

On 1-summability of formal solution of inhomogeneous heat equation

By

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Abstract

We consider the Cauchy problem for the inhomogeneous heat equation, where inhomogeneous term is a formal power series of Gevrey order 1 with respect to t . Under the assumptions of the 1-summability of the inhomogeneous term and a global exponential growth condition with respect to x for its sum, we show the 1-summability of the formal solution of the Cauchy problem by using the integral representation of the 1-sum of the formal solution which is given in terms of the heat kernel.

§ 1. Introduction

We consider the following Cauchy problem for the complex inhomogeneous heat equation

$$(\hat{H}) \quad \begin{cases} \partial_t u(t, x) = \partial_x^2 u(t, x) + \hat{f}(t, x), \\ u(0, x) = \varphi(x) \in \mathcal{O}_x, \end{cases}$$

where $t, x \in \mathbb{C}$ and $\mathcal{O}_x$ denotes the set of holomorphic functions in a neighborhood of $x = 0$. The inhomogeneous term $\hat{f}(t, x) = \sum_{i \geq 0} f_i(x) t^i / i!$ is a formal power series of Gevrey order 1 with respect to t which means that all coefficients $f_i(x)$ are holomorphic in a common closed disc $\overline{D}_\sigma := \{x \in \mathbb{C}; |x| \leq \sigma\}$ for some positive σ and there exist positive constants A and B such that for all i

$$\max_{|x| \leq \sigma} |f_i(x)| \leq AB^i i!^2$$

and we denote it by $\hat{f}(t, x) \in \mathcal{O}_x[[t]]_1$.

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This Cauchy problem has a unique formal power series solution of the form

$$(1.1) \quad \begin{aligned} \hat{u}(t, x) &= \sum_{n \geq 0} \frac{\varphi^{(2n)}(x)}{n!} t^n + \sum_{n \geq 1} \left(\sum_{i=0}^{n-1} f_i^{(2(n-1-i))}(x) \right) \frac{t^n}{n!} \\ &=: \hat{u}_{hom}(t, x; \varphi) + \hat{u}_{inh}(t, x; \hat{f}), \end{aligned}$$

which is divergent in general. Exactly, we can see that $\hat{u}(t, x)$ is the formal power series of Gevrey order 1 with respect to t .

For the divergent solution when the inhomogeneous term $\hat{f}(t, x) \equiv 0$, the problem of k -summability with $k = 1$ for the formal solution $\hat{u}(t, x) = \hat{u}_{hom}(t, x; \varphi)$ was proved by Lutz, Miyake and Schäfke [5], where the definition of k -summability will be given in next section.

Theorem 1.1 ([5]). *Let $S_x(0, \pi; \varepsilon_1, \sigma) = S_x(0; \varepsilon_1) \cup S_x(\pi; \varepsilon_1) \cup \overline{D}_\sigma$, where $S_x(\theta; \varepsilon_1) := \{x \in \mathbb{C}; |\arg x - \theta| < \varepsilon_1/2\}$ and $\varepsilon_1 > 0$. Then the formal solution $\hat{u}(t, x)$ of the homogeneous Cauchy problem $(\hat{H})$ is 1-summable in 0 direction (we denote it by $\hat{u}(t, x) \in \mathcal{O}_x\{t\}_{1,0}$) if and only if the Cauchy data $\varphi(x) \in \mathcal{O}_x$ satisfies the following conditions.*

- (i) *The Cauchy data $\varphi(x)$ can be analytically continued on a region $S_x(0, \pi; \varepsilon_1, \sigma)$.*
- (ii) *The Cauchy data has the exponential growth estimate of order at most 2 there, that is, $|\varphi(x)| \leq C e^{\delta|x|^2}$ ($x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma)$) with some positive constants C and δ for any closed subsector $\overline{S}_x(0, \pi; \varepsilon'_1, \sigma) \subset S_x(0, \pi; \varepsilon_1, \sigma)$.*

In this case, 1-sum of $\hat{u}_{hom}(t, x; \varphi)$ in 0 direction is obtained by

$$(1.2) \quad u_{hom}^0(t, x; \varphi) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \varphi(x+y) e^{-y^2/4t} dy$$

with $|t - \rho_0| < \rho_0$ and $|x| \leq \sigma_0$ for some positive ρ_0 and $\sigma_0 (< \sigma)$.

We remark that 1-sum $u_{hom}^0(t, x; \varphi)$ in a sector $S_t(0, \alpha, \rho)$ with some $\alpha > \pi$ and $\rho > 0$ in t -space is obtained from the analytic continuation in t -variable by rotating the integral path $\mathbb{R}$ to $e^{i\theta}\mathbb{R}$ with $|\theta| < \varepsilon_1/2$.

In the following, we write the conditions (i) and (ii) by

$$\varphi(x) \in \text{Exp}^2(S_x(0, \pi; \varepsilon_1, \sigma))$$

and we call the conditions "1-summability condition" or **1-S-C** (with ε_1 and σ) for short.

We consider the case that $\hat{f}(t, x) \not\equiv 0$ and $\hat{f}(t, x) \in \mathcal{O}_x[[t]]_1$. We may assume that the Cauchy data $\varphi(x) \equiv 0$ without loss of generality. In this case, Michalik [7] gave a sufficient condition for 1-summability of the formal solution $\hat{u}(t, x) = \hat{u}_{inh}(t, x; \hat{f})$.

