

A maximal forcing axiom compatible with weak club guessing

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Abstract

We show there is no maximal forcing axiom compatible with tail club guessing. On the other hand, we may formulate a maximal forcing axiom compatible with a weak club guessing.

Introduction

We formulate a forcing axiom compatible with a tail club guessing in [M]. This note is a continuation to [M]. We show a maximal forcing axiom compatible with tail club guessing does not hold. On the other hand, we may force a maximal forcing axiom compatible with a weak club guessing. Namely, if a ladder system $\langle C_\delta \mid \delta \in A \rangle$ is weak club guessing and a supercompact cardinal exists, then there is a model of set theory where

- (1) $\langle C_\delta \mid \delta \in A \rangle$ remains weak club guessing. Namely, for any club D of ω_1 , there exists $\delta \in A$ such that $D \cap C_\delta$ is infinite.
- (2) Let P be any preorder such that P preserves every stationary subset of ω_1 and that for any $B \subseteq A$ such that the ladder system $\langle C_\delta \mid \delta \in B \rangle$ is weak club guessing, P also preserves the ladder system $\langle C_\delta \mid \delta \in B \rangle$ to be weak club guessing. Then every system $\langle D_i \mid i < \omega_1 \rangle$ of dense subsets of P has a filter which hits every D_i .

§1. No maximal forcing axioms are compatible with TCG

We set our notation.

1.1 Definition A sequence $\langle C_\delta \mid \delta \in A \rangle$ is a *ladder system*, if

- $A \subseteq \{\delta \mid \delta < \omega_1, \delta \text{ is a limit ordinal}\}$.
- For every $\delta \in A$, C_δ is a cofinal subset of δ and is of order type ω .

A ladder system $\langle C_\delta \mid \delta \in A \rangle$ is *tail club guessing*, if for any club $D \subseteq \omega_1$, there exists $\delta \in A$ such that $C_\delta \setminus D$ is finite. A ladder system $\langle C_\delta \mid \delta \in A \rangle$ is *weak club guessing*, if for any club $D \subseteq \omega_1$, there exists $\delta \in A$ such that $C_\delta \cap D$ is infinite. Hence if $\langle C_\delta \mid \delta \in A \rangle$ is tail club guessing, then it is weak club guessing.

Fix a ladder system $\langle C_\delta \mid \delta \in A \rangle$. We write for small sets and positive sets as follows;

- $(TCG) = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X \rangle \text{ fails to be tail club guessing}\}$.
- $(TCG)^+ = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X \rangle \text{ is tail club guessing}\}$.

Similarly,

- $(WCG) = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X \rangle \text{ fails to be weak club guessing}\}$.
- $(WCG)^+ = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X \rangle \text{ is weak club guessing}\}$.

We know of a forcing axiom which is compatible with tail club guessing.

Theorem. ([M]) Let $\langle C_\delta \mid \delta \in A \rangle$ be tail club guessing. Then we may force the following, assuming that a supercompact cardinal exists.

- (1) $\langle C_\delta \mid \delta \in A \rangle$ remains tail club guessing.
- (2) Forcing axiom⁺ holds for the class of partially ordered sets P which are semiproper and $\langle C_\delta \mid \delta \in A \rangle$ - ω -semiproper,

□

On the other hand,

1.2 Proposition. Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system. Let $n_* < \omega$. Then there exists a partially ordered set P and a P -name $\dot{D}$ such that

- (1) P is proper and $(TCG)^+$ -preserving.
- (2) $\Vdash_P \text{"}\dot{D} \text{ is a club in } \omega_1 \text{ such that for all } \delta \in A, |C_\delta \setminus \dot{D}| \geq n_*\text{"}$.

□

We make use of this P in two ways. First, we observe TCG-sequences may get killed at limit stages of iterated forcing.

1.3 Corollary. Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system. Then there exists an iterated forcing $\langle P_n \mid n < \omega \rangle$ such that

- If $\langle C_\delta \mid \delta \in A \rangle$ is tail club guessing, then for all $n < \omega$, $\Vdash_{P_n} \text{"}\langle C_\delta \mid \delta \in A \rangle \text{ remains tail club guessing"}$.
- If P_ω is any limit of the P_n 's, then $\Vdash_{P_\omega} \text{"}\langle C_\delta \mid \delta \in A \rangle \text{ must fail to be tail club guessing"}$.

Second, we put above in terms of forcing axiom. Suppose $\langle C_\delta \mid \delta \in A \rangle$ is tail club guessing. Then no maximal forcing axioms hold for the class of partially ordered sets P which preserve all stationary subsets of ω_1 and all elements of $(TCG)^+$ (i.e. for any $X \subseteq \omega_1$, if $\langle C_\delta \mid \delta \in A \cap X \rangle$ is tail club guessing, then it remains so in the generic extensions of P).

