

Polarized Partition on Successors of Singular Cardinals with the GCH

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Abstract

In this paper, we prove that $\forall \mu < \kappa \left(\begin{smallmatrix} \kappa^{++} \\ \kappa^+ \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \mu \\ \kappa^+ \end{smallmatrix} \right)_\kappa$ is consistent with $2^{\kappa^+} = \kappa^{++}$ for singular κ .

1 Introduction

Definition 1.1 (Erdős–Hajnal–Rado [2]). *For any cardinals $\kappa_0, \kappa_1, \lambda_0, \lambda_1, \theta$,*

$$\left(\begin{smallmatrix} \kappa_0 \\ \kappa_1 \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \lambda_0 \\ \lambda_1 \end{smallmatrix} \right)_\theta$$

means for any $c : \kappa_0 \times \kappa_1 \rightarrow \theta$, there are $H_0 \in [\kappa_0]^{\lambda_0}$ and $H_1 \in [\kappa_1]^{\lambda_1}$ such that $c \upharpoonright H_0 \times H_1$ is a constant function.

We say $\left(\begin{smallmatrix} \kappa_0 \\ \kappa_1 \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \lambda_0 \\ \lambda_1 \end{smallmatrix} \right)_{<\theta}$ iff $\forall \theta' < \theta \left(\begin{smallmatrix} \kappa_0 \\ \kappa_1 \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \lambda_0 \\ \lambda_1 \end{smallmatrix} \right)_{\theta'}$. For this partition relation, we are interested in the case of $\kappa_0 = \lambda_0$ and $\kappa_1 = \lambda_1$. But if $\kappa_0 > 2^{\kappa_1}$ then $\left(\begin{smallmatrix} \kappa_0 \\ \kappa_1 \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \kappa_0 \\ \kappa_1 \end{smallmatrix} \right)_{<cf(\kappa_1)}$ is obviously satisfied. In addition, $\left(\begin{smallmatrix} \kappa \\ \kappa \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \kappa \\ \kappa \end{smallmatrix} \right)_2$ fails for every κ . Therefore we consider a coloring on $\kappa_0 \times \kappa_1$ where κ_0 is in between κ_1^+ and 2^{κ_1} .

For such coloring, the following theorem is known.

Theorem 1.2 (Sierpiński [9] for $\kappa = \omega$; Erdős–Hajnal–Rado [2]). *For any infinite cardinal κ , if $2^\kappa = \kappa^+$,*

$$\left(\begin{smallmatrix} \kappa^+ \\ \kappa \end{smallmatrix} \right) \not\rightarrow \left(\begin{smallmatrix} \kappa^+ \\ \kappa \end{smallmatrix} \right)_2.$$

Let us consider the following question.

Question 1.3. *How about polarized partition on $\kappa^+ \times \kappa$ under the assumption $2^\kappa = \kappa^+$?*

Note that $\left(\begin{smallmatrix} \kappa^+ \\ \kappa \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \kappa \\ \kappa \end{smallmatrix} \right)_{<cf(\kappa)}$ is the maximal form under the $2^\kappa = \kappa^+$. For large λ , we ask whether $\left(\begin{smallmatrix} \kappa^+ \\ \kappa \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} \lambda \\ \kappa \end{smallmatrix} \right)_{<cf(\kappa)}$ is consistent with $2^\kappa = \kappa^+$ or not.

In the case of κ is a limit cardinal, the following are known.

Theorem 1.4 (Baumgartner–Hajnal [1]). *If κ is weakly compact, then*

$$\binom{\kappa^+}{\kappa} \rightarrow \binom{\kappa}{\kappa}_{<\kappa}.$$

Theorem 1.5 (Erdős–Hajnal–Rado [2]). *If κ is singular of cofinality ω , then*

$$\binom{\kappa^+}{\kappa} \rightarrow \binom{\kappa}{\kappa}_{<\omega}.$$

Theorem 1.6 (Shelah [8]). *If κ is singular limit of measurable cardinals, then*

$$\binom{\kappa^+}{\kappa} \rightarrow \binom{\kappa}{\kappa}_{<cf(\kappa)}.$$

On the other hand, for successor cardinals, the following consistency result is known.