Theorem 1.2 ([7]). *Let $g(s, x)$ be the formal 1-Borel transform of $\hat{f}(t, x)$ which is defined by $g(s, x) = \sum_{i \geq 1} f_i(x) s^i / i!^2$, which is convergent at $(s, x) = (0, 0)$. We assume that $g(s, x)$ can be analytically continued to a sector $S_s(0, \varepsilon)$ in s -space by some positive ε and has the exponential growth estimate of order at most 1 there. Moreover, we assume that $g(s, x)$ satisfies 1-S-C with respect to x -variable. Then the formal solution $\hat{u}(t, x)$ of the inhomogeneous Cauchy problem $(\hat{H})$ is 1-summable in 0 direction.*

Under the above same conditions we obtain the following result for 1-sum of $\hat{u}(t, x)$ of the inhomogeneous Cauchy problem $(\hat{H})$ in terms of the integral representation by using heat kernel.

Theorem 1.3. *We assume that $\hat{f}(t, x) \in \mathcal{O}_x\{t\}_{1,0}$ and let $f^0(t, x)$ be 1-sum of $\hat{f}(t, x)$. We assume that there exists a positive constant T such that $f^0(t, x)$ is analytic in $S_T \times \overline{D}_\sigma$, where $S_T = S_t(0, \alpha, T)$ with $\alpha > \pi$. Moreover, we assume that the sum $f^0(t, x)$ satisfies 1-S-C with ε_1 and σ , that is, $f^0(t, x)$ can be analytically continued to $S_x(0, \pi; \varepsilon_1, \sigma)$ with respect to x -variable and $|f^0(t, x)| \leq C e^{\delta|x|^2}$ ($t \in \overline{S}_1$) for $x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma)$ with some positive constants C and δ for any closed subsector $\overline{S}_1 \subset S_T$. Then*

$$(1.3) \quad u_{inh}^0(t, x; f^0) := \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} f^0(t-s, x+y) dy ds$$

with $|t - \rho_0| < \rho_0$ and $|x| \leq \sigma_0$ for some positive constants $\rho_0 (\leq T/2)$ and $\sigma_0 (< \sigma)$ gives 1-sum of $\hat{u}(t, x)$ in 0 direction and satisfies $(\hat{H})$ with $f^0(t, x)$ instead of $\hat{f}(t, x)$.

This paper consists of the following contents. In section 2, we give the definition of k -summability, and in section 3 we give related results for the 1-summability of the formal solution of inhomogeneous Cauchy problem. We prove Theorem 1.3 in section 4, and give a proof of Lemma 4.1 necessary for the proof of the theorem in section 5. In section 6, we give a remark on the integral representation of 1-sum in terms of heat kernel.

§ 2. Definition of k -summability

In this section, we give some notation and definitions in the way of Ramis or Balser (cf. Balser [1] for the details).

For $d \in \mathbb{R}, \beta > 0$ and $\rho (0 < \rho \leq \infty)$, we define a sector $S = S_t(d, \beta, \rho)$ by

$$(2.1) \quad S_t(d, \beta, \rho) := \{t \in \mathbb{C}; |d - \arg t| < \beta/2, 0 < |t| < \rho\},$$

where d, β and ρ are called the direction, the opening angle and the radius of S , respectively. We write $S_t(d, \beta, \infty) = S_t(d; \beta)$ for short.

A closed sector $\overline{S} = \overline{S}_t(d, \beta, \rho)$ is defined by $\overline{S} = \{t \in \mathbb{C}; |d - \arg t| \leq \beta/2, 0 < |t| \leq \rho\}$.

For $k > 0$, we define that $\hat{v}(t, x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ (we say $\hat{v}(t, x)$ is a formal power series of Gevrey order $1/k$) if $v_n(x)$ are holomorphic on a common closed disk $\overline{D}_\sigma$ for some $\sigma > 0$ and there exist some positive constants C and K such that for any n ,

$$(2.2) \quad \max_{|x| \leq \sigma} |v_n(x)| \leq CK^n \Gamma\left(1 + \frac{n}{k}\right),$$

where Γ denotes the gamma function. Here when $v_n(x) \equiv v_n$ (constants) for all n , we use the notation $\mathbb{C}[[t]]_{1/k}$ instead of $\mathcal{O}_x[[t]]_{1/k}$. In the following, we use the similar notation.

Let $k > 0$, $\hat{v}(t, x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ and $v(t, x)$ be an analytic function on $S_t(d, \beta, \rho) \times \overline{D}_\sigma$. Then we define that

$$(2.3) \quad v(t, x) \cong_k \hat{v}(t, x) \quad \text{in } S = S_t(d, \beta, \rho)$$

(we say $v(t, x)$ has the Gevrey asymptotic expansion $\hat{v}(t, x)$ of order k), if for any closed subsector $\overline{S}'$ of S , there exist some positive constants C and K such that for any $N \geq 1$, we have

$$(2.4) \quad \max_{|x| \leq \sigma} \left| v(t, x) - \sum_{n=0}^{N-1} v_n(x)t^n \right| \leq CK^N |t|^N \Gamma\left(1 + \frac{N}{k}\right), \quad t \in \overline{S}'.$$

For $k > 0$, $d \in \mathbb{R}$ and $\hat{v}(t, x) \in \mathcal{O}_x[[t]]_{1/k}$, we say that $\hat{v}(t, x)$ is *k-summable* in d direction, and denote it by $\hat{v}(t, x) \in \mathcal{O}_x\{t\}_{k,d}$, if there exist a sector $S = S_t(d, \beta, \rho)$ with $\beta > \pi/k$ and an analytic function $v(t, x)$ on $S \times \overline{D}_\sigma$ such that $v(t, x) \cong_k \hat{v}(t, x)$ in S .

In the paper, we consider the direction as 0 direction only for simplicity. Therefore we use the notation $\mathcal{O}_x\{t\}_{k,0} = \mathcal{O}_x\{t\}_k$.

