

NORMAL HILBERT POLYNOMIALS

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1. Normal Hilbert Polynomials.

This note is a short summary of my recent work [5]. Throughout this note (A, m) will be a Cohen-Macaulay local ring of dimension $d \geq 2$, and I will be a parameter ideal for A , i.e. I is an m -primary ideal generated by d elements. Assume that A is analytically unramified and A/m is infinite. For an ideal J in A , $\bar{J}$ denotes the integral closure of J , i.e., $\bar{J} = \{x \in A \mid x^n + a_1 x^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in J^i\}$.

It is well known that there exist uniquely determined integers $\bar{e}_0(I), \dots, \bar{e}_d(I)$ such that

$$\text{length}_A(A/\bar{I}^{n+1}) = \bar{e}_0(I) \binom{n+d}{d} - \bar{e}_1(I) \binom{n+d-1}{d-1} + \cdots + (-1)^d \bar{e}_d(I)$$

for all large n . We say that

$$P(I, n) = \bar{e}_0(I) \binom{n+d}{d} - \bar{e}_1(I) \binom{n+d-1}{d-1} + \cdots + (-1)^d \bar{e}_d(I)$$

is the *normal Hilbert polynomial* of I . $\bar{e}_0(I)$ is a well-known number called the multiplicity of I i.e., $\bar{e}_0(I) = e(I) = \text{length}_A(A/I)$. Our purpose of this note is to report some properties of $\bar{e}_1(I)$, $\bar{e}_2(I)$ and $\bar{e}_3(I)$. Our results are contained in the following two theorems.

THEOREM 1. (1) $\bar{e}_1(I) - \text{length}_A(\bar{I}/I) \geq \text{length}_A(\bar{I}^2/I\bar{I})$, and the equality holds if and only if $\bar{I}^{n+2} = I^n \bar{I}^2$ for every $n \geq 0$.

(2) $\bar{e}_2(I) \geq \bar{e}_1(I) - \text{length}_A(\bar{I}/I)$, and the equality holds if and only if $\bar{I}^{n+2} = I^n \bar{I}^2$ for every $n \geq 0$.

THEOREM 2. (1) $\bar{e}_3(I) \geq 0$, and if $\bar{e}_3(I) = 0$, then $\bar{I}^{n+2}$ is contained in I^n for every $n \geq 0$.

(2) Assume that A is Gorenstein and $\bar{I} = m$. Then $\bar{e}_3(I) = 0$ if and only if $\bar{I}^{n+2} = I^n \bar{I}^2$ for every $n \geq 0$.

Ooishi called the number $\bar{e}_1(I) - \text{length}_A(\bar{I}/I)$ the *normal sectional genus* of I and denoted it by $\bar{g}_s(I)$.

In two dimensional case, Huneke remarked in his paper that

$$\begin{aligned} \text{length}_A(A/\bar{I}^{n+1}) &= \text{length}_A(A/I) \binom{n+2}{2} \\ &\quad - \left(\sum_{r \geq 0} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) \right) \binom{n+1}{1} + \sum_{r \geq 1} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) r \end{aligned}$$

for all large n ([3]). Therefore

$$\begin{aligned} \sum_{r \geq 1} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) r &\geq \bar{g}_s(I) = \sum_{r \geq 1} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) \\ &\geq \text{length}_A(\bar{I}^2/I\bar{I}); \end{aligned}$$

thus our Theorem 1 is a natural generalization of this fact to high dimensional case, and the main difficulty is how we reduce the problem to two dimensional case.

2. Key Theorem.

Detailed studies in $\bar{e}_i(I)$'s are based on the following theorem and lemma.

