

Discrete Subgroups of $PU(1, 2; \mathbb{C})$

Shigeyasu KAMIYA

神谷 茂保 (岡山理科大学)

Our aim of this paper is to introduce the following Basmajian-Miner's theorem.

Theorem 1. *Fix a stable basin point (r, r, ε) . Let g be a parabolic element with fixed point ∞ . If f is a loxodromic element with attracting fixed point 0 and repelling fixed point q satisfying*

$$|\lambda(f) - 1| < \varepsilon$$

and

$$\delta(0, q) \geq \frac{\delta(0, g(0))}{r^2} (1 + r^2 + \sqrt{1 + r^2}),$$

then the group $\langle f, g \rangle$ generated by f and g is not discrete.

1. Let $H_{\mathbb{C}}^2$ be complex hyperbolic 2-space. Set p_{∞} to be the point $(0, -1, 1)$ in the boundary $\partial H_{\mathbb{C}}^2$ of $H_{\mathbb{C}}^2$. Since the Heisenberg group acts simply transitively on $\partial H_{\mathbb{C}}^2 - \{p_{\infty}\}$, we may identify the boundary $\partial H_{\mathbb{C}}^2$ with the one-point compactification of the Heisenberg group. We define the map $\phi : \mathbb{C} \times \mathbb{R} \rightarrow \partial H_{\mathbb{C}}^2 - \{p_{\infty}\}$ by

$$\phi(w, t) = \left(\frac{2w}{1 + |w|^2 - it}, \frac{1 - |w|^2 + it}{1 + |w|^2 - it}, 1 \right).$$

We extend ϕ to $\mathbb{C} \times \mathbb{R} \cup \{\infty\}$ by setting $\phi(\infty) = (0, -1, 1)$. This map ϕ defines Heisenberg coordinates (w, t) on $\partial H_{\mathbb{C}}^2$. The space $\partial H_{\mathbb{C}}^2$ with Heisenberg coordinates is called Heisenberg space and is denoted by H_3 . Set $\tilde{H}_3 = H_3 \cup \{\infty\}$. Under this identification, the action of $PU(1, 2; \mathbb{C})$ can be transported to that on $\tilde{H}_3$. We consider the action on $\tilde{H}_3$ of elements of $PU(1, 2; \mathbb{C})$ with fixed point ∞ . *Translation* by (a, y) is given by

$$T_{(a, y)}(w, t) = (w + a, t + y + 2\operatorname{Im}(a\bar{w})),$$

where $a \in \mathbb{C}$, $y \in \mathbb{R}$. Note that $T_{(a, y)}^{-1} = T_{(-a, -y)}$. *Rotation* is of the form

$$(w, t) \mapsto (e^{i\theta}w, t),$$

and (real) dilations look like

$$(w, t) \mapsto (\lambda w, \lambda^2 t),$$

where $\lambda > 0$. We say that an element is a *complex dilation* if it is the product of a dilation and a rotation. If g is loxodromic, then it is conjugate to a unique complex dilation with attracting fixed point at ∞ and repelling fixed point at the origin 0, namely $(w, t) \mapsto (\lambda w, |\lambda|^2 t)$. We assume that $|\lambda(g)| > 1$.

2. For $p = (w, t_1)$ and $q = (w', t_2)$ in H_3 we define the *Cygan metric* $\delta(p, q)$ by

$$\delta(p, q) = [|w - w'|^4 + \{t_1 - t_2 + 2\operatorname{Im}(w\bar{w}')\}^2]^{\frac{1}{4}}.$$

Let B_s denote the ball of radius s on the boundary $\partial H_{\mathbb{C}}^2$ with respect to the Cygan metric. For $0 < r < 1$, the pair of open sets $(B_r, \bar{B}_{1/r}^c)$ is said to be *stable* with respect to a set of elements S in $PU(1, 2; \mathbb{C})$ if any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \bar{B}_{1/r}^c.$$

Let $S(r, \varepsilon)$ denote the family of elements conjugate to complex dilation g with fixed points in B_r and $\iota(B_r) = \bar{B}_{1/r}^c$, and satisfying $|\lambda(g) - 1| < \varepsilon$, where $\lambda(g)$ is the complex dilation factor of g and $|\lambda(g)| > 1$. Note that $S(r, \varepsilon)$ is closed under conjugation by the inversion ι .