We remark that the function $v(t, x)$ above for a *k-summable* $\hat{v}(t, x)$ is unique if it exists. Therefore such a function $v(t, x)$ is called the *k-sum* of $\hat{v}(t, x)$ in 0 direction and we write it $v^0(t, x)$.

§ 3. Related results

Balser [2] gave the necessary and sufficient condition for 1-summability of the formal solution $\hat{u}(t, x) = \hat{u}_{inh}(t, x; \hat{f})$ of the inhomogeneous Cauchy problem $(\widehat{H})$. Here we use the notations.

$$\hat{u}(t, x) = \sum_{j \geq 0} u_{j*}(x) \frac{t^j}{j!} = \sum_{n \geq 0} \hat{u}_{*n}(t) \frac{x^n}{n!} = \sum_{j,n} u_{jn} \frac{t^j}{j!} \frac{x^n}{n!}.$$

Then Balser's result is stated as follows.

Proposition 3.1 ([2]). *The formal solution $\hat{u}(t, x)$ is 1-summable in a direction 0 if and only if $\hat{u}_{*0}(t)$, $\hat{u}_{*1}(t)$ and $\hat{f}(t, x)$ are 1-summable in 0 direction.*

In the above Proposition, we can't know what the 1-summability of $\hat{u}_{*0}(t)$ and $\hat{u}_{*1}(t)$ mean. In the paper of Balser and Loday-Richaud [3], they tried to characterize 1-summability of $\hat{u}_{*0}(t)$ and $\hat{u}_{*1}(t)$ as a property of $\hat{f}(t, x)$ as follows.

For $D_t^{-1}\hat{f}(t, x) = \sum_{j \geq 1, n \geq 0} f_{j-1, n} \frac{t^j}{j!} \frac{x^n}{n!}$ with $D_t^{-1} = \int_0^t$, we put

$$g(t, x) := (\mathcal{L}_t \mathcal{L}_x D_t^{-1} \hat{f})(t, x) = \sum_{j \geq 1, n} f_{j-1, n} t^j x^n,$$

where $\mathcal{L}_t$ is defined by $\mathcal{L}_t t^j = j! t^j$ and $\mathcal{L}_x$ is also same. Moreover, for

$$g(t, t^{1/2}) = \sum_{j, \ell} f_{j-1, 2\ell} t^{j+\ell} + t^{1/2} \sum_{j, \ell} f_{j-1, 2\ell+1} t^{j+\ell},$$

we put

$$\hat{G}(\tau) := ((\mathcal{B}_t^{[1]} g)(t, t^{1/2}))|_{t=\tau^2}$$

which is a formal series in τ -variable. Here $\mathcal{B}_t^{[1]}$ is defined by $\mathcal{B}_t^{[1]}(t^{n+i/2}) = t^{n+i/2}/n!$ for $i = 0, 1$. In this case, since we see

$$\hat{G}(t^{1/2}) = \hat{u}_{*0}(t) + t^{1/2} \hat{u}_{*1}(t),$$

they gave the following proposition.

Proposition 3.2 ([3]). *The series $\hat{u}_{*0}(t)$ and $\hat{u}_{*1}(t)$ are 1-summable in 0 if and only if the series $\hat{G}(\tau)$ associated with $\hat{f}$ is 2-summable in the directions 0 and π .*

Proposition 3.1 was extended to the heat equation with variable coefficient by Balser and Loday-Richaud [3], higher order linear partial differential equation with variable coefficients by Remy [9, 10], semilinear heat equation with variable coefficients by Remy [11], semilinear higher order equation with variable coefficients by Remy [12].

Balser [2] also gave the another necessary and sufficient condition for 1-summability of $\hat{u}(t, x) = \hat{u}_{inh}(t, x; \hat{f})$, which was refined by Michalik [6].

Proposition 3.3 ([2, 6]). *For the inhomogeneous term $\hat{f}(t, x) = \sum_{i \geq 0} f_i(x) t^i / i! \in \mathcal{O}_x[[t]]_1$, we define $g(s, x)$ and $h(\tau)$ by*

$$g(s, x) := \sum_{i \geq 0} f_i(x) s^i / (2i!),$$

which is convergent at $(s, x) = (0, 0)$ and

$$h(\tau) := \partial_\tau \int_0^\tau g((\tau - s)^2, s) ds,$$

respectively. Then the following statements are equivalent.

i) $\hat{u}(t, x)$ is 1-summable in 0 direction.

ii) $h(\tau)$ satisfies **1-S-C**, that is, $h(\tau)$ can be analytically continued to $S_\tau(0, \pi; \varepsilon_1, \sigma)$ and

$$|h(\tau)| \leq C e^{\delta|\tau|^2}, \quad \tau \in \overline{S}_\tau(0, \pi; \varepsilon'_1, \sigma),$$

for all $\varepsilon'_1 < \varepsilon_1$.

Proposition 3.3 was extended to $1/p$ -fractional equations by Michalik [7] and moment partial differential equations by Michalik [8].

§ 4. A proof of Theorem 1.3

First, from the analytic continuation of the integral representation of $u^0(t, x) = u_{inh}(t, x; f^0)$, and the assumption that $f^0(t, x)$ satisfies **1-S-C**, we can see that $u^0(t, x)$ is analytic in a sector $S_t(0, \alpha, \rho) \times \overline{D}_{\sigma_0}$ with $\alpha > \pi$ for some sufficiently small positive constants ρ and σ_0 .

Next, we shall show that $u^0(t, x)$ has the Gevrey asymptotic expansion $\hat{u}(t, x)$ of order 1, which completes a proof of Theorem 1.3(cf. [4]).