THEOREM 3. *There exists a system of generators $x_1, \dots, x_d$ of I such that, if we put $C = A(T)/(\sum_i x_i T_i)$ and $J = IC$, where $A(T) = A[T]_{m[T]}$ with $T = (T_1, \dots, T_d)$ d indeterminates, then*

- (1) $\bar{J}^n \cap A = \bar{I}^n$ for every $n \geq 0$;
- (2) $\bar{J} = \bar{I}C$;
- (3) $\bar{J}^n = \bar{I}^n C \cong \bar{I}^n A(T)/(\sum_i x_i T_i) \bar{I}^{n-1} A(T)$ for all large n ;
- (4) C is normal if A is analytically normal and $\dim A \geq 3$.

LEMMA 4. *Choose a system of minimal generators $x_1, \dots, x_d$ of I , and put $B = A[x_1/x_2]$, $R = R(A, I) = A[It, t^{-1}]$ (the Rees ring of I), $D = R_P$, where $P = (t^{-1}, m)R$, and nD = the maximal ideal of D . Let $h : B \rightarrow D$ be a canonical homomorphism which maps x_1/x_2 to $x_1 t/x_2 t$. Then*

- (0) $nD \cap B = m[x_1/x_2]$.

We here put $C = Bm[x_1/x_2]$ and $J = IC$. Then

- (1) $\bar{J}^n \cap A = \bar{I}^n$ for every $n \geq 0$,
- (2) $\bar{J} = \bar{I}C$ and

(3) $\overline{I^{n+r}} \subseteq \overline{I^n}$ for every $n \geq 0$ if and only if $\overline{J^{n+r}} \subseteq \overline{J^n}$ for every $n \geq 0$.

Applying the theorem (and the lemma), we have the following results.

PROPOSITION 5. *With the same notation as in Theorem 3, we have the following assertions.*

- (1) $\bar{e}_i(I) = \bar{e}_i(J)$ for every $i \leq d-1$;
- (2) $\bar{g}_s(I) = \bar{g}_s(J)$;
- (3) $\overline{I^2 A(T)} / \overline{I \bar{I} A(T)}$ is a submodule of $\overline{J^2 / J \bar{J}}$, in particular $\text{length}_A(\overline{I^2 / I \bar{I}}) \geq \text{length}_C(\overline{J^2 / J \bar{J}})$.
- (4) $\overline{I^{n+r}}$ is contained in I^n for every $n \geq 0$ if and only if $\overline{J^{n+r}}$ is contained in J^n for every $n \geq 0$.

PROOF. We put $z = \sum_i x_i T_i$ for simplicity. (1) and (2) follow from Theorem 3. (3): It is enough to show that $\overline{I^2 A(T)} \cap (\overline{I \bar{I}}, z)A(T) = \overline{I \bar{I} A(T)}$. $\overline{I^2 A(T)} \cap (\overline{I \bar{I}}, z)A(T) = \overline{I \bar{I} A(T)} + \overline{I^2 A(T)} \cap zA(T) = \overline{I \bar{I} A(T)} + \overline{I A(T)}z = \overline{I \bar{I} A(T)}$. (4) follows from Lemma 4(3).

It is known that $\bar{e}_2(I) \geq \bar{g}_s(I)$ if $\dim A = 2$; therefore by the induction on $d = \dim A$, we have

COROLLARY 6. $\bar{e}_2(I) \geq \bar{g}_s(I)$.

As proved in [4, Proposition 10],

$$\begin{aligned}
 (*) \text{length}_A(A / \overline{I^{n+1}}) &\geq \text{length}_A(A / \overline{I^{n+1} \bar{I}^2}) \\
 &= \text{length}_A(A / I) \binom{n+d}{d} - (\text{length}_A(\bar{I} / I) + \text{length}_A(\overline{I^2 / I \bar{I}})) \binom{n+d-1}{d-1} \\
 &\quad + \text{length}_A(\overline{I^2 / I \bar{I}}) \binom{n+d-2}{d-2}
 \end{aligned}$$

for all n . Therefore

$$\begin{aligned}
 \bar{e}_1(I) &\geq \text{length}_A(\bar{I} / I) + \text{length}_A(\overline{I^2 / I \bar{I}}) \text{ i.e.,} \\
 \bar{g}_s(I) &= \bar{e}_1(I) - \text{length}_A(\bar{I} / I) \geq \text{length}_A(\overline{I^2 / I \bar{I}}).
 \end{aligned}$$

We give here the proof of (1) in Theorem 1.

