

QP-Structures of Degree 3 and 4D Topological Field Theory

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Abstract

A BV algebra and a QP-structure of degree 3 is formulated. A QP-structure of degree 3 gives rise to Lie algebroids up to homotopy and its algebraic and geometric structure is analyzed. A new algebroid is constructed, which derives a new topological field theory in 4 dimensions by the AKSZ construction.

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1 Introduction

A BV algebra and a QP-structure has been motivated by the structure of the Batalin-Vilkovisky formalism of a gauge theory[1] and is its mathematical formulation [2]. In case of a topological field theory of Schwarz type, a BV formalism has been reformulated to the AKSZ formulation, which is a clear construction using geometry of a graded manifold [3][4]. Application to higher $n + 1$ dimensions has been formulated and new topological field theories in higher dimensions have been founded by applying this construction [5][6][7].

In $n = 1$, a classical QP-structure is a Poisson structure on a manifold M and is also a Lie algebroid on T^*M from the explicit construction. This is equivalent to the construction of a Poisson structure by the Schouten-Nijenhuis bracket in a classical limit. The topological field theory in two dimensions constructed by the AKSZ formulation [4] is the Poisson sigma model [8][9] and the quantization of this model on disc derives the Kontsevich formula of the deformation quantization on a Poisson manifold [11][12].

In $n = 2$, a classical QP-structure is a Courant algebroid [13][14]. The topological field theory derived in three dimensions is the Courant sigma model [15][16][17].

However structures for higher n , more than 2, have not been understood enough apart from BF theories.

In this paper, we analyze $n = 3$ case. A QP-structure of degree 3 leads us to a new type of algebroid, which is called a **Lie algebroid up to homotopy**. The notion of this algebroid is defined as a homotopy deformation of a Lie algebroid satisfying some integrability conditions. We will prove that a QP-structure of degree 3 on a N-manifold (nonnegatively graded manifold) is equivalent to a Lie algebroid up to homotopy. This QP-structure defines a new natural 4-dimensional topological field theory via the AKSZ construction.

The paper is organized as follows. In section 2, a BV algebra and a QP-structure of degree 3 are formulated. In section 3, a QP-structure of degree 3 is constructed and analyzed. In section 4, examples of QP-structures of degree 3 are listed. In section 5, the AKSZ construction of a topological field theory in four dimensions is formulated and examples are listed. ^c

^cVery recently, Grützmann's paper appears which has overlaps with our paper [18].

2 QP-manifolds and BV Algebras

2.1 Classical QP-manifold

Definition 2.1 *A graded manifold $\mathcal{M}$ is by definition a sheaf of a graded commutative algebra over an ordinary smooth manifold M .*

In the following, we assume the degrees are nonnegative.

The structure sheaf of $\mathcal{M}$ is locally isomorphic to a graded commutative algebra $C^\infty(U) \otimes S(V)$, where U is an ordinary local chart of M , $S(V)$ is the polynomial algebra over V and where $V := \sum_{i \geq 1} V_i$ is a graded vector space such that the dimension of V_i is finite for each i . For example, when $V = V_1$, $\mathcal{M}$ is a vector bundle whose fiber is V_1^* : the dual space of V_1 .

Definition 2.2 *A graded manifold $(\mathcal{M}, \omega)$ equipped with a graded symplectic structure ω of degree n is called a **P-manifold** of degree n .*

In the next section, we will study a concrete P-manifold of degree 3.

The structure sheaf $C^\infty(\mathcal{M})$ of a P-manifold becomes a graded Poisson algebra. The Poisson bracket is defined in the usual manner,

$$\{F, G\} = (-1)^{|F|+1} \iota_{X_F} \iota_{X_G} \omega, \quad (2.1)$$

where $F, G \in C^\infty(\mathcal{M})$, $|F|$ is the degree of F and $X_F := \{F, -\}$ is the Hamiltonian vector field of F . We recall the basic properties of the Poisson bracket,

$$\begin{aligned} \{F, G\} &= -(-1)^{(|F|-n)(|G|-n)} \{G, F\}, \\ \{F, GH\} &= \{F, G\}H + (-1)^{(|F|-n)|G|} G\{F, H\}, \\ \{F, \{G, H\}\} &= \{\{F, G\}, H\} + (-1)^{(|F|-n)(|G|-n)} \{G, \{F, H\}\}, \end{aligned}$$

where n is the degree of the symplectic structure and $F, G, H \in C^\infty(\mathcal{M})$. We remark that the degree of the Poisson bracket is $-n$.

Definition 2.3 *Let $(\mathcal{M}, \omega)$ be a P-manifold of degree n . A function $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$ is called a **Q-structure**, if it is a solution of the **classical master equation**,*

$$\{\Theta, \Theta\} = 0. \quad (2.2)$$

*The triple $(\mathcal{M}, \omega, \Theta)$ is called a **QP-manifold**.*

We define an operator $Q := \{\Theta, -\}$, which is called a homological vector field. From (2.2) we have the cocycle condition,

$$Q^2 = 0,$$

which says that the homological vector field is a coboundary operator on $C^\infty(\mathcal{M})$ and defines a cohomology called the classical BRST cohomology.

2.2 Quantum QP-manifold

Definition 2.4 *A graded manifold is called a quantum BV-algebra if it has an odd Laplace operator Δ , which is a linear operator on $C^\infty(\mathcal{M})$ satisfying $\Delta^2 = 0$, and the graded Poisson bracket is given by*

$$\{F, G\} = (-1)^{|F|} \Delta(FG) - (-1)^{|F|} \Delta(F)G - F\Delta(G), \quad (2.3)$$

where $F, G \in C^\infty(\mathcal{M})$.