Theorem 1.7 (Jensen). *In L , for every infinite κ ,*

$$\binom{\kappa^{++}}{\kappa^+} \not\rightarrow \binom{2}{\kappa^+}_\kappa.$$

In fact, Weak Kurepa Hypothesis over κ^+ gives a such coloring. For successors of regular cardinals, a positive polarized partition is also known:

Theorem 1.8 (Laver [5]). *If κ is regular below some HUGE cardinals then there is a κ -directed closed poset $\mathbb{P}$ which forces that*

$$\binom{\kappa^{++}}{\kappa^+} \rightarrow \binom{\kappa^+}{\kappa^+}_\kappa.$$

We will prove that we can force $2^{\kappa^+} = \kappa^{++}$ without destroying $\binom{\kappa^{++}}{\kappa^+} \rightarrow \binom{\kappa^+}{\kappa^+}_\kappa$ as lemma 3.5. So it is consistent that $\binom{\kappa^{++}}{\kappa^+} \rightarrow \binom{\kappa^+}{\kappa^+}_\kappa$ with $2^{\kappa^+} = \kappa^{++}$. Therefore a case of successors of regular cardinals is solved. However, the following is still open.

Question 1.9. *Is it consistent that $\binom{\kappa^{++}}{\kappa^+} \rightarrow \binom{\kappa^+}{\kappa^+}_\kappa$ for singular κ ?*

As a partial answer for this question, we will show the following theorem.

Theorem 1.10. *If κ is supercompact below HUGE and $2^\kappa = \kappa^+$, there is a poset which forces that*

1. $\forall \mu < \kappa \quad \binom{\kappa^{++}}{\kappa^+} \rightarrow \binom{\mu}{\kappa^+}_\kappa,$
2. κ is strong limit singular,
3. $2^{\kappa^+} = \kappa^{++}.$

2 Polarized Partition and Saturated Ideal

In this paper, if we say that I is an ideal over κ^+ , I denotes κ^+ -complete non-principal ideal over κ^+ .

Definition 2.1. An ideal I is (λ, μ, κ) -saturated if and only if for every $X \in [I^+]^\lambda$, there is a $Y \in [X]^\mu$ such that $\forall Z \in [Y]^\kappa \cap Z \in I^+$.

Note that κ -saturation is $(\kappa, 2, 2)$ -saturation. So this is an extended concept of saturation property.

Theorem 2.2 (Laver [5]). If λ is HUGE and $\kappa < \lambda$ is regular, then there is a κ -directed closed $\mathbb{P}$ which forces that there is a $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturated ideal over κ^+ .

Laver's polarized partition theorem is shown by using $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturation properties. We give a direct proof of Laver's theorem.

Theorem 2.3 (Laver [5]). If $2^\kappa = \kappa^+$ and there is a $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturated ideal over κ^+ , then

$$\binom{\kappa^{++}}{\kappa^+} \rightarrow \binom{\kappa^+}{\kappa^+}_\kappa.$$

Lemma 2.4. Suppose that $2^\kappa = \kappa^+$ and $\langle X_\alpha \mid \alpha < \kappa^{++} \rangle \in {}^{\kappa^{++}}([\kappa^+]^{\kappa^+})$ satisfies $\forall X \in [\kappa^{++}]^\kappa (\bigcap_{\alpha \in X} X_\alpha = \kappa^+)$. Then there is a $\mathcal{Y} \in [\kappa^{++}]^{\kappa^+}$ such that $\bigcap_{\alpha \in \mathcal{Y}} X_\alpha = \kappa^+$.

Proof. Let Ψ be sufficiently large regular and let $M < \mathcal{H}_\Psi$ be an elementary substructure such that:

- $|M| = \kappa^+$.
- $\kappa^+ + 1 \subseteq M$.
- $\overline{X_\alpha}, \kappa^{++} \in M$.
- M is closed under the taking κ -sequence. i.e. ${}^\kappa M \subseteq M$.
- $\delta = M \cap \kappa^{++}$ is an ordinal.