We put

$$\hat{u}(t, x) = \sum_{n \geq 1} \left(\sum_{i=1}^n f_{i-1}^{(2(n-i))}(x) \right) \frac{t^n}{n!} =: \sum_{n \geq 1} u_n(x) t^n \in \mathcal{O}_x[[t]]_1.$$

For all $M \in \mathbb{N}$, we put

$$\begin{aligned} \hat{f}(t, x) &= \sum_{i \geq 1} f_{i-1}(x) \frac{t^{i-1}}{(i-1)!} = \left\{ \sum_{i=1}^M + \sum_{i > M} \right\} f_{i-1}(x) \frac{t^{i-1}}{(i-1)!} \\ &=: \hat{f}_M(t, x) + \hat{f}^M(t, x). \end{aligned}$$

Moreover, we put

$$(4.1) \quad F_M(t, x) := f^0(t, x) - \hat{f}_M(t, x).$$

From the assumptions that $\hat{f}(t, x) = \sum_{i \geq 0} f_i(x) t^i / i! \in \mathcal{O}_x\{t\}_1$ and $f^0(t, x)$ satisfies **1-S-C**, we can prove the following lemma, which will be proved in the next section.

Lemma 4.1.

$$(4.2) \quad |f_i(x)| \leq C_1 K_1^i i!^2 e^{\delta|x|^2}, \quad x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma),$$

$$(4.3) \quad |f_i^{(n)}(x)| \leq C_2 K_2^{i+n} i!^2 n! e^{\tilde{\delta}|x|^2}, \quad x \in \overline{S}_x(0, \pi; \varepsilon''_1, \sigma') \quad (\varepsilon''_1 < \varepsilon'_1, \sigma' < \sigma/2),$$

$$(4.4) \quad |F_M(t, x)| \leq C_3 K_3^M M! |t|^M e^{\delta|x|^2} \quad x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma), \quad t \in \overline{S}_1$$

with some positive constants C_j, K_j ($j = 1, 2, 3$) and $\delta, \tilde{\delta}$ for all i, n, M . Here $\overline{S}_1$ is any closed subsector of the region $\{t \in \mathbb{C}; |t - \rho_0| < \rho_0\}$.

We have the following Taylor formula

$$(4.5) \quad f_{i-1}(x+y) = \sum_{j=0}^{L-1} \frac{f_{i-1}^{(j)}(x)}{j!} y^j + \int_0^y \frac{(y-\eta)^{L-1}}{(L-1)!} f_{i-1}^{(L)}(x+\eta) d\eta$$

for all L

By substituting into the integral representation of $u^0(t, x)$ the expression (4.1) with $M = \lfloor \frac{N-1}{2} \rfloor$ and the Taylor formula (4.5) with $L = N - 2i$ for $1 \leq i \leq \lfloor \frac{N-1}{2} \rfloor$, we have

$$\begin{aligned} u^0(t, x) &= \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} f^0(t-s, x+y) dy ds \\ &= \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} \left\{ \hat{f}_M(t-s, x+y) + F_M(t-s, x+y) \right\} dy ds \\ &= \sum_{1 \leq i \leq \lfloor (N-1)/2 \rfloor} \int_0^t \frac{1}{\sqrt{4\pi s}} \sum_{j=0}^{N-2i-1} \frac{f_{i-1}^{(j)}(x)}{j!} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{4s}} y^j dy \frac{(t-s)^{i-1}}{(i-1)!} ds \\ &+ \sum_{1 \leq i \leq \lfloor (N-1)/2 \rfloor} \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{4s}} \int_0^y \frac{(y-\eta)^{N-2i-1}}{(N-2i-1)!} f_{i-1}^{(N-2i)}(x+\eta) d\eta dy \frac{(t-s)^{i-1}}{(i-1)!} ds \\ &+ \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-y^2/4s} F_M(t-s, x+y) dy ds \\ &=: I_N(t, x) + R_N^1(t, x) + R_N^2(t, x). \end{aligned}$$

Then from the following lemma, we can get the desired result.

Lemma 4.2. For $(t, x) \in \overline{S}_1 \times \overline{D}_{\sigma_0}$ with any closed subsector $\overline{S}_1 \subset \{t \in \mathbb{C}; |t - \rho_0| < \rho_0\}$, we have

$$(4.6) \quad I_N(t, x) = \sum_{n=1}^{\lfloor (N-1)/2 \rfloor} \left(\sum_{i=1}^n f_{i-1}^{(2(n-i))}(x) \right) \frac{t^n}{n!},$$

$$(4.7) \quad \max_{|x| \leq \sigma_0} |R_N^1(t, x)| \leq C_1 K_1^N |t|^{N/2} \Gamma((N+1)/2),$$

$$(4.8) \quad \max_{|x| \leq \sigma_0} |R_N^2(t, x)| \leq C_2 K_2^{\lfloor (N-1)/2 \rfloor} |t|^{\lfloor (N-1)/2 \rfloor + 1} \Gamma\left(\left\lfloor \frac{N-1}{2} \right\rfloor + 1\right)$$

with some positive constants C_1, C_2, K_1 and K_2 for any N .

In fact, when $N = 2m$, since $[(N - 1)/2] = m - 1$, we have

$$\begin{aligned} I_N(t, x) &= \sum_{n=1}^{m-1} u_n(x) t^n, \\ |R_N^1(t, x)| &\leq C_1 K_1^{2m} |t|^m \Gamma(m + 1/2) \leq \tilde{C}_1 \tilde{K}_1^m |t|^m \Gamma(m + 1), \\ |R_N^2(t, x)| &\leq C_2 K_2^{m-1} |t|^m \Gamma(m) \leq \tilde{C}_2 \tilde{K}_2^m |t|^m \Gamma(m + 1) \end{aligned}$$

with $C_1 < \tilde{C}_1, C_2 < \tilde{C}_2$ and $K_1 < \tilde{K}_1$.

