

ON BORSUK-ULAM GROUPS

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ABSTRACT. A Borsuk-Ulam group is a group for which the isovariant Borsuk-Ulam theorem holds. A fundamental question is: which groups are Borsuk-Ulam groups? In this article, we shall recall some properties and previous results on a Borsuk-Ulam group. After that, we provide a new family of Borsuk-Ulam groups. We also pose some open questions.

1. NOTATION AND TERMINOLOGY

Let G be a compact Lie group and V an (orthogonal or unitary) representation space of G . We denote by SV the unit sphere of V , called a G -representation sphere. A G -equivariant map (or G -map for short) $f : X \rightarrow Y$ is a continuous map between G -spaces satisfying

$$f(gx) = gf(x), \quad \forall x \in X, g \in G.$$

It is easy to see that if f is G -equivariant, then

- (1) $f(X^H) \subset Y^H$, so we have the restriction map

$$f^H : X^H \rightarrow Y^H.$$

- (2) $G_x \leq G_{f(x)}$ ($\forall x \in X$).

Definition. A continuous map $f : X \rightarrow Y$ is called a G -isovariant map if f is a G -equivariant map satisfying $G_x = G_{f(x)}$ ($\forall x \in X$).

It is easy to see that $f : X \rightarrow Y$ is G -isovariant if and only if f is a G -equivariant map such that $f|_{G(x)} : G(x) \rightarrow Y$ is injective for any $x \in X$, where $G(x)$ is the orbit of x . Similarly we define an isovariant homotopy as follows.

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Definition. Let f, g be G -isovariant maps. We call f and g isovariantly G -homotopic if there exists a G -isovariant map $H : X \times I \rightarrow Y$, called a G -isovariant homotopy, such that $H(-, 0) = f$ and $H(-, 1) = g$.

Let $[X, Y]_G^{\text{isov}}$ denote the set of G -isovariant homotopy classes of G -isovariant maps.

By the definition of isovariance, we easily see the following.

- (1) Let X and Y be free G -spaces. Then G -equivariance is equivalent to G -isovariance.
- (2) If $f : X \rightarrow Y$ is an injective G -map, then f is G -isovariant.
- (3) If there exists a G -isovariant map $f : X \rightarrow Y$, then $\text{Iso}(X) \subset \text{Iso}(Y)$, where $\text{Iso}(X)$ is the set of isotropy subgroups of X .

Example 1.1. Let $X = G/H$ and $Y = G/K$.

- (1) There exists a G -map $f : G/H \rightarrow G/K$ if and only if $(H) \leq (K)$, i.e., $H \leq aKa^{-1}$ for some $a \in G$.
- (2) There exists a G -isovariant map $f : G/H \rightarrow G/K$ if and only if $(H) = (K)$. In this case, a G -isovariant map f is defined by $f(gH) = gaK, H = aKa^{-1}$.

2. ISOVARIANT MAPS BETWEEN REPRESENTATIONS

The following result says that isovariant maps between representations are essentially same as those between representation spheres.

Proposition 2.1. *Let V, W be (orthogonal) G -representations. The following are equivalent.*

- (1) *There exists a G -isovariant map $f : V \rightarrow W$.*
- (2) *There exists a G -isovariant map $f : V^{G^\perp} \rightarrow W^{G^\perp}$.*
- (3) *There exists a G -isovariant map $f : S(V^{G^\perp}) \rightarrow S(W^{G^\perp})$.*

Here $V^{G^\perp}$ is the orthogonal complement of V^G in V . In particular, if $V^G = W^G = 0$, then there exists a G -isovariant map $f : V \rightarrow W$ if and only if $f : SV \rightarrow SW$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Composing the inclusion i and the projection p with $f : V \rightarrow W$, we have an isovariant map

$$\bar{f} : V^{G^\perp} \xrightarrow{i} V \xrightarrow{f} W \xrightarrow{p} W^{G^\perp}.$$