This M can be taken because we have $2^\kappa = \kappa^+$. Note that δ has cofinality κ^+ since M closed under the taking κ -sequence. We construct a sequence $\langle \beta_\xi \mid \xi < \kappa^+ \rangle$ and $\langle \alpha_\xi \mid \xi < \kappa^+ \rangle$ by the following way:

- $\alpha_0 = \min X_\delta$.
- $\beta_0 =$ the least $\gamma \in \delta$ such that $\alpha_0 \in X_\gamma$.
- $\alpha_\xi = \min \bigcap_{\eta < \xi} X_{\beta_\eta} \cap X_\delta \setminus \{ \alpha_\eta \mid \eta < \xi \}$.
- $\beta_\xi =$ the least $\gamma \in \delta \setminus \sup \{ \beta_\eta \mid \eta < \xi \}$ such that $\{ \alpha_\eta \mid \eta \leq \xi \} \subseteq X_\gamma$.

This construction will be succeed. In every stage, α_ξ can be defined by $|\bigcap_{\eta < \xi} X_{\beta_\eta} \cap X_\delta| = \kappa^+$. β_ξ can also be defined in every $\xi < \kappa^+$. By construction, we have $\{\alpha_\eta \mid \eta \leq \xi\} \subseteq X_\delta$. By assumption for M , we have $\beta := \sup_{\eta < \xi} \beta_\eta < \delta$ and $\{\alpha_\eta \mid \eta \leq \xi\}$ is in M . By elementarity of M ,

$$M \models \exists \delta' > \beta \{\alpha_\eta \mid \eta \leq \xi\} \subseteq X_{\delta'}.$$

So we can take β_ξ as δ' .

As conclusion, let $\mathcal{Y} := \{\beta_\xi \mid \xi < \kappa^+\}$, then

$$\{\alpha_\xi \mid \xi < \kappa^+\} \subseteq \bigcap_{\xi < \kappa^+} X_{\beta_\xi} = \bigcap_{\alpha \in \mathcal{Y}} X_\alpha.$$

So we have $|\bigcap_{\alpha \in \mathcal{Y}} X_\alpha| = \kappa^+$. □

Proof of Theorem 2.3. Let I be $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturated ideal over κ^+ . Let $c : \kappa^{++} \times \kappa^+ \rightarrow \kappa$ be an arbitrary coloring. For every $\alpha < \kappa^{++}$, by κ -completeness of I , there is an η_α such that $X_\alpha = \{\xi < \kappa^+ \mid c(\alpha, \xi) = \eta_\alpha\} \in I^+$.

For simplicity, we may assume that there is an η such that $\eta_\alpha = \eta$ for all $\alpha < \kappa^{++}$. By $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturated, we may assume that $\forall X \in [\kappa^{++}]^\kappa (\bigcap_{\alpha \in X} X_\alpha \in I^+)$. In particular, $\bigcap_{\alpha \in X} X_\alpha$ is of size κ^+ for every $X \in [\kappa^{++}]^\kappa$.

By lemma 2.4, we can pick $H_0 \in [\kappa^{++}]^{\kappa^+}$ such that $H_1 := \bigcap_{\alpha \in H_0} X_\alpha$ is of size κ^+ . We have $c \upharpoonright H_0 \times H_1$ is monochromatic with color η . □

Remark 2.5. We can proof more strong form of theorem 2.3. In fact, under the same assumption, we can take $H_0 \in [\kappa^{++}]^\alpha$ and $H_1 \in [\kappa^+]^{\kappa^+}$ such that $c \upharpoonright H_0 \times H_1$ is monochromatic for every $c : \kappa^{++} \times \kappa^+ \rightarrow \kappa$ and ordinal $\alpha < \kappa^{++}$.

Definition 2.6 (Garti [3]). For any family $\mathcal{A} \subseteq \mathcal{P}(\lambda)$,

1. Coloring $c : \kappa \times \lambda \rightarrow \theta$ is $\mathcal{A}$ -amenable if and only if

$$\forall \alpha < \kappa \exists \eta < \theta \exists X \in \mathcal{A} (\forall \xi \in X c(\alpha, \xi) = \eta).$$

2. $\binom{\kappa_0}{\kappa_1} \rightarrow_{\mathcal{A}} \binom{\lambda_0}{\lambda_1}_\theta$ says that for every $\mathcal{A}$ -amenable coloring $c : \kappa_0 \times \kappa_1 \rightarrow \theta$, there are $H_0 \in [\kappa_0]^{\lambda_0}, H_1 \in [\kappa_1]^{\lambda_1}$ such that $c \upharpoonright H_0 \times H_1$ is monochromatic.