If n is odd, a P-manifold $(\mathcal{M}, \omega)$ has the odd Poisson bracket. If an odd P-manifold $(\mathcal{M}, \omega)$ has a volume form ρ , one can define an odd Laplace operator Δ (See [19]):

$$\Delta F := \frac{1}{2}(-1)^{|F|} \operatorname{div}_\rho X_F.$$

Here a divergence div_ρ is a map from a space of vector fields on $\mathcal{M}$ to $C^\infty(\mathcal{M})$ and is defined by

$$\int_{\mathcal{M}} \operatorname{div}_\rho X F dv = - \int_{\mathcal{M}} X(F) dv,$$

for a vector field X on $\mathcal{M}$. The pair $(\mathcal{M}, \Delta)$ is called a **quantum P-structure**. An odd Laplace operator has degree $-n$.

Definition 2.5 *A function $\Theta \in C^\infty(\mathcal{M})$ with the degree $n + 1$ is called a **quantum Q-structure**, if it satisfies a **quantum master equation***

$$\Delta(e^{\frac{i}{\hbar}\Theta}) = 0, \quad (2.4)$$

where $\hbar$ is a formal parameter. The triple $(\mathcal{M}, \Delta, \Theta)$ is called a **quantum QP-manifold**.

From the definition of an odd Laplace operator, the equation (2.4) is equivalent to

$$\{\Theta, \Theta\} - 2i\hbar\Delta\Theta = 0. \quad (2.5)$$

If we take the limit of $\hbar \rightarrow 0$ in (2.5), which is called a classical limit, the classical master equation $\{\Theta, \Theta\} = 0$ is derived. Since $\Delta^2 = 0$, Δ is also a coboundary operator. This defines a quantum BRST cohomology. Let $\mathcal{O}' = \mathcal{O}e^{\frac{i}{\hbar}\Theta} \in C^\infty(\mathcal{M})$ be a cocycle with respect to Δ . The cocycle condition $\Delta(\mathcal{O}') = \Delta(\mathcal{O}e^{\frac{i}{\hbar}\Theta}) = 0$ is equivalent to

$$\{\Theta, \mathcal{O}\} - i\hbar\Delta\mathcal{O} = 0. \quad (2.6)$$

The solutions of (2.6) are called *observables* in physics. In the classical limit, (2.6) is $\{\Theta, \mathcal{O}\} = Q\mathcal{O} = 0$. $\mathcal{O}$ reduces to an element of a classical BRST cohomology.

3 Structures and homotopy algebroids

In this section, we construct and analyze a classical QP-structure of degree 3 explicitly.

3.1 P-structures

Let $E \rightarrow M$ be a vector bundle over an ordinary smooth manifold M . The shifted bundle $E[1] \rightarrow M$ is a graded manifold whose fiber space has the degree +1. We consider the shifted cotangent bundle $\mathcal{M} := T^*[3]E[1]$. It is a P-manifold of the degree 3 over M ,

$$T^*[3]E[1] \rightarrow \mathcal{M}_2 \rightarrow E[1] \rightarrow M,$$

where $\mathcal{M}_2$ is a certain graded manifold.^d The structure sheaf $C^\infty(\mathcal{M})$ of $\mathcal{M}$ is decomposed into the homogeneous subspaces,

$$C^\infty(\mathcal{M}) = \sum_{i \geq 0} C^i(\mathcal{M}),$$

where $C^i(\mathcal{M})$ is the space of functions of degree i . In particular, $C^0(\mathcal{M}) = C^\infty(M)$: the algebra of smooth functions on the base manifold and $C^1(\mathcal{M}) = \Gamma E^*$: the space of sections of the dual bundle of E .

^dIn fact, $\mathcal{M}_2$ is $E[1] \oplus E^*[2]$, which is derived from the result in the previous sentence of Remark 3.2.

Let us denote by (x, q, p, ξ) a canonical (Darboux) coordinate on $\mathcal{M}$, where x is a smooth coordinate on M , q is a fiber coordinate on $E[1] \rightarrow M$, (ξ, p) is the momentum coordinate on $T^*[3]E[1]$ for (x, q) . The degrees of the variables (x, q, p, ξ) are respectively $(0, 1, 2, 3)$.

Two directions of counting the degree of functions on $T^*[3]E[1]$ are introduced. Roughly speaking, these are the fiber direction and the base direction.

Definition 3.1 (*Bidegree, see also Remark 3.3.3 in [14]*) Consider a monomial $\xi^i p^j q^k$ on a local chart $(U; x, q, p, \xi)$ of $\mathcal{M}$, of which the total degree is $3i + 2j + k$. The **bidegree** of the monomial is, by definition, $(2(i + j), i + k)$.

This definition is invariant under the natural coordinate transformation,

$$\begin{aligned} x'_i &= x'_i(x_1, x_2, \dots, x_{\dim(M)}), \\ q'_i &= \sum_j t_{ij} q_j, \\ p'_i &= \sum_j t_{ij}^{-1} p_j, \\ \xi'_i &= \sum_j \frac{\partial x_j}{\partial x'_i} \xi_j + \sum_{jkl} \left(\frac{\partial t_{jl}^{-1}}{\partial x'_i} t_{lk} + \frac{\partial t_{kl}}{\partial x'_i} t_{lj}^{-1} \right) p_j q_k, \end{aligned}$$

where t is a transition function. Since $T^*[3]E[1]$ is covered by the natural coordinates, the bidegree is globally well-defined (See also Remark 3.2 below.)

