

Completions of generalized inverse $*$ -semigroups¹

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Abstract

It is well known that every inverse semigroup can be embedded both in a (join) complete inverse semigroup and a meet complete inverse semigroup (see [8]). The purpose of this paper is to obtain its generalization for generalized inverse $*$ -semigroups. We succeed the former, that is, each generalized inverse $*$ -semigroup S is embedded in a $*$ -complete, infinitely distributive generalized inverse $*$ -semigroup. Unfortunately, we can not answer for the later. However, we have that S is embedded in $K(S)$ consisting of all cosets of S .

1 Preliminaries

A semigroup S with a unary operation $*$: $S \rightarrow S$ is called a *regular $*$ -semigroup* if it satisfies

$$(i) (x^*)^* = x; \quad (ii) (xy)^* = y^*x^*; \quad (iii) xx^*x = x.$$

Let S be a regular $*$ -semigroup. An idempotent e in S is called a *projection* if $e^* = e$. For a subset A of S , denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular $*$ -semigroup. If $E(S) = P(S)$, S is called an *inverse semigroup*. If eSe is an inverse subsemigroup of S for any $e \in E(S)$, it is called a *locally inverse $*$ -semigroup*. If $E(S)$ forms a subsemigroup of S , it is called an *orthodox $*$ -semigroup*. If S is orthodox and locally inverse, it is called a *generalized inverse $*$ -semigroup*. It is well known that S is a generalized inverse $*$ -semigroup if and only if $E(S)$ satisfies the identity $xyzw = xzyw$.

Result 1.1. [4] *Let S be a regular $*$ -semigroup. Then we have*

- (1) $E(S) = P(S)^2$.
- (2) For any $a \in S$ and $e \in P(S)$, $a^*ea \in P(S)$.

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(3) Each $\mathcal{L}$ -class and $\mathcal{R}$ -class contains one and only one projection.

Let S be a regular $*$ -semigroup. Define a relation $\leq$ on S as follows:

$$a \leq b \iff a = eb = bf \text{ for some } e, f \in P(S).$$

Result 1.2. [5] *Let a and b be elements of a regular $*$ -semigroup S . Then the following statements are equivalent:*

- (1) $a \leq b$,
- (2) $aa^* = ba^*$ and $a^*a = b^*a$,
- (3) $aa^* = ab^*$ and $a^*a = a^*b$,
- (4) $a = aa^*b = ba^*a$.

The relation $\leq$, defined above, is a partial order on S which preserves the $*$ -operation. We call $\leq$ the *natural order* on S . It is well known that S is a locally inverse $*$ -semigroup if and only if $\leq$ is compatible.

Proposition 1.3. *Let S be a regular $*$ -semigroup. Then S is a generalized inverse $*$ -semigroup if and only if $xey \leq xy$ for any $x, y \in S$ and $e \in P(S)$*

Let $(P, \leq)$ be a partial order set. A subset Q of P is said to be an *order ideal* if $x \leq y \in Q$ implies $x \in Q$. For $x \in P$, $[x] = \{y \in P : y \leq x\}$ is the smallest order ideal of P containing x , which is called the *principal order ideal* of P containing x .

Proposition 1.4. *Let S be a regular $*$ -semigroup. Then $P(S)$ is an order ideal of S . Moreover, if S is orthodox, then $E(S)$ is an order ideal.*

Let S and T be regular $*$ -semigroups. A mapping $\theta : S \rightarrow T$ is called a $*$ -homomorphism if, for any $a, b \in S$,

$$\theta(ab) = \theta(a)\theta(b) \text{ and } \theta(a^*) = \theta(a)^*.$$

The following properties are well known.

Result 1.5. *Let $\theta : S \rightarrow T$ be a $*$ -homomorphism between regular $*$ -semigroups.*

- (1) *If $e \in E(S)$, then $\theta(e) \in E(T)$.*
- (2) *If $e \in P(S)$, then $\theta(e) \in P(T)$.*
- (3) *If U is a regular $*$ -subsemigroup of S , then $\theta(U)$ is a regular $*$ -subsemigroup of T .*
- (4) *If V is a regular $*$ -subsemigroup of T , then $\theta^{-1}(V)$ is a regular $*$ -subsemigroup of S .*
- (5) *The mapping θ is order-preserving.*

The notation and terminology are those of [7] and [8], unless otherwise stated.