The above lemma gives a polarized partition for amenable colorings:

Corollary 2.7. If $2^\kappa = \kappa^+$ and $\mathcal{A} \subseteq [\kappa^+]^{\kappa^+}$ has κ^+ -completeness (or every intersection of κ -many sets in $\mathcal{A}$ have size κ^+), $\binom{\kappa^{++}}{\kappa^+} \rightarrow_{\mathcal{A}} \binom{\kappa^+}{\kappa^+}_\kappa$ holds.

In particular, $\binom{\kappa^{++}}{\kappa^+} \rightarrow_{Club(\kappa^+)} \binom{\kappa^+}{\kappa^+}_\kappa$ holds.

Proof. By the similar proof of Theorem 2.3. □

Theorem 2.8 (Garti [3]). If $2^{\aleph_1} = \aleph_2$ holds, then

$$\binom{\aleph_2}{\aleph_1} \not\rightarrow_{Club(\aleph_1)} \binom{\aleph_2}{\aleph_1}_2.$$

By these observations, we have

$$\text{GCH implies } \left(\begin{smallmatrix} \aleph_2 \\ \aleph_1 \end{smallmatrix} \right) \not\rightarrow_{\text{Club}(\aleph_1)} \left(\begin{smallmatrix} \aleph_2 \\ \aleph_1 \end{smallmatrix} \right)_2 \text{ but } \left(\begin{smallmatrix} \aleph_2 \\ \aleph_1 \end{smallmatrix} \right) \rightarrow_{\text{Club}(\aleph_1)} \left(\begin{smallmatrix} \aleph_1 \\ \aleph_0 \end{smallmatrix} \right).$$

Question 2.9. *Is it consistent that there is a $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturated ideal over κ^+ for singular κ ?*

3 Proof of Main Theorem

Definition 3.1. *We say that $\mathbb{P}$ satisfies $(*)_\kappa$ if and only if there is a $\langle \mathbb{F}_\eta \mid \eta < \kappa \rangle$ such that $\mathbb{F}_\eta$ is κ -directed and $\mathbb{P} = \bigcup_{\eta < \kappa} \mathbb{F}_\eta$.*

Here, X is κ -directed iff every subset of X of size $< \kappa$ have a common extension. Clearly, $(*)_\kappa$ implies κ -c.c. Many Prikry-type forcing satisfies $(*)_\kappa$. $(*)_\kappa$ poset preserves saturation property for ideal over κ^+ as follows.

Lemma 3.2. *For regular cardinals $\mu < \kappa$, suppose that*

1. *There is a (κ^{++}, μ, μ) -saturated ideal I over κ^+ .*
2. *$\mathbb{P}$ is a poset which satisfies $(*)_\kappa$ and preserves $\mu < \kappa$ and κ .*

Then

$$\mathbb{P} \Vdash \check{I} \text{ generates } (\check{\kappa}^{++}, \check{\mu}, \check{\mu})\text{-saturated ideal}$$

Proof. Let $\langle \mathbb{F}_\eta \mid \eta < \kappa \rangle$ witnesses to $(*)_\kappa$. Let $\dot{I}$ be a $\mathbb{P}$ -name which denotes an ideal generated by $\dot{I}$.

Since $\mathbb{P}$ has the κ^+ -c.c., $\mathbb{P}$ forces $\dot{I}$ is $\check{\kappa}^+$ -complete.

Consider a p and $\langle \dot{X}_\alpha \mid \alpha < \kappa^{++} \rangle$ such that

$$p \Vdash \dot{X}_\alpha \in \dot{I}^+.$$

It is enough to show that there is a $q \leq p$ such that $q \Vdash \exists H \in [\check{\kappa}^{++}]^{\check{\mu}} \bigcap_{\alpha \in H} \dot{X}_\alpha \in \dot{I}^+$.

