+^3)^{\circ}$, then

$$C_{\tilde{x}} = -{}_1a_{\gamma} - ({}_1a_1 \cdot {}_{\gamma}a_{\gamma})^{\frac{1}{2}}, \quad (8)$$

where $\cdot$ is the ordinary multiplication of numbers.

THEOREM 3.1. *Given system (LV) with $a_{ij} + a_{ji} < 0$ ($i, j \in \mathbf{Z}_3$). If the set I satisfies (G1), (G2) and (G3), then the set I is positively invariant and the every orbit from $(\mathbf{R}_+^3)^{\circ}$ has an ω -limit in the set I .*

Proof We consider the function S as follows:

$$S(x) := \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3},$$

on $\mathbf{R}_+^3$. Thus the differential of S is

$$\dot{S} = x_1 + x_2 + x_3 + xA^tx. \quad (9)$$

Define a plane S_c for $c > 0$ and a quadratic surface Q as follows:

$$S_c := \{x \in \mathbf{R}_+^3 : S(x) = c\},$$

$$Q := \{x \in \mathbf{R}_+^3 : \dot{S} = 0\}.$$

Remark that the surface Q includes all equilibrium points of the system (LV). For enough large $l \in \mathbf{R}$ and the matrix E_1 whose all elements are equal to 1,

$$\dot{S} \geq x_1 + x_2 + x_3 - l x E_1^t x = (x_1 + x_1 + x_2)\{1 - l(x_1 + x_2 + x_3)\}.$$

Thus for an enough small $c' > 0$,

$$\dot{S}|_{x \in S_{c'}} > 0 \quad \text{on} \quad (\mathbf{R}_+^3)^{\circ}.$$

And for the identity matrix E_0 ,

$$\dot{S} \leq x_1 + x_2 + x_3 - x E_0^t x = x_1(1 - x_1) + x_2(1 - x_2) + x_3(1 - x_3).$$

Thus for an enough large $C' > 0$,

$$\dot{S}|_{x \in S_{c'}} < 0 \quad \text{on} \quad (\mathbf{R}_+^3)^\circ.$$

If for some $c > 0$

$$Q \cap S_c \cap \mathbf{R}_+^3 = \emptyset,$$

then for an arbitrary initial value $\psi(0) \in \mathbf{R}_+^3 \cap S_c$ and arbitrary $t > 0$,

$$\psi(t) \in \{x : S(x) > c\} \quad \text{or} \quad \psi(t) \in \{x : S(x) < c\}.$$

Here we denote the minimum value and maximum value of c such that $Q \cap S_c \cap \mathbf{R}_+^3 \neq \emptyset$ by $C_{\min}$ and $C_{\max}$ respectively.

Now γ_i and a_{ij} ($i, j \in \mathbf{Z}_3$) are fixed. The quadratic curve $Q \cap S_c$ does not change the type of it (but may degenerate).

Hence we shall consider the value of c such that this quadratic curve $Q \cap S_c$ and the compact set $S_c \cap \mathbf{R}_+^3$ have a common point.

In the case of (G1) and (G2), we shall consider the common set of $S_c \cap \partial \mathbf{R}^3$ and $Q \cap S_c$, where $\partial \mathbf{R}^3 := \{x \in \mathbf{R}^3 : \prod_i x_i = 0\}$. We may assume $x_3 = 0$ without loss of generality. So if

$$\{x \in \mathbf{R}_+^3 : x \in Q \cap S_c \text{ and } x_3 = 0\} \subset \{x \in \mathbf{R}_+^3 : x \in S_c \cap \partial \mathbf{R}^3\},$$

then $Q \cap S_c \cap \mathbf{R}_+^3 \neq \emptyset$.

It becomes a simple problem where the solution of the quadratic equation exist.

In the case of (G3), we can not determine the value of $C_{\min}$ and $C_{\max}$ in this way. But just on $C_{\min}$ or $C_{\max}$, the following equality holds.

$$s \nabla S_c = \nabla Q|_{\{x: S=C_{\min} \text{ or } C_{\max}\}}, \quad (10)$$

where $s (\neq 0) \in \mathbf{R}$.