PROPOSITION 7. (1) $\bar{g}_s(I) \geq \text{length}_A(\overline{I^2 / I \bar{I}})$.

(2) $\bar{g}_s(I) = \text{length}_A(\bar{I}^2/I\bar{I})$ if and only if $\overline{I^{n+2}} = I^n \bar{I}^2$ for every $n \geq 0$.

PROOF. (2) We may assume that A/m is infinite. We use the induction on $d = \dim A$. If $d = 2$, the assertion clearly holds. So assume that $d > 2$. if part follows from (*). only if part: Choose a system of generators $x_1, \dots, x_d$ of I satisfying the conditions of Theorem 3, and put $z = \sum_i x_i T_i$, $C = A(T)/zA(T)$ and $J = IC$. Since $\text{length}_A(\bar{I}^2/I\bar{I}) \leq \text{length}_C(\bar{J}^2/J\bar{J}) \leq \bar{g}_s(J) = \bar{g}_s(I)$ by Proposition 5, we have $\text{length}_C(\bar{J}^2/J\bar{J}) = \bar{g}_s(J)$ and $\bar{J}^2 = \bar{I}^2 C$. Thus $\overline{J^{n+2}} = J^n \bar{J}^2$ for every $n \geq 0$. Then $\overline{I^{n+2}A(T)} = \overline{I^{n+2}A(T)}$ is contained in $I^n \bar{I}^2 A(T) + zA(T)$, and hence $\overline{I^{n+2}A(T)} = \overline{I^{n+2}A(T)} \cap (I^n \bar{I}^2 A(T) + zA(T)) = I^n \bar{I}^2 A(T) + z \overline{I^{n+1}A(T)}$. By the induction on n , $\overline{I^{n+2}A(T)} = I^n \bar{I}^2 A(T)$, and therefore $\overline{I^{n+2}} = I^n \bar{I}^2$.

As we remarked in [4, Proposition 3],

(**) if $\overline{I^{n+2}} = I^n \bar{I}^2$ for every $n \geq 0$, then $G = R'/t^{-1}R'$ is Cohen-Macaulay.

Since

$$\begin{aligned} \text{length}_A(A/\overline{I^{n+1}}) &= \text{length}_A(A/I^{n+1}) \\ &\quad - \sum_{0 \leq r \leq n} \text{length}_A(I^{n-r} \overline{I^{r+1}}/I^{n-r+1} \bar{I}^r) \end{aligned}$$

and

$$\begin{aligned} &\text{length}_A(I^{n-r} \overline{I^{r+1}}/I^{n-r+1} \bar{I}^r) \\ &\leq \text{length}_A((\overline{I^{r+1}}/I\bar{I}^r) \otimes (I^{n-r}/I^{n-r+1})) \\ &= \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r) \binom{n-r+d-1}{d-1} \\ &= \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r) \binom{n+d-1}{d-1} - r \binom{n-r+d-2}{d-2} + \text{lower degree terms}, \end{aligned}$$

we have

$$\begin{aligned} \bar{e}_1(I) &\leq \sum_{r \geq 0} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r) \text{ i.e.,} \\ \bar{g}_s(I) &= \bar{e}_1(I) - \text{length}_A(\bar{I}/I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r). \end{aligned}$$

PROPOSITION 8. (1) $\bar{g}_s(I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r)$.

(2) If $\text{depth}_M R' \geq d$, then $\bar{g}_s(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r)$ and $\bar{e}_2(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r)r$, where $M = (t^{-1}, It)R$.

3. $\bar{e}_2(I)$.