The space $C^n(\mathcal{M})$ is uniquely decomposed into the homogeneous subspaces with respect to the bidegree,

$$C^n(\mathcal{M}) = \sum_{2i+j=n} C^{2i,j}(\mathcal{M}).$$

Since $C^{2,0}(\mathcal{M}) = \Gamma E$ and $C^{0,2}(\mathcal{M}) = \Gamma \wedge^2 E^*$, we have

$$C^2(\mathcal{M}) = \Gamma E \oplus \Gamma \wedge^2 E^*.$$

Remark 3.2 The P -manifold $T^*[3]E[1]$ is regarded as a shifted manifold of $T^*[2]E[1]$. The structure sheaf is also a shifted sheaf of the one on $T^*[2]E[1]$. In particular, the space $C^{2i,j}$ is the shifted space of $C^{i,j}$ on $T^*[2]E[1]$.

For the canonical coordinate on $\mathcal{M}$, the symplectic structure has the following form:

$$\omega = \delta x^i \delta \xi_i + \delta q^a \delta p_a,$$

and the associated Poisson bracket has the following expression:

$$\{F, G\} = F \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial \xi_i} G - F \frac{\overleftarrow{\partial}}{\partial \xi_i} \frac{\overrightarrow{\partial}}{\partial x^i} G + F \frac{\overleftarrow{\partial}}{\partial q^a} \frac{\overrightarrow{\partial}}{\partial p_a} G - F \frac{\overleftarrow{\partial}}{\partial p_a} \frac{\overrightarrow{\partial}}{\partial q^a} G,$$

where $F, G \in C^\infty(\mathcal{M})$ and $\frac{\overrightarrow{\partial}}{\partial \phi}$ and $\frac{\overleftarrow{\partial}}{\partial \phi}$ are the right and left differentiations, respectively. Note that the degree of the symplectic structure is +3 and the one of the Poisson bracket is -3. The bidegree of the Poisson bracket is $(-2, -1)$, that is,

$$\{(2i, j), (2k, l)\} = (2(i + k) - 2, j + l - 1),$$

where $(2i, j) \dots$ are functions with the bidegree $(2i, j)$.

3.2 Q-structures

We consider a (classical) Q-structure, Θ , on the P-manifold. It is required that Θ has degree 4. That is, $\Theta \in C^4(\mathcal{M})$. Because $C^4(\mathcal{M}) = C^{4,0}(\mathcal{M}) \oplus C^{2,2}(\mathcal{M}) \oplus C^{0,4}(\mathcal{M})$, the Q-structure is uniquely decomposed into

$$\Theta = \theta_2 + \theta_{13} + \theta_4,$$

where the bidegrees of the substructures are $(4, 0)$, $(2, 2)$ and $(0, 4)$, respectively. In the canonical coordinate, Θ is the following polynomial:

$$\Theta = f_1^i{}_a(x) \xi_i q^a + \frac{1}{2} f_2^{ab}(x) p_a p_b + \frac{1}{2} f_3^a{}_{bc}(x) p_a q^b q^c + \frac{1}{4!} f_{4abcd}(x) q^a q^b q^c q^d, \quad (3.7)$$

and the substructures are

$$\begin{aligned} \theta_2 &= \frac{1}{2} f_2^{ab}(x) p_a p_b, \\ \theta_{13} &= f_1^i{}_a(x) \xi_i q^a + \frac{1}{2} f_3^a{}_{bc}(x) p_a q^b q^c, \\ \theta_4 &= \frac{1}{4!} f_{4abcd}(x) q^a q^b q^c q^d, \end{aligned}$$

where f_1 - f_4 are structure functions on M . By counting the bidegree, one can easily prove that the classical master equation $\{\Theta, \Theta\} = 0$ is equivalent to the following three identities:

$$\{\theta_{13}, \theta_2\} = 0, \quad (3.8)$$

$$\frac{1}{2} \{\theta_{13}, \theta_{13}\} + \{\theta_2, \theta_4\} = 0, \quad (3.9)$$

$$\{\theta_{13}, \theta_4\} = 0. \quad (3.10)$$

The conditions (3.8), (3.9) and (3.10) are equivalent to

$$f_1^i f_2^{ba} = 0, \quad (3.11)$$

$$f_1^k{}_c \frac{\partial f_2^{ab}}{\partial x^k} + f_2^{da} f_3^b{}_{cd} + f_2^{db} f_3^a{}_{cd} = 0, \quad (3.12)$$

$$f_1^k{}_b \frac{\partial f_1^i{}_a}{\partial x^k} - f_1^k{}_a \frac{\partial f_1^i{}_b}{\partial x^k} + f_1^i{}_c f_3^c{}_{ab} = 0, \quad (3.13)$$

$$f_1^k{}_{[d} \frac{\partial f_3^a{}_{bc]}{\partial x^k} + f_2^{ae} f_{4bcde} - f_3^a{}_e [b f_3^e{}_{cd}] = 0, \quad (3.14)$$

$$f_1^k{}_{[a} \frac{\partial f_{4bcde]}{\partial x^k} + f_3^f{}_{[ab} f_{4cde]f} = 0, \quad (3.15)$$

where $[b \ c \ d \ \dots]$ is a skewsymmetrization with respect to indices $b, c, d, \dots$, etc.

3.3 Lie algebroid up to homotopy

In this section we study an algebraic structure associated with the QP-structure in 3.1 and 3.2.