For positive real numbers r and r' with $r < 1/2$, we define $\varepsilon(r, r')$ by

$$(*) \quad \varepsilon(r, r') = \sup_{\alpha} \min\{\alpha, \varepsilon(r, r', \alpha)\},$$

where

$$\varepsilon(r, r', \alpha) = \sqrt{2 + \left(\frac{1 - (4 + \alpha)r^2}{1 - (3 + \alpha)r^2}\right)^2 \left(\frac{1 - 2r^2}{1 - r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - \sqrt{2},$$

and the supremum is over all real numbers α satisfying

$$\alpha < \frac{1 - 4r^2}{2r^2}.$$

Lemma 2 (Stable Basin Theorem). *Given positive real numbers r and r' with $r < 1/2$, the pair of open sets $(B_r, \bar{B}_{1/r}^c)$ is stable with respect to the family $S(r, \varepsilon(r, r'))$, where $\varepsilon(r, r')$ is given by (*). Furthermore, if $g \in S(r, \varepsilon(r, r'))$, then $\delta(0, g(0)) < \delta(0, a_g)$, where a_g is any fixed point of g .*

Sketch of the Proof. Set $s = 1/r$. Let g be an element conjugate to a complex dilation with an attracting fixed point $a_g \in B_r$ and a repelling fixed point $r_g \in \bar{B}_s^c$.

It is seen that

$$\begin{aligned} \delta(0, g(0)) &= \delta(0, h^{-1}\tilde{g}h(0)) \\ &= \delta(0, TG\tilde{g}G^{-1}T^{-1}(0)) \\ &= \delta(T^{-1}(0), G\tilde{g}G^{-1}T^{-1}(0)) \\ &\leq k\delta(G^{-1}(T^{-1}(0)), G^{-1}(G\tilde{g}G^{-1}T^{-1}(0))) \\ &\leq k\delta(h(0), \tilde{g}h(0)) \leq k\frac{r'}{k} = r', \end{aligned}$$

where k depends on r and r' .

We explain the above proof more precisely. Since $S(r, \varepsilon(r, r'))$ is closed under conjugation by ι (inversion), it suffices to determine conditions which guarantee that $g(0) \in B_{r'}$ for all $g \in S(r, \varepsilon(r, r'))$ in order to establish the pair $(B_{r'}, \bar{B}_{1/r'})^c$ is stable under $S(r, \varepsilon(r, r'))$. We may assume that $g(0) \neq 0$. In particular, $a_g \neq 0$.

(1) $h = h(g)$: "normalizing element"; $a_g \mapsto 0$, $r_g \mapsto \infty$.

(2) $h = G_{(\gamma, y)}^{-1} T_{a_g}^{-1}$, where

T_{a_g} : Heisenberg translation; $0 \mapsto a_g$,

$G_{(\gamma, y)}$: parabolic element with fixed point 0, $G_{(\gamma, y)}(\infty) = \iota(\gamma, y) = T_{a_g}^{-1}(r_g)$.

(3) $(\gamma, y) = \iota(T_{a_g}^{-1}(r_g)) \in B_{\frac{1}{s-r}}$.

(4) $\tilde{g} = hgh^{-1}$; $\tilde{g}(0) = 0, \tilde{g}(\infty) = \infty$.

To simplify notation, set $T = T_{a_g}$ and $G = G_{(\gamma, y)}$.

(5)

$$\begin{aligned} \delta(0, g(0)) &= \delta(0, h^{-1}\tilde{g}h(0)) \\ &= \delta(0, TG\tilde{g}G^{-1}T^{-1}(0)) \\ &= \delta(T^{-1}(0), G\tilde{g}G^{-1}T^{-1}(0)). \end{aligned}$$

We estimate how much G^{-1} distorts the distance from $T^{-1}(0)$ to $G\tilde{g}G^{-1}T^{-1}(0)$.

(6) $T^{-1}(0) \in B_r$.

(7) $G\tilde{g}G^{-1}(0) = 0, G\tilde{g}G^{-1}(\infty) = G(\infty)$.

Assume that there exists a parameter $\alpha > 0$, for which $||\lambda(g)| - 1| < \alpha$ and $r < \frac{1}{\sqrt{3+\alpha}}$.

(8) $\delta(0, G(\infty)) = \delta(0, \iota(\gamma, y)) > \frac{1}{s-r} = \frac{1-r^2}{r}$.

(9) $G\tilde{g}G^{-1}(T^{-1}(0)) \in B_l$, where $l = \frac{(1+\alpha)(1-r^2)r}{1-(3+\alpha)r^2}$.

(10) Since $l \geq r$, $T^{-1}(0), G\tilde{g}G^{-1}T^{-1}(0) \in B_l$.

(11) $\delta(0, g(0)) = \delta(T^{-1}(0), G\tilde{g}G^{-1}T^{-1}(0)) \leq k\delta(h(0), \tilde{g}h(0))$.

Next we estimate $\delta(h(0), \tilde{g}h(0))$.