In this section, we shall give the proof of (2) in Theorem 1. By Corollary 6, the assertion remained to be proved is the following

PROPOSITION 9. $\bar{e}_2(I) = \bar{g}_s(I)$ if and only if $\overline{I^{n+2}} = I^n \bar{I}^2$ for every $n \geq 0$.

If part of the above proposition clearly holds by (*). Only if part follows from the following proposition.

PROPOSITION 10. Assume that $d \geq 3$, and choose a system of generators $x_1, \dots, x_d$ satisfying the conditions of Theorem 1, and put $z = \sum_i x_i T_i$, $C = A(T)/zA(T)$ and $J = IC$. If $\overline{J^{n+2}} = J^n \bar{J}^2$ for every $n \geq 0$, then $\overline{I^{n+2}} = I^n \bar{I}^2$ for every $n \geq 0$.

PROPOSITION 11. Let r be either 1 or 2. Then the following assertions are equivalent.

- (1) $\overline{I^{n+r}} = I^n \bar{I}^r$ for every $n \geq 0$.
- (2) $[H_N^i(R')]_j = 0$ for $i + j \geq r + 1$.
- (3) $[H_N^i(R')]_j = 0$ for $i + j = r + 1$.

Proof of Proposition 10: Assume that $d \geq 3$, and choose a system of generators $x_1, \dots, x_d$ satisfying the conditions of Theorem 3, and put $z = \sum_i x_i T_i$, $C = A(T)/zA(T)$, $J = IC$, as in Theorem 3. $S = A(T) \otimes R (= R(A(T), IA(T)))$, $S' = A(T) \otimes R' (= R'(A(T), IA(T)))$, $N' = ItS$, $M' = (t^{-1}, It)S'$, $F = S/zS (= R(C, J))$ and $F' = \sum \bar{I}^n C t^n (= R'(C, J))$. Suppose that $[H_{N'}^i(F')]_j = 0$ for $i + j = 3$. We shall prove $[H_N^i(R')]_j = 0$ for $i + j = 3$. Since $A \rightarrow A(T)$ is faithfully flat, $H_N^i(R') = H_{N'}^i(S')$; thus it is sufficient to prove that $[H_{N'}^i(S')]_j = 0$ for $i + j = 3$. We first prove that $[H_{N'}^{i-1}(S'/ztS')]_{j+1} = 0$ for $3 \leq i \leq d$ and $j \geq 3 - i$. If this is proved, then $([H_N^i(S')]_j = [H_{N'}^i(S'(-1))]_{j+1} \xrightarrow{zt} [H_{N'}^i(S')]_{j+1})$ is injective; since every element of $H_{N'}^i(S')$ is annihilated by some power of zt , $[H_N^i(S')]_j$ must be 0. Since $\dim F'/(S'/ztS') = 0$, we have $H_{N'}^1(S'/ztS') = F'/(S'/ztS')$ and $H_{N'}^i(S'/ztS') = H_{N'}^i(F')$ for $i \geq 2$. Therefore, for $3 \leq i \leq d$, $H_{N'}^{i-1}(S'/ztS')_{j+1} = H_{N'}^{i-1}(F')_{j+1} = 0$, since F' is a Cohen-Macaulay ring. We next prove that $H_{N'}^2(S')_1 = 0$. It is known that $H_{N'}^2(S')_0 = 0$; therefore

$0 = H_{N'}^2(S')_0 \xrightarrow{zt} H_{N'}^2(S')_1 \longrightarrow H_{N'}^2(S'/ztS')_1 = 0$; hence $H_{N'}^2(S')_1 = 0$. It is also known that $H_{M'}^0(S') = H_{M'}^1(S') = 0$ and $H_{M'}^i(S') = H_{N'}^i(S')_i = 0, 1$. Therefore $H_{N'}^i(S') = 0$ for $i = 0, 1$.

4. $\bar{e}_3(I)$.