Since $\nabla S_c = (\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3})$, and Q is quadratic, the point $\tilde{x} \in \mathbf{R}^3$ which holds (10) are less than two or equal two. For those points, the just value of C such that $\tilde{x} \in S_c$ is $C_{\min}$ or $C_{\max}$.

We shall calculate in the state as follows:

$$\nabla Q = g + x D_x g = (1, 1, 1) + (x_1, x_2, x_3)(A + {}^t A),$$

where we put the system (LV), $\dot{x}_i = \gamma_i x_i g_i$ ($i \in \mathbf{Z}_3$) and $g := (g_1, g_2, g_3)$.

From the equation (10),

$$\tilde{x} = \{s(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}) - (1, 1, 1)\}(A + {}^t A)^{-1}. \quad (11)$$

Since $\tilde{x} \in Q$,

$$(1, 1, 1)\tilde{x} + \frac{1}{2}\tilde{x}(A + {}^t A){}^t \tilde{x} = 0. \quad (12)$$

By (11) and (12),

$$s^2 = {}_1 a_1 / \gamma a_\gamma. \quad (13)$$

When $\tilde{x} \in \mathbf{R}_+^3$, we may put $\tilde{c} := \{c : \tilde{x} \in S_c\}$.

Thus

$$\tilde{c} = (\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}){}^t \tilde{x} = -(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3})(A + {}^t A)^{-1} {}^t (1, 1, 1) - ({}_1 a_1 \cdot \gamma a_\gamma)^{\frac{1}{2}},$$

where we put $s = -(a_1/\gamma a_\gamma)^{\frac{1}{2}}$ by (13), because if $s = (a_1/\gamma a_\gamma)^{\frac{1}{2}}$ then $\bar{c} \leq 0$. The value of $\bar{c}$ is $C_{\min}$ or $C_{\max}$ in the case. $\square$

For arbitrary $\gamma'_i > 0$ ($i \in \mathbf{Z}_3$), we shall consider the set:

$$I_{S_\gamma} := \{x \in \mathbf{R}_+^3 : C_{\min} \leq \frac{x_1}{\gamma'_1} + \frac{x_2}{\gamma'_2} + \frac{x_3}{\gamma'_3} \leq C_{\max}\}.$$

Here the value of $C_{\min}$ and $C_{\max}$ are the minimum and the maximum value respectively such that

$$Q_\gamma \cap S_\gamma c \cap \mathbf{R}_+^3 \neq \emptyset,$$

where

$$S_\gamma c := \{x \in \mathbf{R}^3 : S_\gamma := \frac{x_1}{\gamma'_1} + \frac{x_2}{\gamma'_2} + \frac{x_3}{\gamma'_3} = c\},$$

$$\dot{S}_\gamma = \frac{\gamma_1}{\gamma'_1} x_1 + \frac{\gamma_2}{\gamma'_2} x_2 + \frac{\gamma_3}{\gamma'_3} x_3 + x \operatorname{diag}\left\{\frac{\gamma_1}{\gamma'_1}, \frac{\gamma_2}{\gamma'_2}, \frac{\gamma_3}{\gamma'_3}\right\} A^t x,$$

$$Q_\gamma := \{x \in \mathbf{R}_+^3 : \dot{S}_\gamma = 0\}.$$

Here $\operatorname{diag}\{\cdot\}$ is a diagonal matrix with components $\cdot$. Remark that all equilibrium points of the system (LV) are included in Q_γ , too.

COROLLARY 3.2. *Consider the system as in Theorem 3.1. For an arbitrary vector $\gamma' := (\gamma'_1, \gamma'_2, \gamma'_3)$ ($\gamma'_i > 0$ $i \in \mathbf{Z}_3$), we put*

$$I := \bigcap_{\gamma} I_{S_\gamma}.$$

Then the set I is positively invariant, and every orbit from $(\mathbf{R}_+^3)^\circ$ has an ω -limit in the set I .

Remark that in the case of (G3), it is well known that if the Q_γ is elliptic, then the x^* is exactly a global attractor on $(\mathbf{R}_+^3)^\circ$.

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