Definition 3.3 *Let $Q = \theta_2 + \theta_{13} + \theta_4$ be a Q -structure on $T^*[3]E[1]$, where $(\theta_2, \theta_{13}, \theta_4)$ is the unique decomposition of Θ . We call the quadruple $(E; \theta_2, \theta_{13}, \theta_4)$ a **Lie algebroid up to homotopy**, in shorthand, Lie algebroid u.t.h.*

We should study the algebraic properties of the Lie algebroid up to homotopy. Let us define a bracket product by

$$[e_1, e_2] := \{\{\theta_{13}, e_1\}, e_2\}, \quad (3.16)$$

where $e_1, e_2 \in \Gamma E$. By the bidegree counting, ΓE is closed under this bracket. The bracket is not necessarily a Lie bracket, but it is still skewsymmetric:

$$\begin{aligned} [e_1, e_2] &= \{\{\theta_{13}, e_1\}, e_2\}, \\ &= \{\theta_{13}, \{e_1, e_2\}\} + \{e_1, \{\theta_{13}, e_2\}\}, \\ &= -\{\{\theta_{13}, e_2\}, e_1\} = -[e_2, e_1], \end{aligned}$$

where $\{e_1, e_2\} = 0$ is used. A bundle map $\rho : E \rightarrow TM$ which is called an anchor map is defined by the following identity:

$$\rho(e)(f) := \{\{\theta_{13}, e\}, f\},$$

where $f \in C^\infty(M)$. The bracket and the anchor map satisfy the algebroid conditions (A0) and (A1) below:

$$(A0) \quad \rho[e_1, e_2] = [\rho(e_1), \rho(e_2)],$$

$$(A1) \quad [e_1, f e_2] = f[e_1, e_2] + \rho(e_1)(f)e_2,$$

where the bracket $[\rho(e_1), \rho(e_2)]$ is the usual Lie bracket on ΓTM . The bracket (3.16) does not satisfy the Jacobi identity in general. So we should study its Jacobi anomaly, which characterizes the algebraic structure of the Lie algebroid u.t.h. The structures θ_{13} , θ_2 and θ_4 define the three operations:

- $\delta(-) := \{\theta_{13}, -\}$; a de Rham type derivation on $\Gamma \wedge E^*$,
- $(\alpha_1, \alpha_2) := \{\{\theta_2, \alpha_1\}, \alpha_2\}$; a symmetric pairing on E^* , where $\alpha_1, \alpha_2 \in \Gamma E^*$,
- $\Omega(e_1, e_2, e_3, e_4) := \{\{\{\{\theta_4, e_1\}, e_2\}, e_3\}, e_4\}$; a 4-form on E .

Remark that $\delta\delta \neq 0$ in general. Because the degree of the pairing is -2 , it is $C^\infty(M)$ -valued. The pairing induces a symmetric bundle map $\partial : E^* \rightarrow E$ which is defined by the equation, $(\alpha_1, \alpha_2) = \langle \partial\alpha_1, \alpha_2 \rangle$, where $\langle -, - \rangle$ is the canonical pairing of the duality of E and E^* . Since $\langle \alpha, e \rangle = \{\alpha, e\}$, we have

$$\partial\alpha = -\{\theta_2, \alpha\}.$$

By direct computation, we obtain

$$\frac{1}{2}\{\{\{\{\theta_{13}, \theta_{13}\}, e_1\}, e_2\}, e_3\} = [[e_1, e_2], e_3] + (\text{cyclic permutations}),$$

and

$$\{\{\{\{\theta_2, \theta_4\}, e_1\}, e_2\}, e_3\} = -\partial\Omega(e_1, e_2, e_3).$$

From Eq. (3.9), we get an explicit formula of the Jacobi anomaly,

$$(A2) \quad [[e_1, e_2], e_3] + (\text{cyclic permutations}) = \partial\Omega(e_1, e_2, e_3).$$

In a similar way, we obtain the following identities:

$$(A3) \quad \rho\partial = 0,$$

$$(A4) \quad \rho(e)(\alpha_1, \alpha_2) = (\mathcal{L}_e\alpha_1, \alpha_2) + (\alpha_1, \mathcal{L}_e\alpha_2),$$

(A5) $\delta\Omega = 0$,

where $\mathcal{L}_e(-) := \{\{\theta_{13}, e\}, -\}$ is the Lie type derivation which acts on E^* . Axioms (A3) and (A4) are induced from Eq. (3.8) and (A5) is from Eq. (3.10).

The fundamental relations (3.11)–(3.15) correspond to Axioms (A1)–(A5)^e. Thus, the notion of the Lie algebroid up to homotopy is characterized by the algebraic properties (A1)–(A5). One concludes that

The classical algebra associated with the QP-manifold $(T^[3]E[1], \Theta)$ is the space of sections of the vector bundle E with the operations $([\cdot, \cdot], \rho, \partial, \Omega)$ satisfying (A1)–(A5).*

In the next section, we will study some special examples of Lie algebroid u.t.h.s.

Remark 3.4 If the pairing is nondegenerate, then the bundle map ∂ is bijective and then from (A3) we have $\rho = 0$.

Remark 3.5 (Higher Courant-Dorfman brackets) We define a bracket on $C^\infty(\mathcal{M})$ by

$$[-, -]_{CD} := \{\{\Theta, -\}, -\},$$

which is called a Courant-Dorfman (CD) bracket. It is well-known that $[\cdot, \cdot]_{CD}$ is a Loday bracket ([20]). Since the degree of the CD-bracket is -2 , the total space of degree $i \leq 2$,

$$C^2(\mathcal{M}) \oplus C^1(\mathcal{M}) \oplus C^0(M)$$

is closed under the CD-bracket, in particular, the top space $C^2(\mathcal{M}) = \Gamma(E \oplus \wedge^2 E^*)$ is a subalgebra. If $\theta_2 = 0$, the CD-bracket on $E \oplus \wedge^2 E^*$ has the following form,

$$[e_1 + \beta_1, e_2 + \beta_2]_{CD} = [e_1, e_2] + \mathcal{L}_{e_1}\beta_2 - i_{e_2}\delta\beta_1 + \Omega(e_1, e_2),$$

where $\beta_1, \beta_2 \in \Gamma\wedge^2 E^*$. This CD-bracket is regarded as a higher analogue of Courant-Dorfman's original bracket (cf. [13]). We refer the reader to Hagiwara [21] and Sheng [22] for the detailed study of the higher CD-brackets.