When $N = 2m + 1$, since $[(N - 1)/2] = m$, we have

$$\begin{aligned} I_N(t, x) &= \sum_{n=1}^m u_n(x) t^n = \sum_{n=1}^{m-1} u_n(x) t^n + u_m(x) t^m, \\ |R_N^1(t, x)| &\leq C_1 K_1^{2m+1} |t|^{m+1/2} \Gamma(m + 1) \leq \tilde{C}_1 \tilde{K}_1^m |t|^m \Gamma(m + 1), \quad (\cdot \cdot |t| < 2\rho_0) \\ |R_N^2(t, x)| &\leq C_2 K_2^m |t|^{m+1} \Gamma(m + 1) \leq \tilde{C}_2 \tilde{K}_2^m |t|^m \Gamma(m + 1). \end{aligned}$$

Here we remark that for $|x| \leq \sigma$

$$\begin{aligned} |u_m(x) t^m| &\leq \left| \left(\sum_{i=1}^m f_{i-1}^{(2(m-i))}(x) \right) \frac{t^m}{m!} \right| \\ &\leq \sum C_2 K_2^{(i-1)+2(m-i)} (i-1)!^2 (2(m-i))! e^{\tilde{\delta}|x|^2} \frac{|t|^m}{m!} \\ &\leq \tilde{C}_2 \tilde{K}_2^m m! |t|^m. \end{aligned}$$

From the above observations, we see that $u^0(t, x)$ has the Gevrey asymptotic expansion $\hat{u}(t, x)$ of order 1.

Before giving a proof of Lemma 4.2, we give a formula for $a > 0$ and $b \geq 0$

$$\int_0^\infty e^{-y^2/a} y^b dy = \frac{1}{2} a^{\frac{b+1}{2}} \Gamma\left(\frac{b+1}{2}\right).$$

Let us show Lemma 4.2.

First, we show the equality (4.6). We put $I_j(s) := \int_{-\infty}^\infty e^{-y^2/4s} y^j dy$. When j is odd, $I_j(s) = 0$. When $j = 2n$, we have

$$I_{2n}(s) = 2 \int_0^\infty e^{-y^2/4s} y^{2n} dy = (4s)^{n+1/2} \Gamma(n + 1/2).$$

Therefore by noticing $(2n)! = 2^{2n}\Gamma(n + 1/2)n!/\sqrt{\pi}$, we have

$$\begin{aligned}
I_N(t, x) &= \sum_{i=1}^{[(N-1)/2]} \sum_{n=0}^{[(N-1)/2]-i} \frac{f_{i-1}^{(2n)}(x)}{(2n)!} \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} s^n ds \frac{4^n \Gamma(n + 1/2)}{\sqrt{\pi}} \\
&= \sum_{i=1}^{[(N-1)/2]} \sum_{n=0}^{[(N-1)/2]-i} f_{i-1}^{(2n)}(x) \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} \frac{s^n}{n!} ds \\
&= \sum_{i=1}^{[(N-1)/2]} \sum_{n=0}^{[(N-1)/2]-i} f_{i-1}^{(2n)}(x) \frac{t^{n+i}}{(n+i)!} \\
&= \sum_{i=1}^{[(N-1)/2]} \sum_{n=i}^{[(N-1)/2]} f_{i-1}^{(2(n-i))}(x) \frac{t^n}{n!} \\
&= \sum_{n=1}^{[(N-1)/2]} \sum_{i=1}^n f_{i-1}^{(2(n-i))}(x) \frac{t^n}{n!}.
\end{aligned}$$

Next, by using the inequality (4.4), we show the inequality (4.8).

We have

$$\begin{aligned}
r_N^2(t, s, x) &:= \left| \int_{-\infty}^{\infty} e^{-y^2/4s} F_M(t-s, x+y) dy \right| \quad (M = \lfloor \frac{N-1}{2} \rfloor) \\
&\leq 2CK^{[(N-1)/2]} \Gamma(\lfloor \frac{N-1}{2} \rfloor + 1) |t-s|^{[(N-1)/2]} e^{2\delta|x|^2} \int_0^{\infty} e^{-c_1 y^2/4|s|} e^{2\delta y^2} dy
\end{aligned}$$

for some positive c_1 , where we use the inequality $|x+y|^2 \leq 2(|x|^2 + |y|^2)$. Here since there exists $c_2 > 0$ such that $c_1 - 8\delta|s| \geq c_2$ for sufficiently small $|s|$, we have

$$\int_0^{\infty} e^{-c_1 y^2/4|s|} e^{2\delta y^2} dy \leq \int_0^{\infty} e^{-c_2 y^2/4|s|} dy = \sqrt{\frac{4|s|}{c_2}} \frac{\sqrt{\pi}}{2}.$$

Therefore we have

$$\begin{aligned}
\max_{|x| \leq \sigma_0} |R_N^2(t, x)| &\leq \tilde{C} K^{[(N-1)/2]} \Gamma(\lfloor \frac{N-1}{2} \rfloor + 1) \left| \int_0^t |t-s|^{[(N-1)/2]} ds \right| \\
&\leq \tilde{C} K^{[(N-1)/2]} \Gamma(\lfloor \frac{N-1}{2} \rfloor + 1) |t|^{[(N-1)/2]+1}.
\end{aligned}$$

Finally, by using the inequality (4.3), we show the inequality (4.7).