Composing the inclusion j and the normalization map with $\bar{f}$, we have an isovariant map

$$\bar{\bar{f}} : S(V^{G^\perp}) \xrightarrow{j} V^{G^\perp} \setminus \{0\} \xrightarrow{\bar{f}} W^{G^\perp} \setminus \{0\} \xrightarrow{\text{norm.}} S(W^{G^\perp}).$$

$$(1) \Leftarrow (2) \Leftarrow (3)$$

Let $g : S(V^{G^\perp}) \rightarrow S(W^{G^\perp})$ be an isovariant map. By the radial extension, we have an isovariant map

$$\tilde{g} : V^{G^\perp} \rightarrow W^{G^\perp}.$$

By adding the zero map to $\bar{g}$, we have an isovariant map

$$h := \tilde{g} \oplus 0 : V = V^{G^\perp} \oplus V^G \rightarrow W^{G^\perp} \oplus W^G = W.$$

□

By further arguments, we also obtain

Proposition 2.2. *When $V^G = W^G = 0$, there is a one-to-one correspondence*

$$[V, W]_G^{\text{isov}} \cong [SV, SW]_G^{\text{isov}}.$$

We here provide some examples. Let $G = C_n = \langle c \rangle$ be a cyclic group of order n , where c is a generator of C . Consider the irreducible representations of C . Let

$$U_k (= \mathbb{C}) \quad (0 \leq k \leq n-1)$$

denote the irreducible representation with the linear action:

$$c \cdot z = \xi_n^k z \quad (z \in U_k), \quad \xi_n = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right).$$

Assume $n = pq$, where p, q are distinct primes and $G = C_{pq}$.

Example 2.3. If $(k, pq) = (l, pq) = 1$, then there exist a G -isovariant map $f : SU_k \rightarrow SU_l$.

In fact, fix s such that $ks \equiv 1 \pmod{pq}$. We define a map f by

$$f(z) = z^{sl}, \quad z \in SU_k.$$

Then one can check that

- (1) f is G -equivariant,
- (2) G acts freely on SU_k and SU_l .

Hence f is G -isovariant.

Further arguments show that the degree of maps classifies isovariant homotopy classes, and we have

$$[U_k, U_l]_{C_{pq}}^{\text{isov}} \cong [SU_k, SU_l]_{C_{pq}}^{\text{isov}} \cong \mathbb{Z},$$

and the representatives are given by

$$f_m(z) = z^{sl+mpq}, \quad z \in SU_k, \quad m \in \mathbb{Z}.$$

See [3], [4] for the detail.

Example 2.4. There do not exist isovariant maps $f : U_p \rightarrow U_q$ and $g : U_1 \rightarrow U_q$.

In fact, if $f : X \rightarrow Y$ is an isovariant map, then $\text{Iso}(X) \subset \text{Iso}(Y)$. However

$$\text{Iso}(U_p) = \{C_p, G\} \not\subset \text{Iso}(U_q) = \{C_q, G\}$$

and

$$\text{Iso}(U_1) = \{1, G\} \not\subset \text{Iso}(U_q) = \{C_q, G\}.$$

Example 2.5. There exists an isovariant map $f : U_1 \rightarrow U_p \oplus U_q$.

In fact there are isovariant maps

$$f_{\alpha, \beta} : SU_1 \rightarrow S(U_p \oplus U_q)$$

defined by

$$f_{\alpha, \beta}(z) = (z^{(1+\alpha q)p}, z^{(1+\beta p)q}), \quad \alpha, \beta \in \mathbb{Z}, \quad z \in SU_1.$$

These are isovariant maps since

$$G_{f_{\alpha, \beta}(z)} = G_{z^{(1+\alpha q)p}} \cap G_{z^{(1+\beta p)q}} = 1 \quad (z \in SU_1).$$

In this case, the multidegree classifies isovariant maps and one sees

$$[U_1, U_p \oplus U_q]_{C_{pq}}^{\text{isov}} \cong [SU_1, S(U_p \oplus U_q)]_{C_{pq}}^{\text{isov}} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

See [3], [4] for the detail.

Example 2.6. There does not exist a G -isovariant map $f : U_1 \oplus U_1 \rightarrow U_p \oplus U_q$.