^eActually, the axiom (A0) depends on (A1) and (A2).

4 Examples and twisting transformations

4.1 The cases of $\theta_2 = \theta_4 = 0$

In this case, the bracket (3.16) satisfies (A0), (A1) and the Jacobi identity. Therefore, the bundle $E \rightarrow M$ becomes a Lie algebroid:

Definition 4.1 ([23]) *A Lie algebroid over a manifold M is a vector bundle $E \rightarrow M$ with a Lie algebra structure on the space of the sections $\Gamma(E)$ defined by the bracket $[e_1, e_2]$ for $e_1, e_2 \in \Gamma(E)$ and an anchor map $\rho : E \rightarrow TM$ satisfying (A0) and (A1) above.*

We take $\{e_a\}$ as a local basis of ΓE and let a local expression of an anchor map be $\rho(e_a) = f^i_{1a}(x) \frac{\partial}{\partial x^i}$ and a Lie bracket be $[e_b, e_c] = f^a_{3bc}(x) e_a$. The Q-structure Θ associated with the Lie algebroid E is defined as a function on $T^*[3]E[1]$,

$$\Theta := \theta_{13} := f^i_{1a}(x) \xi_i q^a + \frac{1}{2} f^a_{3bc}(x) p_a q^b q^c,$$

which is globally well-defined. Conversely, if we consider $\Theta := \theta_{13}$, the classical master equation induces the Lie algebroid structure on E .

Let us consider the case that the bundle is a vector space on a point. A Lie algebroid over a point $\mathfrak{g} \rightarrow \{pt\}$ is a Lie algebra $\mathfrak{g}$. The P-manifold over $\mathfrak{g} \rightarrow \{pt\}$ is isomorphic to $\mathfrak{g}^*[2] \oplus \mathfrak{g}[1]$ and the structure sheaf is the polynomial algebra over $\mathfrak{g}[2] \oplus \mathfrak{g}^*[1]$,

$$C^\infty(\mathcal{M}) = S(\mathfrak{g}) \otimes \bigwedge^\bullet \mathfrak{g}^*.$$

The bidegree is defined by the natural manner,

$$C^{2i,j}(\mathcal{M}) = S^i(\mathfrak{g}) \otimes \bigwedge^j \mathfrak{g}^*.$$

The Q-structure associated with the Lie bracket on $\mathfrak{g}$ is

$$\theta_{13} = \frac{1}{2} f^a_{bc} p_a q^b q^c \cong \frac{1}{2} f^a_{bc} p_a \otimes (q^b \wedge q^c), \quad (4.17)$$

where $p. \in \mathfrak{g}$, $q. \in \mathfrak{g}^*$ and f^a_{bc} is the structure constant of the Lie algebra.

4.2 The cases of $\theta_2 \neq 0$ and $\theta_4 = 0$

In this case, the bracket induced by θ_{13} still satisfies the Jacobi identity.

We assume that $\mathfrak{g}$ is semi-simple. Then the dual space $\mathfrak{g}^*$ has a metric, $(\cdot, \cdot)_{K^{-1}}$, which is the inverse of the Killing form on $\mathfrak{g}$. The metric inherits the following invariant condition from the Killing form:

$$(\mathcal{L}_p q_1, q_2)_{K^{-1}} + (q_1, \mathcal{L}_p q_2)_{K^{-1}} = 0, \quad (4.18)$$

where $\mathcal{L}_p(-)$ is the canonical coadjoint action of $\mathfrak{g}$ to $\mathfrak{g}^*$. Eq. (4.18) is a linear version of (A4). Thus, we obtain a Q-structure,

$$\Theta := k^{ab} p_a p_b + \frac{1}{2} f^a{}_{bc} p_a q^b q^c, \quad (4.19)$$

where $k^{ab} p_a p_b := (\cdot, \cdot)_{K^{-1}}$.

4.3 Non Lie algebra example

We consider the cases that the Jacobi identity is broken. Let $(\mathfrak{g}, [\cdot, \cdot], (\cdot, \cdot)_K)$ be a vector space (not necessarily Lie algebra) equipped with a skewsymmetric bracket $[\cdot, \cdot]$ and an invariant metric $(\cdot, \cdot)_K$. The metric induces a bijection $K : \mathfrak{g} \rightarrow \mathfrak{g}^*$ which is defined by the identity,

$$(p_1, p_2)_K = \langle K p_1, p_2 \rangle.$$

We define a map from $\mathfrak{g}^*$ to $\mathfrak{g}$ by $\partial := K^{-1}$ and define a 4-form by,

$$\Omega(p_1, p_2, p_3, p_4) := \left([[p_1, p_2], p_3] + \text{cyclic permutations}, p_4 \right)_K.$$

Remark 4.2 *The 4-form above is considered to be a higher analogue of the Cartan 3-form $([p_1, p_2], p_3)_K$.*