1.4 Corollary. Let $\langle C_\delta \mid \delta \in A \rangle$ be tail club guessing. Let forcing axiom hold for the class of partially ordered sets P such that P are proper and that for any $B \subseteq A$ such that $\langle C_\delta \mid \delta \in B \rangle$ is tail club guessing, then $\Vdash_P \text{"}\langle C_\delta \mid \delta \in B \rangle \text{ remains to be tail club guessing"}$. Then we have a contradiction.

□

Proof of proposition 1.2. Let $p \in P$, if $p = (\alpha^p, D^p)$ such that

- (1) $\alpha^p < \omega_1$.
- (2) $D^p \subseteq \alpha^p + 1$, $\alpha^p \in D^p$ and D^p is closed.
- (3) For all $\delta \in A$ with $\delta \leq \alpha^p$, $|C_\delta \setminus D^p| \geq n_*$.

For $p, q \in P$, let $q \leq p$, if

- (4) $\alpha^p \leq \alpha^q$.
- (5) $D^p = D^q \cap (\alpha^p + 1)$.

Claim. (Dense) For any $p \in P$ and any η with $\alpha^p < \eta < \omega_1$, there exists $q \leq p$ such that $\alpha^q = \eta$.

Proof. Let $\alpha^q = \eta$ and $D^q = D^p \cup \{\eta\}$. Then this $q = (\alpha^q, D^q)$ works.

□

Claim. P is proper and σ -Baire.

Proof. Let θ be a sufficiently large regular cardinal. Let N be a countable elementary substructure of H_θ with $P \in N$. Let $p \in N \cap P$. We want $q \leq p$ such that q is (P, N) -generic. Let $\delta = N \cap \omega_1$. We construct a (P, N) -generic sequence $\langle p_n \mid n < \omega \rangle$ such that $p_0 \leq p$ and that $|(\alpha^{p_0} \cap C_\delta) \setminus D^{p_0}| \geq n_*$. Let $\alpha^q = \delta$ and $D^q = \bigcup \{D^{p_n} \mid n < \omega\} \cup \{\delta\}$. Then this $q = (\alpha^q, D^q)$ works.

□

Claim. If $X \subseteq \omega_1$ such that $\langle C_\delta \mid \delta \in A \cap X \rangle$ is tail club guessing, then $\Vdash_P \text{"}\langle C_\delta \mid \delta \in A \cap X \rangle \text{ remains to be tail club guessing"}$.

Proof. Suppose $p \Vdash_P \dot{C}$ is a club in ω_1 . Want $q \leq p$ and $\delta \in A \cap X$ such that $q \Vdash_P "C_\delta \setminus \dot{C} \text{ is finite}"$. To this end, take an $\in$ -chain $\langle N_i \mid i < \omega_1 \rangle$ in H_θ , where θ is sufficiently large. Since $\langle C_\delta \mid \delta \in A \cap X \rangle$ is tail club guessing, there exists $\delta \in A \cap X$ such that $C_\delta \setminus \{N_i \cap \omega_1 \mid i < \omega_1\}$ is finite. By renaming, we have an $\in$ -chain $\langle N_n \mid n < \omega \rangle$ in H_θ such that $\{N_n \cap \omega_1 \mid n < \omega\}$ is an end-segment of C_δ . We may assume that $P, p, \dot{C} \in N_0$ and $|C_\delta \cap (\alpha^p, N_0 \cap \omega_1)| \geq n_*$. We construct a descending sequence $\langle q_n \mid n < \omega \rangle$ of conditions such that

- $q_0 \leq p$.
- $|(\alpha^{q_0} \cap C_\delta) \setminus D^{q_0}| \geq n_*$.
- $q_n \in N_{n+1} \cap P$ is (P, N_n) -generic.

Let $\alpha^q = \delta$ and $D^q = \bigcup \{D^{q_n} \mid n < \omega\} \cup \{\delta\}$. Then $q \Vdash_P "N_n \cap \omega_1 \in \dot{C}"$ for all $n < \omega$. Hence this q works.

Claim. Let G be P -generic over V . Let $\dot{D} = \bigcup \{D^p \mid p \in G\}$. Then $\dot{D}$ is a club in ω_1 and for all $\delta \in A$, $|C_\delta \setminus \dot{D}| \geq n_*$. □

Proof of corollary 1.3. Iteratively force clubs D_n in ω_1 so that for all $\delta \in A$ and all $n < \omega$, $|C_\delta \setminus D_n| \geq n$. Then let $D = \bigcap \{D_n \mid n < \omega\}$. If ω_1 gets preserved, then D is a club in ω_1 such that for all $\delta \in A$, $C_\delta \setminus D$ is infinite. Hence $\langle C_\delta \mid \delta \in A \rangle$ fails to be tail club guessing. If ω_1 gets collapsed, then this entails the same conclusion.