If there is an isovariant map, then the isovariant Borsuk-Ulam theorem stated in the next section shows

$$\dim U_1 \oplus U_1 - \dim(U_1 \oplus U_1)^{C_p} \leq \dim U_p \oplus U_q - \dim(U_p \oplus U_q)^{C_p}$$

$$\parallel$$

$$4 - 0 = 4$$

$$\parallel$$

$$4 - 2 = 2.$$

This is a contradiction.

Remark. There is a G -map $f : S(U_1 \oplus U_1) \rightarrow S(U_p \oplus U_q)$. In fact there are G -maps $f_i : SU_1 \rightarrow SU_i$ defined by $f_i(z) = z^i$ for $i = p$ and q . Taking join of f_p and f_q , one obtains a G -map $f = f_p * f_q : S(U_1 \oplus U_1) \rightarrow S(U_p \oplus U_q)$.

Thus one can finally see

Proposition 2.7. *Let $G = C_{pq}$, and V, W G -representations. There exists a G -isovariant map $V \rightarrow W$ if and only if*

$$\begin{cases} \dim V - \dim V^H \leq \dim W - \dim W^H \\ \dim V^H - \dim V^G \leq \dim W^H - \dim W^G \end{cases}$$

for $H = C_p, C_q$.

See [2] for the detail.

Question (unsolved). How about C_n for an arbitrary n ?

3. BORSUK-ULAM TYPE THEOREM FOR ISOVARIANT MAPS

In this section we discuss a Borsuk-Ulam type theorem for isovariant maps, which provides non-existence results on isovariant maps as mentioned in the previous section.

The Borsuk-Ulam theorem due to Borsuk [1] is generalized in various ways (see [6], [7]). The following is one of them. Let C_p be a cyclic group of prime order p and assume that C_p acts freely on spheres S^m and S^n .

Theorem 3.1 (mod p Borsuk-Ulam theorem).

If there exists a C_p -map ($\iff C_p$ -isovariant map) $f : S^m \rightarrow S^n$, then $m \leq n$, (or equivalently, if $m > n$, there does not exist a C_p -map $f : S^m \rightarrow S^n$).

Wasserman first studied the isovariant version of the Borsuk-Ulam theorem and introduced the notion of the Borsuk-Ulam group.

Definition (Wasserman). A compact Lie group G is called a *Borsuk-Ulam group* (BUG) if the following statement holds:

For any pair of G -representations V and W , if there is a G -isovariant map $f : V \rightarrow W$, then the Borsuk-Ulam inequality:

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

Proposition 3.2 ([8]). C_p and S^1 are BUGs.

The following are fundamental properties of Borsuk-Ulam groups.

Proposition 3.3 ([8]).

- (1) *If $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ is exact and H, K are BUGs, then G is also a BUG.*
- (2) *A quotient group of a BUG is also a BUG.*

Question (unsolved). Is a subgroup of a BUG also a BUG?

Using this result repeatedly, we have

Corollary 3.4. *If*

$$1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G$$

and H_i/H_{i-1} are BUGs ($1 \leq i \leq r$), then G is a BUG.

We have the following.

Theorem 3.5 (Isovariant Borsuk-Ulam theorem). *Any solvable compact Lie group G is a BUG.*

Proof. As is well-known, G is solvable if and only if there exists a composition series

$$1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G$$

such that $H_i/H_{i-1} = C_p$ or S^1 . By Proposition 3.4, G is a BUG. $\square$

So the next question is: how about non-solvable case? Wasserman also found non-solvable examples of BUGs using the prime condition.

Definition (Prime condition (PC)). (1) We say that a finite simple group G satisfies the prime condition (PC) if

$$\sum_{p|o(g)} \frac{1}{p} \leq 1$$

holds for any $g \in G$, where $o(g)$ is the order of g , and the sum is taken over all prime divisors of $o(g)$.

(2) We say that a finite group G satisfies (PC) if for a composition series

$$1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G,$$

each simple H_i/H_{i-1} satisfies (PC) in the sense of (1).

Theorem 3.6 ([8]). *If a finite group G satisfies (PC), then G is a BUG.*

Remark. In the proof of [8], the fact that a cyclic group C is a BUG is used.