In general, it is known that

$$(-1)^d \bar{e}_d(I) = F(0) = \sum_i (-1)^i \text{length}_A([H_N^i(R')]_0).$$

If $d = 3$, then $\bar{e}_3(I) = \text{length}_A([H_N^3(R')]_0) = \text{length}_A(H^2(X, O_X))$, because $[H_N^2(R')]_0 = [H_M^2(R')]_0 = 0$. Therefore by Proposition 5,

$$(***) \bar{e}_3(I) \geq 0.$$

It is then natural to ask when $\bar{e}_3 = 0$. It follows from [4, Appendix 2] that there exists a canonical graded homomorphism $\alpha : H_N^d(R') \longrightarrow H_m^d(A)[t, t^{-1}]$. We denote by α_j the graded part of degree j of α . Then we have

LEMMA 12. $\alpha_j = 0$ (i.e., $[H_M^d(R')]_j = [H_N^d(R')]_j$) if and only if $\overline{I^{n+d-1+j}} \subseteq I^n$ for every $n \geq 0$.

Proof of Theorem 2. (1): By Proposition 5, we may assume that $d = \dim A = 3$. Then the assertion follows from (***) and Lemma 12. (2): This follows from the following proposition.

PROPOSITION 13. Assume that A is a Gorenstein local ring and $\bar{I}^2 = I\bar{I}$. If $\overline{I^{n+2}}$ and $m\overline{I^{n+1}}$ are contained in I^n for every $n \geq 0$, then $\text{length}_A(\bar{I}^2/I\bar{I}) = 1$ and $\overline{I^{n+2}} = I^n \bar{I}^2$ for every $n \geq 0$.

PROOF. Since $m\overline{I^{n+1}} \supseteq I^n$, we have $(I^n : m)/I^n \supseteq (\overline{I^{n+1}} + I^n)/I^n = \overline{I^{n+1}}/I^n \bar{I}$, and hence $\text{length}_A(\overline{I^{n+1}}/I^n \bar{I}) \leq \text{length}_A((I^n : m)/I^n) = \binom{n-1+d-1}{d-1}$, because A is Gorenstein. Therefore

$$\begin{aligned} & \text{length}_A(A/\overline{I^{n+1}}) \\ &= \text{length}_A(A/I^{n+1}) - \text{length}_A((I^n \bar{I}/I^{n+1}) - \text{length}_A(\overline{I^{n+1}}/I^n \bar{I}) \\ &\geq \text{length}_A(A/I) \binom{n+d}{d} - \text{length}_A(\bar{I}/I) \binom{n+d-1}{d-1} - \binom{n-1+d-1}{d-1} \\ &= \text{length}_A(A/I) \binom{n+d}{d} - (\text{length}_A(\bar{I}/I) + 1) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}. \end{aligned}$$

(We have already proved in [4, Proposition 10] that $\text{length}_A((I^n \bar{I}/I^{n+1}) = \text{length}_A(\bar{I}/I) \binom{n+d-1}{d-1}$.) Thus by (*), $\text{length}_A(\bar{I}^2/I\bar{I}) = 1$ and $\text{length}_A(A/\overline{I^{n+1}}) = \text{length}_A(A/I) \binom{n+d}{d} - (\text{length}_A(\bar{I}/I) + 1) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}$, and in particular, $\overline{I^{n+1}} = I^{n-1} \bar{I}^2$.

It is natural to ask whether the assertion (2) in Theorem 2 is true for any parameter ideals.

CONJECTURE: Assume that A is Gorenstein and $d = \dim A \geq 3$. Then $\bar{e}_3(I) = 0$ if and only if $\overline{I^{n+2}} = I^n \bar{I}^2$ for every $n \geq 0$.

Assume that $d = 3$ and $\bar{e}_3(I) = 0$: By Proposition 11, if $[H_M^2(R')]_1 (= [H_N^2(R')]_1) = 0$, then $\overline{I^{n+2}} = I^n \bar{I}^2$ for every $n \geq 0$.

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