(12) $h(0) = G^{-1}T^{-1}(0) = G^{-1}(-a_g)$.

(13) $h(0) \in B_{\left(\frac{1-r^2}{1-2r^2}\right)\delta(0, a_g)}$.

(14) $\delta(h(0), \tilde{g}h(0)) < \frac{r'}{k}$.

If $r < 1/2$ and $\varepsilon < \varepsilon(r, r')$, a triple of non-negative numbers (r, r', ε) is called a *basin point*. If $r' \leq r$, we call (r, r', ε) a *stable basin point*.

Define the *real cross ratio* $||[q_1, q_2, q_3, q_4]||$ by

$$||[q_1, q_2, q_3, q_4]|| = \frac{\delta^2(q_3, q_1)\delta^2(q_4, q_2)}{\delta^2(q_4, q_1)\delta^2(q_3, q_2)}.$$

It is easy to show that this real cross ratio is invariant under $PU(1, 2; \mathbb{C})$.

Lemma 3. Suppose f and g are conjugate to complex dilations with fixed points $\{q_1, q_2\}, \{q_3, q_4\}$, respectively. If necessary, interchange the roles of q_3 and q_4 so that the real cross ratio $[[q_1, q_2, q_3, q_4]]$ is less than 1. Then f and g can be normalized by an element $h \in PU(1, 2; \mathbb{C})$ as follows.

- (1) hfh^{-1} has fixed points $0, \infty$,
- (2) hgh^{-1} has fixed points at Cygan distance r and $1/r$ from 0 , where

$$r = [[q_1, q_2, q_3, q_4]]^{1/4}.$$

Proof. As in the proof of Lemma 2, there exists an element h_1 in $PU(1, 2; \mathbb{C})$ such that $h_1(q_1) = 0$ and $h_1(q_2) = \infty$. Take a complex dilation h_2 with its dilation factor $\{\delta(0, h_1(q_3))\delta(0, h_1(q_4))\}^{-1/2}$. Set $h = h_2h_1$. Then it follows that $hfh^{-1}(0) = 0$, $hfh^{-1}(\infty) = \infty$. Also we see that hgh^{-1} has fixed points $h_2h_1(q_3)$, $h_2h_1(q_4)$. We have

$$\delta(0, h_2(h_1(q_3)))^4 = \frac{\delta(0, h_1(q_3))^2}{\delta(0, h_1(q_4))^2}$$

and

$$\delta(0, h_2(h_1(q_4)))^4 = \frac{\delta(0, h_1(q_4))^2}{\delta(0, h_1(q_3))^2}.$$

Using these, we obtain

$$\begin{aligned} \frac{\delta(0, h_1(q_3))^2}{\delta(0, h_1(q_4))^2} &= \frac{\delta(0, h_1(q_3))^2 \delta(h_1(q_4, \infty))^2}{\delta(0, h_1(q_4))^2 \delta(h_1(q_3, \infty))^2} \\ &= \frac{\delta^2(q_3, q_1) \delta^2(q_4, q_2)}{\delta^2(q_4, q_1) \delta^2(q_3, q_2)} \\ &= [[q_1, q_2, q_3, q_4]] = r^4. \end{aligned}$$

Thus $\delta(0, h_2(h_1(q_3))) = r$ and $\delta(0, h_2(h_1(q_4))) = 1/r$.

Lemma 4. Let f and g be loxodromic elements of $PU(1, 2; \mathbb{C})$ with fixed points a_f, r_f, a_g, r_g , respectively. If there exists a stable basin point (r, r, ε) such that

$$[[a_f, r_f, a_g, r_g]] < r^4,$$

and

$$\max\{|\lambda(f) - 1|, |\lambda(g) - 1|\} < \varepsilon,$$

then either f and g commute, or the group $\langle f, g \rangle$ is not discrete.

Proof. Suppose that f and g do not commute. Therefore we may assume that f has an attracting fixed point 0 and a repelling fixed point ∞ . By Lemma 3, it is possible to normalize so that a_g is in B_r and r_g is in $\overline{B}_{1/r}^c$. Consider the sequence of f -conjugates

$$g_1 = gf g^{-1}, g_2 = g_1 f g_1^{-1}, \dots, g_k = g_{k-1} f g_{k-1}^{-1}, \dots$$