From the inequality (4.3), we have

$$\begin{aligned}
|r_{N1}^1(y, x)| &:= \left| \int_0^y \frac{(y-\eta)^{N-2i-1}}{(N-2i-1)!} f_{i-1}^{(N-2i)}(x+\eta) d\eta \right| \\
&\leq CK^{N-i-1} (i-1)!^2 e^{2\delta(|x|^2 + |y|^2)} |y|^{N-2i}.
\end{aligned}$$

Therefore since

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{-y^2/4s} r_{N1}^1(y, x) dy \right| &\leq C' K^{N-i} (i-1)!^2 \int_0^{\infty} e^{-c_2 y^2/4|s|} y^{N-2i} dy \\ &\leq C'' K'^{N-i} (i-1)!^2 |s|^{(N+1)/2-i} \Gamma((N+1)/2-i) \end{aligned}$$

with some positive constants C', C'' and K' for $|x| \leq \sigma_0$, we have

$$\begin{aligned} \max_{|x| \leq \sigma_0} |R_N^1(t, x)| &\leq \sum_i C'' K'^{N-i} (i-1)! \Gamma((N+1)/2-i) \left| \int_0^t |s|^{N/2-i} (t-s)^{i-1} ds \right| \\ &\leq C'' K'^N \sum_i (i-1)! \Gamma((N+1)/2-i) \times |t|^{N/2} \\ &\leq C_1 K_1^N |t|^{N/2} \Gamma((N+1)/2). \end{aligned}$$

§ 5. Proof of Lemma 4.1

We give a proof of Lemma 4.1.

First, we consider the formal 1-Borel transform of $\hat{f}(t, x) = \sum_{i \geq 0} f_i(x) t^i / i!$

$$g(s, x) := (\hat{B}_1 \hat{f})(s, x) = \sum_{i \geq 0} f_i(x) \frac{s^i}{i!^2},$$

which is convergent in $|s| < r$ and $|x| \leq \sigma$ for some $r > 0$. Then from the assumptions that $\hat{f}(t, x) \in \mathcal{O}_x\{t\}_1$ and that $f^0(t, x)$ satisfies 1-**S-C**, we can show that $g(s, x)$ is analytic in the region $(D_r \cup S_s(0; \varepsilon)) \times S_x(0, \pi; \varepsilon_1, \sigma)$ for sufficiently small $\varepsilon > 0$ and has the estimate

$$(5.1) \quad |g(s, x)| \leq C e^{\delta_1 |s| + \delta_2 |x|^2}$$

with some positive constants C, δ_1 and δ_2 for $s \in (\overline{D}_{r'} \times \overline{S}_2)$ and $x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma)$, where $r' < r$ and $\overline{S}_2$ is any closed subsector $S_s(0; \varepsilon)$.

Indeed, from the assumption that $\hat{f}(t, x) \in \mathcal{O}_x\{t\}_1$, we see that $g(s, x)$ is analytic in $(D_r \cup S_s(0; \varepsilon)) \times \overline{D}_\sigma$ and has the exponential growth estimate of order at most 1 with respect to s , that is, $\max_{|x| \leq \sigma} |g(s, x)| \leq C_1 e^{\delta_1 |s|}$ for $s \in \overline{S}_2$.

Let us consider 1-Borel transform of $f^0(t, x)$

$$\tilde{g}(s, x) := \frac{-1}{2\pi i} \int_{\gamma} f^0(t, x) e^{\frac{s}{t}} \frac{dt}{t}$$

for $s \in S_s(0; \varepsilon)$ and $x \in \overline{D}_\sigma$, where the path γ denotes the path from the origin along $\arg t = (\varepsilon + \pi)/2$ to some point t_1 with a positive ε , then along the circle $|t| = |t_1|$ to the ray $\arg t = -(\varepsilon + \pi)/2$, and back to the origin along this ray such that $\gamma \subset$

$S_T = S_t(0, \alpha, T)$ with $\alpha > \pi + \varepsilon$. Then from the assumptions that $\hat{f}(t, x) \in \mathcal{O}_x\{t\}_1$ and that $f^0(t, x)$ is 1-sum of $\hat{f}(t, x)$, we have $\tilde{g}(s, x) = g(s, x)$ in a neighborhood at $(s, x) = (0, 0)$. Moreover, since $f^0(t, x)$ satisfies 1-**S-C**, we see that $\tilde{g}(s, x)$ is analytic in $(D_r \cup S_s(0; \varepsilon)) \times S_x(0, \pi; \varepsilon_1, \sigma)$, and has the estimate

$$|\tilde{g}(s, x)| = |g(s, x)| \leq C e^{\delta_1|s| + \delta_2|x|^2}$$

by some positive constants C, δ_1 and δ_2 for $s \in (\overline{D}_{r'} \times \overline{S}_2)$ and $x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma)$.

Next, by using the inequality (5.1), we have

$$|f_i(x)/i!| = |\partial_s^i g(0, x)| = \left| \frac{i!}{2\pi\sqrt{-1}} \oint_{|\tau|=r'} \frac{g(\tau, x)}{\tau^{i+1}} d\tau \right| \leq \frac{C_1 i!}{r^i} e^{\delta_2|x|^2}$$

with some positive constant C_1 for $x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma)$, which gives a proof of (4.2).