2 *-Compatibility relations and infinitely distributive semigroups

Let S be a regular $*$ -semigroup. For any $s, t \in S$, the left $*$ -compatibility relation is defined by

$$s \sim_l^* t \Leftrightarrow st^* \in P(S),$$

the right $*$ -compatibility relation is defined by

$$s \sim_r^* t \Leftrightarrow s^*t \in P(S),$$

and the compatibility relation is defined by

$$s \sim^* t \Leftrightarrow s^*t, st^* \in P(S).$$

A subset A of S is said to be $*$ -compatible if $a \sim^* b$ for all $a, b \in A$.

Lemma 2.1. *Let S be a regular $*$ -semigroup and let $s, t \in S$. Then $s \sim^* t$ if and only if the greatest lower bound $s \wedge t$ of s and t exists and*

$$s \wedge t = st^*t = ts^*t = ts^*s = st^*s = ss^*t = tt^*s.$$

Lemma 2.2. *Let S be a locally inverse $*$ -semigroup, and let $s, t, u, v \in S$. Then*

- (1) $s \leq t, u \leq v$ and $t \sim^* v$ implies that $s \sim^* u$.
- (2) $[s]$ is a $*$ -compatible order ideal of S .

If S is a generalized inverse $$ -semigroup, then*

- (3) $s \sim^* t$ and $u \sim^* v$ implies that $su \sim^* tv$.

Lemma 2.3. *Let S be a locally inverse $*$ -semigroup and let A and B be non-empty subsets of projections and idempotents, respectively. Then we have the following:*

- (1) *If $\bigwedge A$ exists, then it is a projection.*
- (2) *If $\bigvee A$ exists, it is a projection.*

Moreover, let S be a generalized inverse $$ -semigroup. Then*

- (3) *If $\bigwedge B$ exists, it is an idempotent.*
- (4) *If $\bigvee B$ exists, it is an idempotent.*

Lemma 2.4. *Let S be a locally inverse $*$ -semigroup and let A be a non-empty subset of S such that $\bigvee A$ exists. Then any two elements of A are $*$ -compatible.*

A regular $*$ -semigroup is said to be *left infinitely distributive* if, whenever A is a non-empty subset of S for which $\bigvee A$ exists, then $\bigvee sA$ exists for any element $s \in S$ and $s(\bigvee A) = \bigvee sA$. *Right infinitely distributive* is defined analogously. Also a semigroup which is both left and right infinitely distributive is called *infinitely distributive*. We say that a regular $*$ -semigroup is $*$ -complete if every its non-empty $*$ -compatible subset has a join.

Proposition 2.5. *Let S be a locally inverse $*$ -semigroup and $A = \{a_i : i \in I\}$ a non-empty subset of S .*

(1) *If $\bigvee a_i$ exists then $\bigvee a_i^* a_i$ exists and $(\bigvee a_i)^* (\bigvee a_i) = \bigvee a_i^* a_i$.*

(2) *If $\bigvee a_i$ exists then $\bigvee a_i a_i^*$ exists and $(\bigvee a_i) (\bigvee a_i)^* = \bigvee a_i a_i^*$.*

Theorem 2.6. *Let S be a infinitely distributive locally inverse $*$ -semigroup. If A and B are non-empty subsets of S such that $\bigvee A$, $\bigvee B$ and $\bigvee AB$ exist, then $\bigvee AB = (\bigvee A)(\bigvee B)$.*

3 Join completions

Let A be a subset of a regular $*$ -semigroup S . It is said to be $*$ -permissible if it is a $*$ -compatible order ideal of S . The set of all $*$ -permissible subsets of S is denoted by $C^*(S)$.

Lemma 3.1. *Let S be a regular $*$ -semigroup and A its $*$ -permissible subset. Then*

$$A^* A = \{a^* a : a \in A\} \quad \text{and} \quad A A^* = \{a a^* : a \in A\}$$

are both order ideals.

Lemma 3.2. *Let S be a regular $*$ -semigroup. If A is a $*$ -permissible subset of S which satisfies $AA = A$, then it is a subset of $E(S)$. Moreover, A satisfies $A^* = A$, it is a subset of $P(S)$.*

Now, we have the main theorem.

Theorem 3.3. *Let S be a generalized inverse $*$ -semigroup. Then $C^*(S)$ is a $*$ -complete, infinitely distributive generalized inverse $*$ -semigroup. And the mapping $\iota : S \rightarrow C^*(S)$ ($s \mapsto [s]$) is an injective $*$ -homomorphism. Moreover, every element of $C^*(S)$ is a join of non-empty subset of $\iota(S)$.*

Theorem 3.4. *If $\theta : S \rightarrow T$ be a $*$ -homomorphism to a $*$ -complete, infinitely distributive generalized inverse $*$ -semigroup, then there exists a unique join-preserving $*$ -homomorphism $\phi : C^*(S) \rightarrow T$ such that $\phi \iota = \theta$.*

Now we can obtain that the category of $*$ -complete, infinitely distributive generalized inverse $*$ -semigroups together with join-preserving $*$ -homomorphisms is a reflective subcategory of the category of generalized inverse $*$ -semigroups and $*$ -homomorphism.