Example 3.7. Alternating groups $A_5, A_6, \dots, A_{11}$ satisfy (PC), and hence BUGs. But A_n , $n \geq 12$, does not satisfy (PC). In fact A_n , $n \geq 12$, has an element of order $30 = 2 \cdot 3 \cdot 5$ and $1/2 + 1/3 + 1/5 = 31/30 > 1$.

Question (unsolved). Is A_n a BUG for $n \geq 12$?

Example 3.8. $PSL(2, p)$ satisfies (PC) for p : prime ≤ 53 ; hence a BUG. But $PSL(2, 59)$, $PSL(2, 61)$ do not satisfy (PC). Indeed there are infinitely many primes p such that $PSL(2, p)$ does not satisfy (PC).

4. A NEW FAMILY OF BORSUK-ULAM GROUPS

In this section G is a finite group. Let $\mathbb{F}_q$ be a finite field of order $q = p^r$, p : prime. Recall

$$\begin{aligned} PSL(2, q) &= SL(2, q)/\{\pm I\} \\ &= \{A \in M_2(\mathbb{F}_q) \mid \det A = 1\}/\{\pm I\}. \end{aligned}$$

Remark. $PSL(2, 2^r) = SL(2, 2^r)$.

Also recall:

- (1) If $q = p^r \geq 4$, then $PSL(2, q)$ is simple. On the other hand $PSL(2, 2) \cong S_3$ and $PSL(2, 3) \cong A_4$, which are non-simple.
- (2) $|PSL(2, q)| = \begin{cases} q(q-1)(q+1) & p = 2 \\ \frac{1}{2}q(q-1)(q+1) & p : \text{odd prime.} \end{cases}$

We introduce the Möbius condition in [5] and show the following.

Theorem 4.1 ([5]). *$PSL(2, q)$ is a BUG for any $q = p^r$.*

As a corollary,

Corollary 4.2. *$SL(2, q)$, $GL(2, q)$, $PGL(2, q)$ are BUGs.*

Proof. These are shown from the following exact sequences.

$$\begin{aligned} 1 \rightarrow \{\pm I\} \rightarrow SL(2, q) \rightarrow PSL(2, q) \rightarrow 1 \\ 1 \rightarrow SL(2, q) \rightarrow GL(2, q) \xrightarrow{\det} \mathbb{F}_q^* \rightarrow 1 \end{aligned}$$

$$(F_q^* \cong C_{q-1})$$

$$PGL(2, q) = GL(2, q)/\text{center}$$

$$(\text{center} = \{aI \mid a \in \mathbb{F}_q^*\} \cong \mathbb{F}_q^*).$$

□

As seen before, $PSL(2, 59)$, $PSL(2, 61)$ etc. do not satisfy (PC). Our result provides the first example to be a BUG not satisfying (PC).

Finally we announce the following result which will be proved in the forthcoming paper. Let $\text{Syl}_p(G)$ denote a p -Sylow subgroup of G .

Theorem 4.3 (N-U). *If G satisfies one of the following conditions, then G is a BUG.*

- (1) $\text{Syl}_2(G)$ is a cyclic group C_{2^r} of order 2^r .
- (2) $\text{Syl}_2(G)$ is a dihedral group D_{2^r} of order 2^r ($r \geq 2$). As a convention, $D_4 = C_2 \times C_2$.
- (3) $\text{Syl}_2(G)$ is a generalized quaternion group Q_{2^r} of order 2^r ($r \geq 3$).
- (4) $\text{Syl}_2(G)$ is abelian and $\text{Syl}_p(G)$ is cyclic for every odd prime p .

Example 4.4.

- (1) $PSL(2, q)$, q : odd, is an example of (2).
- (2) $SL(2, q)$, q : odd, is an example of (3).
- (3) $SL(2, 2^r)$ is an example of (4).
- (4) A finite group with periodic cohomology is an example of (1), (3) or (4).

For the proof, we use the fact that $PSL(2, q)$ is a BUG and several deep results of finite group theory.

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