By using the inequality (4.2), we have

$$|f_i^{(n)}(x)| = \left| \frac{n!}{2\pi i} \oint_{|\xi-x|=c(x)} \frac{f_i(\xi)}{(\xi-x)^{n+1}} d\xi \right| \leq n! C_1 K_1^i i!^2 e^{\delta_2|x|^2} / (c(x))^n$$

for $x \in \overline{S}_x(0, \pi; \varepsilon''_1, \sigma')$ with $\varepsilon''_1 < \varepsilon'_1$, $\sigma' < \sigma/2$ and $\tilde{\delta}_2 > 0$, where $c(x) = \sigma'$ if $|x| \leq \sigma'$ and $c(x) = c_0|x|$ for some $c_0 > 0$ if $|x| \geq \sigma'$ and $x \in \overline{S}_x(0, \pi; \varepsilon''_1, \sigma')$. Here, if $|x| \leq \sigma'$, then $1/(c(x))^n = 1/\sigma'^n$. If $|x| \geq \sigma'$, then $1/(c(x))^n = 1/(c_0|x|)^n \leq 1/(c_0\sigma')^n$. Therefore we have

$$|f_i^{(n)}(x)| \leq C_2 K_2^{i+n} n! i!^2 e^{\tilde{\delta}_2|x|^2}$$

for $x \in \overline{S}_x(0, \pi; \varepsilon''_1, \sigma')$, which gives a proof of (4.3).

Finally, we give a proof of (4.4). We put

$$G_M(s, x) := g(s, x) - \sum_{i=0}^{M-1} f_i(x) \frac{s^i}{i!^2}$$

for $s \in (D_r \cup S_s(0; \varepsilon)) \times S_x(0, \pi; \varepsilon_1, \sigma)$. Then we have $G_M(s, x) = D_s^{-M} \partial_s^M g(s, x)$, where $D_s^{-1} = \int_0^s$. By using the inequality (5.1), we have

$$|\partial_s^M g(s, x)| = \left| \frac{M!}{2\pi i} \oint_{|\tau-s|=c(s)} \frac{g(\tau, x)}{(\tau-s)^{M+1}} d\tau \right| \leq C_1 K_1^M M! e^{\tilde{\delta}_1|s| + \delta_2|x|^2}$$

by some positive constants C_1, K_1 and $\tilde{\delta}_1$ for $(s, x) \in (\overline{D}_{r'} \cup \overline{S}_2) \times \overline{S}_x(0, \pi; \varepsilon'_1, \sigma)$ with any closed subsector $\overline{S}_2 \subset S_s(0; \varepsilon)$. Here $c(s) = r' (< r/2)$ if $|s| \leq r'$, and $c(s) = c_1|s|$ for some $c_1 > 0$ if $|s| \geq r'$ and $s \in \overline{S}_2$.

Therefore we have for $s \in \overline{S}_2$ and $x \in \overline{S}_x(0, \pi; \varepsilon'_1, \sigma)$

$$\begin{aligned} |G_M(s, x)| &= \left| \int_0^s \frac{(s-p)^{M-1}}{(M-1)!} \partial_s^M g(p, x) dp \right| \\ &\leq \int_0^1 \frac{|s|^M (1-q)^{M-1}}{(M-1)!} C_1 K_1^M M! e^{\tilde{\delta}_1 p|s| + \delta_2|x|^2} dq = C_1 K_1^M |s|^M e^{\tilde{\delta}_1|s| + \delta_2|x|^2}. \end{aligned}$$

We remark that $F_M(t, x)$ is given by the analytic continuation of 1-Laplace transform of $G_M(s, x)$

$$F_M(t, x) = (\mathcal{L}_1 G_M)(t, x) := \frac{1}{t} \int_0^{\infty(0)} e^{-\frac{s}{t}} G_M(s, x) ds$$

for $|t - \rho_0| < \rho_0$ and $x \in S_x(0, \pi; \varepsilon_1, \sigma)$ for sufficiently small $\rho_0 > 0$. Here the path $\int_0^{\infty(d)}$ takes from 0 to ∞ along $\arg s = d$. Therefore for $t \in \bar{S}_1 \subset \{t \in \mathbb{C}; |t - \rho_0| < \rho_0\}$ and $x \in \bar{S}_x(0, \pi; \varepsilon'_1, \sigma)$ we have

$$\begin{aligned} |F_M(t, x)| &= \left| \frac{1}{t} \int_0^{\infty(0)} e^{-\frac{s}{t}} G_M(s, x) ds \right| = \left| \int_0^{\infty(-\arg t)} e^{-u} G_M(ut, x) du \right| \\ &\leq \int_0^\infty e^{-cv} C_1 K_1^M |vt|^M e^{\tilde{\delta}_1 |vt| + \delta_2 |x|^2} dv \quad (c, \tilde{\delta}_1 > 0) \\ &\leq C_1 K_1^M |t|^M e^{\delta_2 |x|^2} \int_0^\infty e^{-\tilde{c}v} v^M dv \quad (\tilde{c} = c - 2\tilde{\delta}_1 \rho_0 > 0) \\ &\leq \tilde{C}_1 \tilde{K}_1^M |t|^M e^{\delta_2 |x|^2} M! \quad (\tilde{C}_1, \tilde{K}_1 > 0). \end{aligned}$$

§ 6. Remark on the integral representation of 1-sum

We could not obtain the integral representation of 1-sum in terms of the heat kernel under the assumption of the inhomogeneous term $\hat{f}(t, x) \in \mathcal{O}_x[[t]]_1$. However, we can get the integral representation of 1-sum if the assumption is relaxed.