Axioms (A0)–(A4) obviously hold on $\mathfrak{g}$. We check (A5). It suffices to show (3.10). Let us denote by $\{-, p_1, p_2, \dots, p_n\}$ the n-fold bracket $\{\dots\{-, p_1\}, p_2\}, \dots, p_n\}$. We already have (3.8) and (3.9). From $\{\theta_{13}, \{\theta_{13}, \theta_{13}\}\} = 0$ and (3.9), we have $\{\theta_{13}, \{\theta_2, \theta_4\}\} = 0$. Since $\{\theta_{13}, \theta_2\} = 0$, this is equal to $\{\theta_2, \{\theta_3, \theta_4\}\} = 0$ up to sign. This gives $\{\{\theta_2, \{\theta_3, \theta_4\}\}, p_1, \dots, p_5\} = 0$ for any $p_1, \dots, p_5$. From $\{\theta_2, p\} = 0$, we have

$$\{\theta_2, \{\{\theta_3, \theta_4\}, p_1, \dots, p_5\}\} = 0.$$

Since $K^{-1} = -\{\theta_2, -\}$ is bijective, we get

$$\{\{\theta_3, \theta_4\}, p_1, \dots, p_5\} = 0,$$

which yields the desired relation $\{\theta_3, \theta_4\} = 0$.

Proposition 4.3 *The triple $(\mathfrak{g}, \partial, \Omega)$ is a Lie algebra(oid) up to homotopy.*

4.4 Twisting by 3-form and the cases of $\theta_2 = 0$ and $\theta_4 \neq 0$

We introduce the notion of twisting transformation by 3-form before studying the cases of $\theta_2 = 0$. Given a Q-structure Θ and a 3-form $\phi \in C^{0,3}(\mathcal{M})$, there exists the second Q-structure which is defined by the canonical transformation,

$$\Theta^\phi := \exp(X_\phi)(\Theta), \tag{4.20}$$

where $X_\phi := \{\phi, -\}$ is the Hamiltonian vector field of ϕ . The transformation (4.20) is called a **twisting by 3-form**, or simply twisting. By a direct computation, we obtain

$$\begin{aligned} \theta_2^\phi &= \theta_2, \\ \theta_{13}^\phi &= \theta_{13} - \{\theta_2, \phi\}, \\ \theta_4^\phi &= \theta_4 - \{\theta_{13}, \phi\} + \frac{1}{2}\{\{\theta_2, \phi\}, \phi\}, \end{aligned}$$

where $\Theta^\phi = \theta_2^\phi + \theta_{13}^\phi + \theta_4^\phi$ and $X_\phi^{i \geq 3}(\Theta) = 0$. The twisting by 3-form defines an equivalence relation on the Q-structures.

We notice that θ_2 is an invariant for the twisting. If $\theta_2 = 0$, then θ_{13} is an invariant and

$$\theta_4^\phi = \theta_4 - \delta\phi,$$

where $\delta\phi = \{\theta_{13}, \phi\}$. This leads us

Proposition 4.4 *The class of Q-structures which have no θ_2 is classified into $H_{dR}^4(\bigwedge^* E^*, \delta)$ by the twisting by 3-form.*

5 AKSZ Construction of Topological Field Theory in 4 Dimensions

5.1 General Theory

In this section, we consider the AKSZ construction of a topological field theory in 4 dimensions.

For a graded manifold $\mathcal{N}$, let $\mathcal{N}|_0$ be the degree zero part.

Let X be a manifold in 4 dimensions and M be a manifold in d dimensions. Let $(\mathcal{X}, D)$ be a differential graded (dg) manifold $\mathcal{X}$ with a D -invariant nondegenerate measure μ , such that $\mathcal{X}|_0 = X$, where D is a differential on $\mathcal{X}$. $(\mathcal{M}, \omega, \Theta)$ is a QP-manifold of degree 3 and $\mathcal{M}|_0 = M$. A degree $\deg(-)$ on $\mathcal{X}$ is called the *form degree* and a degree $\text{gh}(-)$ on $\mathcal{M}$ is called the *ghost number*^f. Let $\text{Map}(\mathcal{X}, \mathcal{M})$ be a space of smooth maps from $\mathcal{X}$ to $\mathcal{M}$. $|-| = \deg(-) + \text{gh}(-)$ is the degree on $\text{Map}(\mathcal{X}, \mathcal{M})$ and called the *total degree*. A QP-structure on $\text{Map}(\mathcal{X}, \mathcal{M})$ is constructed from the above data.

Since $\text{Diff}(\mathcal{X}) \times \text{Diff}(\mathcal{M})$ naturally acts on $\text{Map}(\mathcal{X}, \mathcal{M})$, D and Q induce homological vector fields on $\text{Map}(\mathcal{X}, \mathcal{M})$, $\hat{D}$ and $\check{Q}$.

Two maps are introduced. An *evaluation map* $\text{ev} : \mathcal{X} \times \mathcal{M}^{\mathcal{X}} \longrightarrow \mathcal{M}$ is defined as

$$\text{ev} : (z, \Phi) \longmapsto \Phi(z),$$

where $z \in \mathcal{X}$ and $\Phi \in \mathcal{M}^{\mathcal{X}}$.

A *chain map* $\mu_* : \Omega^\bullet(\mathcal{X} \times \mathcal{M}) \longrightarrow \Omega^\bullet(\mathcal{M})$ is defined as $\mu_* F = \int_{\mathcal{X}} \mu F$ where $F \in \Omega^\bullet(\mathcal{X} \times \mathcal{M})$ and $\int_{\mathcal{X}} \mu$ is an integration on $\mathcal{X}$ by the D -invariant measure μ . It is an usual integral for the even degree parts and the Berezin integral for the odd degree parts.