Note that the fixed points of g_k are $a_k = g_{k-1}(0)$ and $r_k = g_{k-1}(\infty)$. We shall prove by induction that the sequence $\{g_k\}$ are distinct and contained in $S(r, \varepsilon)$. It is clear that f and g are contained in $S(r, \varepsilon)$. Since (r, r, ε) is a stable basin point, Lemma 2 implies that $a_{g_1} = g(0) \in B_r$ and $r_{g_1} = g(\infty) \in \overline{B}_{1/r}^c$. Noting that $\lambda(g_1) = \lambda(f)$, we see that g_1 is an element in $S(r, \varepsilon)$. Since $g(0) \neq 0$, f and g_1 are distinct. Now assume that $g_1, g_2, \dots, g_k \in S(r, \varepsilon)$ are distinct with $\lambda(g_i) = \lambda(f)$ and with fixed points $\{a_{g_i}\}$ having the property that $\delta(a_{g_i}, 0)$ is minimal in the fixed point set of g_i . Moreover, assume that $\delta(a_{g_{i+1}}, 0) < \delta(a_{g_i}, 0)$ for $i = 1, 2, \dots, k-1$. By Lemma 2, $\delta(g_k(0), 0) = \delta(a_{g_{k+1}}, 0) < \delta(a_{g_k}, 0)$ and $r_k \in \overline{B}_{1/r}^c$. Hence it follows by induction that all the $\{g_k\}$ are distinct and contained in $S(r, \varepsilon)$. Since B_r and $\overline{B}_{1/r}^c$ have disjoint, compact closures, there exists a subsequence $\{g_{k_i}\}$ such that $\{a_{k_i}\} \rightarrow a_\infty$ and $\{r_{k_i}\} \rightarrow r_\infty \neq a_\infty$. Noting that a loxodromic element is determined by its dilation factor and two fixed points, we conclude that $\{g_k\} \rightarrow g_\infty$ in $PU(1, 2; \mathbb{C})$, where g_∞ is the unique element with fixed points a_∞ and r_∞ and $\lambda(g_\infty) = \lambda(f)$. Thus the group $\langle f, g \rangle$ is not discrete.

Lemma 5. *Let g be a parabolic element with its fixed point ∞ . Let $q \in H_3$ with $\delta(0, q) > \delta(0, g(0))$. Then*

$$|[0, q, g(0), g(q)]|^{\frac{1}{2}} \leq \left(1 + \frac{\delta(0, q)}{\delta(0, q) - \delta(0, g(0))}\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right).$$

Proof. It follows from the triangle inequality that

$$\delta(g(0), q) \geq \delta(0, q) - \delta(0, g(0)).$$

Since $\delta(0, g(0)) = \delta(0, g^{-1}(0))$ and g is an isometry,

$$\delta(0, g(0)) = \delta(0, g^{-1}(0)) \geq \delta(0, q) - \delta(0, g(q)).$$

The triangle inequality also implies

$$\delta(q, g(q)) \geq \delta(0, q) - \delta(0, g(q)).$$

Hence we obtain

$$\begin{aligned} |[0, q, g(0), g(q)]|^{\frac{1}{2}} &= \frac{\delta(0, g(0))\delta(q, g(q))}{\delta(0, g(q))\delta(q, g(0))} \\ &\leq \left(\frac{\delta(q, g(q))}{\delta(0, g(q))}\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right) \\ &\leq \left(\frac{\delta(0, g(q)) + \delta(0, q)}{\delta(0, g(q))}\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right) \\ &\leq \left(\frac{\delta(0, q)}{\delta(0, g(q))} + 1\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right) \\ &\leq \left(\frac{\delta(0, q)}{\delta(0, g(q)) - \delta(0, g(0))} + 1\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right). \end{aligned}$$

We are ready to prove Theorem 1.

Proof of Theorem 1.

If $q = \infty$, then the group $\langle f, g \rangle$ generated by f and g is not discrete. Therefore we assume that q is a finite point. Our assumption implies that $\delta(0, q) > \delta(0, g(0))$. Using Lemma 5, we have

$$|[0, q, g(0), g(q)]|^{\frac{1}{2}} \leq \left(1 + \frac{\delta(0, q)}{\delta(0, q) - \delta(0, g(0))}\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right) \leq r^2.$$

Set $h = gfg^{-1}$. We see that the fixed points of h are $g(0)$ and $g(q)$ and that the dilation factor of h is equal to that of f . It is clear that the fixed points of f and h are distinct. Hence $fh \neq hf$. By Lemma 4, the group $\langle f, h \rangle$ is not discrete. Thus $\langle f, g \rangle$ is not discrete.

3. Parker [5] gave a similar condition for a subgroup of $PU(1, n; \mathbb{C})$ to be discrete. The author would like to discuss the relation between Theorem 1 and Parker's Theorem in a subsequent paper.

References

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Shigeyasu Kamiya

Department of Mechanical Engineering,

Okayama University of Science,

1-1 Ridai-cho Okayama 700, Japan

e-mail: kamiya@mech.ous.ac.jp