We consider the Cauchy problem

$$(H) \quad \begin{cases} \partial_t u(t, x) = \partial_x^2 u(t, x) + f(t, x), \\ u(0, x) = 0, \end{cases}$$

where we assume that the inhomogeneous term $f(t, x) = \sum_{i \geq 0} f_i(x) t^i / i!$ is convergent in t -variable and entire in x -variable, and has the following estimate

$$(6.1) \quad \max_{|t| \leq \rho} |f(t, x)| \leq C e^{\delta |x|}, \quad x \in \mathbb{C}$$

by some positive constants ρ, C and δ . Then the formal solution $\hat{u}(t, x)$, which is given by

$$\hat{u}(t, x) = \sum_{n \geq 1} \left(\sum_{i=1}^n f_{i-1}^{(2(n-i))}(x) \right) \frac{t^n}{n!} =: \sum_{n \geq 1} u_n(x) t^n,$$

is a convergent series in t -variable. (This holds if $f(t, x)$ has the exponential growth estimate of order at most 2 instead of the condition (6.1).) In fact, from the condition (6.1), we have the following inequalities.

$$\begin{aligned} |f_i(x)| &\leq C_1 K_1^i i! e^{\delta |x|} \quad (x \in \mathbb{C}), \\ |f_i^{(\ell)}(x)| &\leq C_2 K_2^{i+\ell} i! e^{\delta |x|} \quad (x \in \mathbb{C}) \end{aligned}$$

by some positive constants C_1, C_2, K_1 and K_2 . Therefore since $|u_n(x)| \leq C_3 K_3^n e^{\delta|x|}$ by some C_3 and K_3 , the formal solution $\hat{u}(t, x)$ is convergent in t -variable.

In the following, we write the formal solution $\hat{u}(t, x)$ by $u(t, x)$.

For the solution $u(t, x)$, which is a convergent series in $|t| \leq \rho$ and $x \in \mathbb{C}$, we put

$$\begin{aligned} u(t, x) &= \sum_{n \geq 1} \left(\sum_{i=1}^n f_{i-1}^{(2(n-i))}(x) \right) \frac{t^n}{n!} = \sum_{i \geq 1} \left(\sum_{n \geq i} f_{i-1}^{(2(n-i))}(x) \frac{t^n}{n!} \right) \\ (6.2) \quad &= \sum_{i \geq 1} D_t^{-i} \left(\sum_{n \geq 0} f_{i-1}^{(2n)}(x) \frac{t^n}{n!} \right) =: \sum_{i \geq 1} D_t^{-i} U_i(t, x), \end{aligned}$$

where $D_t^{-1} = \int_0^t$. Here, for each i , $U_i(t, x)$ is a convergent series (in fact, entire) in t -variable and satisfies the following Cauchy problem.

$$(H_i) \quad \begin{cases} \partial_t U(t, x) = \partial_x^2 U(t, x), \\ U(0, x) = f_{i-1}(x). \end{cases}$$

Therefore by restricting in the region $O_t := \{t \in \mathbb{C}; |t - \rho_0| < \rho_0\}$ for sufficiently small ρ_0 , we have the following integral representation of $U_i(t, x)$ in terms of the heat kernel.

$$U_i(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}} f_{i-1}(x+y) dy.$$

Then when $t \in O_t$, we have

$$\begin{aligned} u(t, x) &= \sum_{i \geq 1} D_t^{-i} U_i(t, x) = \sum_{i \geq 1} \int_0^t \frac{(t-s)^i}{(i-1)!} U_i(s, x) ds \\ &= \int_0^t \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4s}} \sum_{i \geq 1} f_{i-1}(x+y) \frac{(t-s)^i}{(i-1)!} dy ds \\ &= \int_0^t \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4s}} f(t-s, x+y) dy ds. \end{aligned}$$

References

- [1] Balser, W., *From Divergent Power Series to Analytic Functions*, Springer Lecture Notes, No. **1582**, 1994.
- [2] —, Power series solutions of the inhomogeneous heat equation. In: Recent Trends in Microlocal Analysis, *Kôkyûroku RIMS* **1412** (2005), 151–159.
- [3] Balser W. and Loday-Richaud, M., Summability of solutions of the heat equation with inhomogeneous thermal conductivity in two variables, *Adv. Dyn. Syst. Appl.* **4** (2009), no. 2, 159–177.

- [4] Ichinobe K. and Miyake M., Experimental observation on k -summability of divergent solutions of the heat equation with $k > 1$, *Kôkyûroku RIMS* **2101** (2019), 13–22.
- [5] Lutz D. A., Miyake M. and Schäfer R., On the Borel summability of divergent solutions of the heat equation, *Nagoya Math. J.* **154** (1999), 1–29.
- [6] Michalik S., Summability of formal solutions to the n -dimensional inhomogeneous heat equation, *J. Math. Anal. Appl.* **347** (2008), no. 1, 323–332.
- [7] —, Multisummability of formal solutions of inhomogeneous linear partial differential equations with constant coefficients, *J. Dyn. Control Syst.* **18** (2012), no. 1, 103–133.
- [8] —, Analytic and summable solutions of inhomogeneous moment partial differential equations, *Funkcial. Ekvac.* **60** (2017), no. 3, 325–351.
- [9] Remy P., Gevrey order and summability of formal series solutions of some classes of inhomogeneous linear partial differential equations with variable coefficients, *J. Dyn. Control Syst.* **22** (2016), no. 4, 693–711.
- [10] —, Gevrey order and summability of formal series solutions of certain classes of inhomogeneous linear integro-differential equations with variable coefficients, *J. Dyn. Control Syst.* **23** (2017), no. 4, 853–878.
- [11] —, On the summability of the solutions of the inhomogeneous heat equations with a power-law nonlinearity and variable coefficients, *J. Math. Anal. Appl.* **494** (2021), no. 2, Paper No. 124656, 12 pp.
- [12] —, Summability of formal power series solutions of a certain class of inhomogeneous partial differential equations with a polynomial semilinearity and variable coefficients, *Results Math.* **76** (2021), no. 3, Paper No. 188, 27 pp.