For each $\alpha < \kappa^{++}$, $\eta < \kappa$, let $A_\alpha^\eta := \{ \xi < \kappa^+ \mid \exists q \in \mathbb{F}_\eta (q \leq p \wedge q \Vdash \xi \in \dot{X}_\alpha) \}$. Since $\bigcup_{\eta < \kappa} A_\alpha^\eta \in I^+$ and I is κ^+ -complete, there is an η_α such that $A_\alpha^{\eta_\alpha} \in I^+$. And $A_\alpha := A_\alpha^{\eta_\alpha}$. Then there are $Z \in [\kappa^{++}]^{\kappa^{++}}$ and $\eta < \kappa$ such that $\eta = \eta_\alpha$ for all $\alpha \in Z$. By the (κ^{++}, μ, μ) -saturation for I , we can pick $H \in [Z]^\mu$ such that $\bigcap_{\alpha \in H} A_\alpha \in I^+$.

Claim 3.3. *For every $\xi \in \bigcap_{\alpha \in H} A_\alpha$, there is a $q \leq p$ such that $q \Vdash \xi \in \bigcap_{\alpha \in \check{H}} \dot{X}_\alpha$.*

Proof of Claim. By the definition of A_α , for each $\alpha \in H$, we can pick $q_\alpha \in \mathbb{F}_\eta$ such that $q_\alpha \Vdash \xi \in \dot{X}_\alpha$. Since each q_α are in $\mathbb{F}_\eta$, there is a $q \leq q_\alpha$ for every α . q is an extension of p and forces that $\xi \in \bigcap_{\alpha \in \check{H}} \dot{X}_\alpha$. $\square$

We show that there is a $q \leq p$ which forces that $\bigcap_{\alpha \in \check{H}} \dot{X}_\alpha \in \dot{I}$. Consider a set $A = \{ q \leq p \mid \exists Z_q \in I (q \Vdash \bigcap_{\alpha \in \check{H}} \dot{X}_\alpha \subseteq \check{Z}_q) \}$. Note that A is the set of all $q \leq p$ which forces $\bigcap_{\alpha \in \check{H}} \dot{X}_\alpha$ is $\dot{I}$ -measure zero. Let $\mathcal{A} \subseteq A$ be a maximal antichain below p . By the κ^+ -c.c., $\mathcal{A}$ has size at most κ . So $Z := \bigcup_{q \in \mathcal{A}} Z_q$ is also I -measure zero.

Therefore $(\bigcap_{\alpha \in H} A_\alpha) \setminus Z \neq \emptyset$. Pick $\xi \in (\bigcap_{\alpha \in H} A_\alpha) \setminus Z$, by claim, there is a $q \leq p$ which forces that $\xi \in \bigcap_{\alpha \in \check{H}} \dot{X}_\alpha$. Clearly, q is incompatible with any element in $\mathcal{A}$. By the maximality of $\mathcal{A}$, $q \Vdash \bigcap_{\alpha \in \check{H}} \dot{X}_\alpha \in \dot{I}^+$. $\square$

Corollary 3.4. In $V^{\mathbb{P}}$, $\left(\kappa^{++}\right)_{\kappa} \rightarrow \left(\mu\right)_{\kappa}$ holds.

Proof. Let G be an arbitrary $(V, \mathbb{P})$ -generic. We discuss in $V[G]$. Let $\hat{I}$ be an ideal generated by I . By lemma, $\hat{I}$ satisfies (κ^{++}, μ, μ) -saturated.

Take any coloring $c : \kappa^{++} \times \kappa^+ \rightarrow \kappa$. For each $\alpha < \kappa^{++}$, there is a η_α such that $X_\alpha = \{\xi < \kappa^+ \mid c(\alpha, \xi) = \eta_\alpha\}$ in $\hat{I}^+$. We may assume that there is an η such that $\eta_\alpha = \eta$ for every $\alpha < \kappa^{++}$.

By the saturation property of $\hat{I}$, there is an $H_0 \in [\kappa^{++}]^\mu$ such that $H_1 := \bigcap_{\alpha \in H_0} X_\alpha \in I^+$. Trivially, $c \upharpoonright H_0 \times H_1$ is monochromatic with color η . □

Lemma 3.5 (Folklore). Suppose that $\left(\kappa^{++}\right)_{\kappa} \rightarrow \left(\mu\right)_{\kappa}$ holds for some $\mu < \kappa^{++}$ and $2^{\kappa^+} > \kappa^{++}$. Then there is a poset $\mathbb{P}$ such that

1. $\mathbb{P}$ forces $2^{\kappa^+} = \kappa^{++}$ and $\left(\kappa^{++}\right)_{\kappa} \rightarrow \left(\mu\right)_{\kappa}$,
2. $\mathbb{P}$ preserves all cardinals below κ^{++} .