A (classical) P-structure on $\text{Map}(\mathcal{X}, \mathcal{M})$ is defined as follows:

Definition 5.1 *For a graded symplectic form ω on $\mathcal{M}$, a graded symplectic form ω on $\text{Map}(\mathcal{X}, \mathcal{M})$ is defined as $\omega := \mu_* \text{ev}^* \omega$.*

We can confirm that ω satisfies the definition of a graded symplectic form because $\mu_* \text{ev}^*$ preserves nondegeneracy and closedness. Thus ω is a P-structure on $\text{Map}(\mathcal{X}, \mathcal{M})$ and induces

^fThe ghost number $\text{gh}(-)$ is the degree $|-|$ on $\mathcal{M}$ in section 2.

a graded Poisson bracket $\{-, -\}$ on $\text{Map}(\mathcal{X}, \mathcal{M})$. Since $|\mu_*\text{ev}^*| = -4$, $|\omega| = -1$ and $\{-, -\}$ on $\text{Map}(\mathcal{X}, \mathcal{M})$ has degree 1 and an odd Poisson bracket.

Next we define a Q-structure S on $\text{Map}(\mathcal{X}, \mathcal{M})$. S is called a *BV action* and consists of two parts $S = S_0 + S_1$. S_0 is constructed as follows: Let ω be the odd symplectic form on $\mathcal{M}$. We take a fundamental form ϑ such that $\omega = -d\vartheta$ and define $S_0 := \iota_{\hat{D}}\mu_*\text{ev}^*\vartheta$. $|S_0| = 0$ because $\mu_*\text{ev}^*$ has degree -4 . S_1 is constructed as follows: We take a Q-structure Θ on $\mathcal{M}$ and define $S_1 := \mu_*\text{ev}^*\Theta$. S_1 also has degree 0.

We can prove that S is a Q-structure on $\text{Map}(\mathcal{X}, \mathcal{M})$, since

$$\{\Theta, \Theta\} = 0, \iff \{S, S\} = 0 \quad (5.21)$$

from the definition of S_0 and S_1 .

A quantum version is

$$\Delta(e^{\frac{i}{\hbar}\Theta}) = 0 \iff \hat{\Delta}(e^{\frac{i}{\hbar}S}) = 0, \quad (5.22)$$

where $\hat{\Delta}$ is an odd Laplace operator on $\text{Map}(\mathcal{X}, \mathcal{M})$. The infinitesimal form of the right hand side in (5.22) is $\{S, S\} - 2i\hbar\hat{\Delta}S = 0$, which is called a *quantum master equation*.⁹

The following theorem has been confirmed [3]:

Theorem 5.2 *If $\mathcal{X}$ is a dg manifold and $\mathcal{M}$ is a QP-manifold, the graded manifold $\text{Map}(\mathcal{X}, \mathcal{M})$ has a QP-structure.*

Definition 5.3 *A topological field theory in 4 dimensions is a triple $(\mathcal{X}, \mathcal{M}, S)$, where $\mathcal{X}$ is a dg manifold with $\dim \mathcal{X}|_0 = 4$, $\mathcal{M}$ is a QP-manifold with the degree 3, and S is a BV action with the total degree 0.*

In order to interpret this theory as a ‘physical’ topological field theory, we must take $\mathcal{X} = T[1]X$. Then we can confirm that a QP-structure on $\text{Map}(\mathcal{X}, \mathcal{M})$ is equivalent to the AKSZ formulation of a topological field theory [4][10]. We set $\mathcal{X} = T[1]X$ from now.

In ‘physics’, a quantum field theory is constructed by quantizing a classical field theory. First we consider a Q-structure $\{\cdot, \cdot\}$ and a classical P-structure S such that

$$\{S, S\} = 0.$$

⁹Discussion for an odd Laplace operator is too naive. In general, the quantum master equation has an obstruction expressed by the modular class [24]. We must regularize an odd Laplace operator and a quantum BV action.

Next we define a quantum P-structure $\hat{\Delta}$ and confirm that

$$\tilde{\Delta}(e^{\frac{i}{\hbar}S}) = 0.$$

Finally we calculate a partition function

$$Z = \int_{\mathcal{L}} e^{\frac{i}{\hbar}S},$$

on a Lagrangian submanifold $\mathcal{L} \subset \text{Map}(\mathcal{X}, \mathcal{M})$. Quantization is not discussed in this paper.

5.2 Local Coordinate Expression and Examples

A general theory in the previous subsection is applied to the local coordinate expression in section 3.1 and a known topological field theory in 4 dimensions is obtained as a special case and a new nontrivial topological field theory is constructed. Let us take a manifold X in 4 dimensions and a manifold M in d dimensions. Let $E[1]$ is a graded vector bundle on M . We take $\mathcal{X} = T[1]X$ and $\mathcal{M} = T^*[3]E[1]$.

Let (σ^μ, θ^μ) be a local coordinate on $T[1]X$. σ^μ is a local coordinate on the base manifold X and θ^μ is one on the fiber of $T[1]X$, respectively. Let $\mathbf{x}^i$ be a smooth map $\mathbf{x}^i : X \rightarrow M$ and ξ_i be a section of $T^*[1]X \otimes \mathbf{x}^*(T^*[3]M)$, $\mathbf{q}^a$ be a section of $T^*[1]X \otimes \mathbf{x}^*(E[1])$ and $\mathbf{p}_a$ be a section of $T^*[1]X \otimes \mathbf{x}^*(T^*[3]E\mathbf{x}[1])$. These are called *superfields*. The exterior derivative d is taken as a differential D on X . From d , a differential $\mathbf{d} = \theta^\mu \frac{\partial}{\partial \sigma^\mu}$ on $\mathcal{X}$ is induced.