Theorem 3.5. *The function $S \mapsto C^*(S)$ is the object part of a functor from the category of generalized inverse $*$ -semigroups and $*$ -homomorphisms to the category of $*$ -complete, infinitely distributive generalized inverse $*$ -semigroups and join-preserving $*$ -homomorphisms.*

4 Cosets of generalized inverse $*$ -semigroups

Let S be a regular $*$ -semigroup and X its subset. We call

$$[X]^\uparrow = \{s \in S : x \leq s \text{ for some } x \in X\}$$

the closure of X in S . If $X = \{x\}$ consists a single element, we denote it by $[x]^\uparrow$, which is called the principal closure containing x . A subset is said to be closed if it is equal to its closure.

Let S be a generalized inverse $*$ -semigroup. A non-empty subset H of S is called a coset if HH^*H , and the set of all cosets of S is denoted by $K(S)$. We first remark to justify the use of the term coset.

Proposition 4.1. *Let A be a non-empty subset of a group G . Then $A = AA^*A (= aa^{-1}A)$ if and only if A is a coset of a subgroup of G .*

A further justification for the term coset comes from the theory of representation of generalized inverse $*$ -semigroups.

Proposition 4.2. *Let $\theta : S \rightarrow \mathcal{GI}(X; \Omega)$ be a representation of a generalized inverse $*$ -semigroup S . Let $x, y \in X$ and put $H_{x,y} = \{s \in S : \theta(s)(x) = y\}$. Then if $H_{x,y}$ is non-empty, it is a coset.*

We give another characterization of cosets in the sense of Dubreil [1]. For non-empty subsets A and B of a semigroup S , define

$$A \cdot B = \{s \in S : Bs \subseteq A\} \text{ and } A \cdot .B = \{s \in S : sB \subseteq A\}.$$

If $B = \{b\}$, we denote each by $A \cdot B$ and $A \cdot .B$.

Lemma 4.3. *Let S be a generalized inverse $*$ -semigroup. If A is a coset and $A \cdot .B [A \cdot B]$ is a non-empty subset of S , then $A \cdot B [A \cdot .B]$ is a coset.*

Theorem 4.4. *Let H be a non-empty subset of a generalized inverse $*$ -semigroup S . Then the following statements are equivalent:*

- (1) H is a coset,
- (2) $H \cdot .s \cap H \cdot .t \neq \emptyset \Rightarrow H \cdot .s = H \cdot .t$ for any $s, t \in S^1$,
- (3) $H \cdot s \cap H \cdot t \neq \emptyset \Rightarrow H \cdot s = H \cdot t$ for any $s, t \in S^1$,
- (4) $xu, vu, vy \in H \Rightarrow xy \in H$ for any $x, y \in S$ and $u, v \in S^1$.

Let S be a generalized inverse $*$ -semigroup. We now introduce a new binary operation on $K(S)$ and it becomes a generalized inverse $*$ -semigroup with respect to the operation. It is clear that the intersection of any non-empty set of cosets is either empty or a coset. For a non-empty subset X of S , we define $j(X)$ to be the intersection of all cosets containing X , that is, the smallest coset containing X . Define a binary operation $\otimes$ and a unary operation $*$ on S as follows:

$$A \otimes B = j(AB) \text{ and } (A)^* = A^*.$$

Theorem 4.5. *Let S be a generalized inverse $*$ -semigroup. Then $K(S)(\otimes, *)$ is a generalized inverse $*$ -semigroup.*

Proposition 4.6. *Let S be a generalized inverse $*$ -semigroup and $s \in S$. Then $[s]^\uparrow$ is a coset.*

Proposition 4.7. *Let S be a generalized inverse $*$ -semigroup. Then, for any $A, B \in K(S)$,*

$$A \leq B \Rightarrow A \supseteq B.$$

Now, we can immediately obtain the following theorem.

Theorem 4.8. *Let S be a generalized inverse $*$ -semigroup. Then the mapping $\iota : S \rightarrow K(S)$ ($s \mapsto [s]^\uparrow$) is an injective $*$ -homomorphism, and each element of $K(S)$ is the union of a non-empty subset of $\iota(S)$.*

Remark. We showed that, for $A, B \in K(S)$, $A \leq B$ implies $A \supseteq B$ in Proposition 4.7. However, we do not know where the converse is true or not. If it is true, we can change "the union" to "the meet" in Theorem 4.8.

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