Proof. Let $\mathbb{P} := \text{coll}(\kappa^{++}, 2^{\kappa^+})$. Note that $\mathbb{P}$ is κ^{++} -closed. Clearly $\mathbb{P}$ forces that $2^{\kappa^+} = \kappa^{++}$. It is enough to show that $\mathbb{P}$ preserves polarized partition relation. Let $p \in \mathbb{P}$ and $\dot{c} \in V^{\mathbb{P}}$ be such that:

$$p \Vdash \dot{c} : \check{\kappa}^{++} \times \check{\kappa}^+ \rightarrow \check{\kappa}.$$

By induction on $\alpha < \kappa^{++}$, we define a sequence $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ and coloring $f : \kappa^{++} \times \kappa^+ \rightarrow \kappa$ such that:

- $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ is decreasing sequence in $\mathbb{P}$ and $p_0 \leq p$.
- $p_\alpha \Vdash \dot{c}(\check{\alpha}, \check{\xi}) = \check{f}(\check{\alpha}, \check{\xi})$ for every $\xi < \kappa^+$.

By κ^{++} -closedness, we can construct this sequence. By $\left(\kappa^{++}\right)_{\kappa} \rightarrow \left(\mu\right)_{\kappa}$, there are $H_0 \in [\kappa^{++}]^\mu$ and $H_1 \in [\kappa^+]^{\kappa^+}$ such that $f \upharpoonright H_0 \times H_1$ is monochromatic.

Let $\alpha := (\sup H_0) + 1 < \kappa^{++}$. Then

$$p_\alpha \Vdash \dot{c} \upharpoonright \check{H}_0 \times \check{H}_1 = \check{f} \upharpoonright \check{H}_0 \times \check{H}_1 \text{ is monochromatic.}$$

□

We recall about Prikry forcing in [7]. For a normal ultrafilter U over κ , Prikry forcing $\mathbb{P}_U$ is $[\kappa]^{<\omega} \times U$ ordered by $\langle a, A \rangle \leq \langle b, B \rangle$ iff a is an end-extension of b (i.e. $a \cap \max(b) = b$), $A \subseteq B$ and $a \setminus b \subseteq B$. Prikry forcing preserves all cardinals but forces that $cf(\kappa) = \omega$.

Lemma 3.6. $\mathbb{P}_U$ satisfies $(*)_\kappa$.

Proof. For each $a \in [\kappa]^{<\omega}$, let $\mathbb{F}_a := \{\langle a, A \rangle \mid A \in U\}$, which is κ -directed. Clearly, $\mathbb{P} = \bigcup_{a \in [\kappa]^{<\omega}} \mathbb{F}_a$. □

Proof of Theorem 1.10. First, by Laver's theorem and indestructibility for supercompactness, there is a poset $\mathbb{P}$ which forces that there is $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturated ideal over κ^+ and κ remains supercompact.

In $V^{\mathbb{P}}$, Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$ -name which denotes Prikry forcing over κ . By $\dot{\mathbb{Q}}$ satisfies that $(*)_{\kappa}$ in $V^{\mathbb{P}}$ and lemma 3.2, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models \forall \mu < \kappa \left(\kappa^{++} \rightarrow \binom{\mu}{\kappa^+}_{\kappa} \right)$ holds and $cf(\kappa) = \omega$. Further, by lemma 3.5, we can forces that $2^{\kappa^+} = \kappa^{++}$ without destroying polarized partition relation. So this proof is done. $\square$

Lemma 3.7. *If κ is supercompact below HUGE, then there is a poset which forces that*

1. $\left(\kappa^{++} \right)_{\kappa^+} \rightarrow \left(\lambda \right)_{\kappa}$ for every $\lambda < \kappa$,
2. $2^{\kappa^+} = \kappa^{++}$,
3. κ is strong limit singular with uncountable cofinality.