Then a BV action S has the following expression:

$$\begin{aligned} S &= S_0 + S_1, \\ S_0 &= \int_{\mathcal{X}} \mu (\xi_i d\mathbf{x}^i - \mathbf{p}_a d\mathbf{q}^a), \\ S_1 &= \int_{\mathcal{X}} \mu (f_1^i{}_a(\mathbf{x}) \xi_i \mathbf{q}^a + \frac{1}{2} f_2^{ab}(\mathbf{x}) \mathbf{p}_a \mathbf{p}_b + \frac{1}{2} f_3^a{}_{bc}(\mathbf{x}) \mathbf{p}_a \mathbf{q}^b \mathbf{q}^c + \frac{1}{4!} f_{4abcd}(\mathbf{x}) \mathbf{q}^a \mathbf{q}^b \mathbf{q}^c \mathbf{q}^d). \end{aligned}$$

Nonabelian BF theory. Let Θ be a Q-structure (4.17) for a Lie algebra $\mathfrak{g}$. $\xi_i d\mathbf{x}^i = 0$ since $M = \{pt\}$. If we define a curvature $\mathbf{F}^a = d\mathbf{q}^a - \frac{1}{2} f^a{}_{bc} \mathbf{q}^b \mathbf{q}^c$, a Q-structure is

$$S = \int_{\mathcal{X}} \mu (-\mathbf{p}_a \mathbf{F}^a),$$

which is equivalent to a BV formalism for a nonabelian BF theory in 4 dimensions.

Topological Yang-Mills Theory. We take a nondegenerate Killing form $(\cdot, \cdot)_K$ for a Lie algebra $\mathfrak{g}$ and consider the Q-structure (4.19). A topological field theory constructed from (4.19) is

$$S = \int_{\mathcal{X}} \mu (-\mathbf{p}_a \mathbf{F}^a + k^{ab} \mathbf{p}_a \mathbf{p}_b).$$

This is equivalent to a topological Yang-Mills theory,

$$S = -\frac{1}{4} \int_{\mathcal{X}} \mu k_{ab} \mathbf{F}^a \mathbf{F}^b,$$

if we delete $\mathbf{p}_a$ by the equations of motion.

Nonassociative BF Theory. Let us take a non Lie algebra $(\mathfrak{g}, [\cdot, \cdot], (\cdot, \cdot)_K)$ in section 4.3. If we take $M = \{pt\}$ and $\mathcal{M} = \mathfrak{g}^*[2] \oplus \mathfrak{g}[1]$, $(\mathfrak{g}, [\cdot, \cdot], (\cdot, \cdot)_K)$ leads a QP-structure with degree 3. In the canonical basis, it is expressed as

$$\begin{aligned} f_1^i{}_a(x) &= 0, & f_2^{ab}(x) &= K^{ab}, \\ f_3^a{}_{bc}(x) &= f^a{}_{bc}, & f_{4abcd}(x) &= K_{ae}^{-1} f^e{}_{f[b} f^f{}_{cd]}, \end{aligned}$$

where $K^{ab} = (p_a, p_b)$ is nondegenerate and $[p_a, p_b] = f^c{}_{ab} p_c$ is a nonassociative bracket and does not satisfy the Jacobi identity. The AKSZ construction derives a new nontrivial topological field theory in 4 dimensions. A BV action S has the following expression:

$$\begin{aligned} S &= \int_{\mathcal{X}} \mu (-\mathbf{p}_a d\mathbf{q}^a + \frac{1}{2} K^{ab} \mathbf{p}_a \mathbf{p}_b + \frac{1}{2} f^a{}_{bc} \mathbf{p}_a \mathbf{q}^b \mathbf{q}^c + \frac{1}{4!} K_{ae}^{-1} f^e{}_{f[b} f^f{}_{cd]} \mathbf{q}^a \mathbf{q}^b \mathbf{q}^c \mathbf{q}^d) \\ &= -\frac{1}{4} \int_{\mathcal{X}} \mu (K_{ab} \mathbf{F}^a \mathbf{F}^b + \frac{1}{3!} K_{ae}^{-1} f^e{}_{f[b} f^f{}_{cd]} \mathbf{q}^a \mathbf{q}^b \mathbf{q}^c \mathbf{q}^d). \end{aligned}$$

It is easily confirmed that $\{S, S\} = 0$.

Topological 3-brane on $Spin(7)$ -structure. Let (M, Ω) be an 8-dimensional $Spin(7)$ -manifold. Here Ω is a $Spin(7)$ 4-form, which satisfies $d\Omega = 0$ and the selfdual condition $\Omega = *\Omega$. A $Spin(7)$ structure is defined as the subgroup of $GL(8)$ to preserve Ω . The Q-structure on (TM, Ω) is given by

$$\Theta = \xi_i q^i + \frac{1}{4!} \Omega_{ijkl}(x) q^i q^j q^k q^l. \quad (5.23)$$

The BV action S for (5.23) defines the same theory as the topological 3-brane analyzed in [25].

6 Conclusions and Discussion

We have defined a BV algebra and a QP-structure of degree 3. A QP-structure of degree 3 has been constructed explicitly and a Lie algebroid u.t.h. has been defined as its algebraic and geometric structure. A general theory of the AKSZ construction of a topological field theory has been expressed and a new topological field theory in four dimensions has been constructed from a QP-structure.

Quantization of this theory and analysis of a Lie algebroid u.t.h. will shed light on a super Poisson geometry and a quantum field theory. They are future problems.

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