Proof. Note that Magidor's forcing in [6] satisfies $(*)_{\kappa}$. We can do the same proof for theorem 1.10. $\square$

Lemma 3.8. *Suppose that there is a $(\kappa^{++}, \kappa^{++}, \kappa)$ -saturated ideal over κ^+ and κ is supercompact, $2^{\kappa} = \kappa^+$. Then there is a poset which forces that*

1. $\left(\aleph_{\omega+2} \right)_{\aleph_{\omega+1}} \rightarrow \left(\aleph_n \right)_{\aleph_{\omega}}$ for every $n < \omega$,
2. $\kappa = \aleph_{\omega}$,
3. $2^{\aleph_{\omega+1}} = \aleph_{\omega+2}$.

Proof. Let U be a normal ultrafilter over κ and $j : V \rightarrow M$ be an elementary embedding induced by U . By $2^{\kappa} = \kappa^+$, there is an $(M, \text{Coll}((\kappa^{++})^M, < j(\kappa)))$ -generic G in V . Then G induces a Gitik's forcing in [4] which forces $\kappa = \aleph_{\omega}$. Note that Gitik's forcing satisfies $(*)_{\kappa}$. We can do the same proof for theorem 1.10 again. $\square$

Note that we don't need a HUGE cardinal to give such a polarized partition over $\aleph_1 \times \aleph_2$.

Lemma 3.9 (Zhang [10]). *Suppose that there is a presaturated ideal I over κ^+ . For every $n < \omega$,*

$$\left(\kappa^{++} \right)_{\kappa^+} \rightarrow \left(n \right)_{\kappa^+}_{\kappa}.$$

In particular, if there is a presaturated ideal over $\aleph_1$ then $\left(\aleph_2 \right)_{\aleph_1} \rightarrow \left(n \right)_{\aleph_1}_{\aleph_0}$ for every $n < \aleph_0$.

Proof. Let G be an arbitrary $(V, \mathbb{P})$ -generic and $j : V \rightarrow M \subseteq V[G]$ be generic ultrapower induced by G .

Since I is presaturated, we have

- $\text{crit}(j) = \kappa^+$.
- $j(\kappa^+) = (\kappa^{++})^V$.

Let $c : \kappa^{++} \times \kappa^+ \rightarrow \kappa$ be an arbitrary coloring.

Let $A := j''\kappa^{++}$. $|A| = (\kappa^{++})^V$. Since $\text{cf}((\kappa^{++})^V) > \kappa$, we can take unbounded subset $B \subseteq A$ and $\eta < \kappa$ such that

$$\forall \xi \in B(j(c)(\xi, (\kappa^+)^V) = \eta).$$

Pick any finite sequence $j(\beta_0), \dots, j(\beta_{n-1}) \in B$. Note that $\langle \beta_0, \dots, \beta_{n-1} \rangle \in V$.

We construct a sequence $\langle \alpha_\xi \mid \xi < \kappa^+ \rangle$ below $(\kappa^+)^V$ by induction. Suppose that $\langle \alpha_\xi \mid \xi < \eta \rangle$ has been defined. Let $\alpha := \sup_{\xi < \eta} \alpha_\xi < \kappa^+$. In M , the following holds:

$$(\kappa^+)^V > \alpha \wedge \forall i < n(j(c)(j(\beta_i), (\kappa^+)^V) = j(\eta)).$$

So $M \models \exists \zeta > \alpha \forall i < n(j(c)(j(\beta_i), \zeta) = j(\eta))$. By elementarity,

$$\exists \zeta > \alpha \forall i < n(c(\beta_i, \zeta) = \eta).$$

Let α_η be defined as such ζ .

Clearly, $c \upharpoonright \{\beta_i \mid i < n\} \times \{\alpha_\xi \mid \xi < \kappa^+\}$ is monochromatic. $\square$

The same proof shows the following lemma.

Lemma 3.10. *Suppose that there are presaturated ideal I over κ^+ and regular $\mu < \kappa$ such that $\langle I^+, \subseteq \rangle$ is μ^+ -Baire. Then*

$$\binom{\kappa^{++}}{\kappa^+} \rightarrow \binom{\mu}{\kappa^+}_\kappa.$$

Remark 3.11. *In [10], Zhang proved it is consistent that $\binom{\aleph_2}{\aleph_1} \rightarrow \binom{\aleph_0}{\aleph_1}_{\aleph_0}$ fails but there is a presaturated ideal over $\aleph_1$.*

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