Proof of corollary 1.4 is the same. Argue in V and get a sequence $\langle D_n \mid n < \omega \rangle$ of clubs in ω_1 . □

§2. A maximal forcing axiom is compatible with WCG

We have seen that there is no maximal forcing axiom compatible with tail club guessing (TCG). But a weak club guessing (WCG) admits maximal one.

2.1 Definition. Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system. Let $\mathcal{F}$ denote the set of all cofinal subsequences of C_δ (viewed as sequences of length ω) for all $\delta \in A$. Let $\text{Seq}^\omega(\omega_1)$ denote the set of all sequences $\langle a_n \mid n < \omega \rangle$ such that each a_n is a countable subset of ω_1 . Hence we have $\mathcal{F} \subseteq \text{Seq}^\omega(\omega_1)$. Let P be a preorder, we say P is $\mathcal{F}$ -limsup-semiproper, if for all sufficiently large regular cardinals θ and all $\in$ -chains $\langle N_n \mid n < \omega \rangle$ in H_θ with $P, \langle C_\delta \mid \delta \in A \rangle \in N_0$, if $\langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}$, then for any $p \in N_0 \cap P$, there exists $q \leq p$ such that for infinitely many $n < \omega$, q is (P, N_n) -semi-generic.

Similarly, we say P is $\mathcal{F}$ -liminf-semiproper, if q is (P, N_n) -semi-generic for all but finitely many $n < \omega$.

Lastly, we say P is $\mathcal{F}$ -generic-limsup-semiproper, if $q \Vdash_P "N_n[G] \cap \omega_1^V = N_n \cap \omega_1^V"$ for infinitely many $n < \omega$.

Hence we are looking at the set $\{n < \omega \mid N[G] \cap \omega_1^V = N \cap \omega_1^V\}$ in $V[G]$ which might be infinite and may or may not be in V .

For the notion of ω -stationary sets $S \subseteq \text{Seq}^\omega(K) = \{\langle a_n \mid n < \omega \rangle \mid a_n \in [K]^\omega \text{ for all } n < \omega\}$, may see [M]. They are analogously formulated as the stationary sets in $[K]^\omega$.

2.2 Proposition. Let P be a preorder.

- If P is ω_1 -closed, then P is ω -semiproper.
- If P is ω -semiproper, then P is $\mathcal{F}$ -liminf-semiproper.
- If P is $\mathcal{F}$ -liminf-semiproper, then P is $\mathcal{F}$ -limsup-semiproper.
- If P is $\mathcal{F}$ -limsup-semiproper, then P is $\mathcal{F}$ -generic-limsup-semiproper.
- If $\mathcal{F}$ is ω -stationary and P is $\mathcal{F}$ -generic-limsup-semiproper, then P preserves ω_1 .

2.3 Definition. Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system. We formulate (*proper*, $\langle C_\delta \mid \delta \in A \rangle$ -*limsup-semiproper*, *full*) -*Reflection Principle* (abusively, $\mathcal{F}$ -*RP*) as follows; Given any $\langle K, S, \theta, a \rangle$ such that

- K is a set with $K \supseteq \omega_1$.
- $S \subseteq \text{Seq}^\omega(K)$.
- θ is a regular cardinal with $K \in H_{|\text{TC}(K)|^+} \in H_{(2^{|\text{TC}(K)|})^+} \in H_\theta$.
- $a \in H_\theta$.

There exists $\langle D, \langle N_i \mid i < \omega_1 \rangle \rangle$ such that

- D is a club in ω_1 .
- $\langle N_i \mid i < \omega_1 \rangle$ is an $\in$ -chain in H_θ with $a \in N_0$.
- For any $\delta \in Y^*(D) = \{\delta \in A \mid |C_\delta \cap D| = \omega\}$, let $\langle C_\delta(k_\delta(m)) \mid m < \omega \rangle$ enumerate $\{C_\delta(k) \mid C_\delta(k) \in D\}$. Then there exists $n_\delta < \omega$ such that we have either (1) or (2).

- (1) $\langle N_{C_\delta(k_\delta(m))} \cap K \mid n_\delta \leq m < \omega \rangle \in S$.
- (2) For any strictly increasing sequence $\langle m_l \mid l < \omega \rangle$ of natural numbers with $n_\delta \leq m_0$ and for any $\in$ -chain $\langle M_l \mid l < \omega \rangle$ with $\langle M_l \mid l < \omega \rangle \supseteq_{\omega_1} \langle N_{C_\delta(k_\delta(m_l))} \mid l < \omega \rangle$, we have $\langle M_l \cap K \mid l < \omega \rangle \notin S$.

We might call $i_\delta = C_\delta(k_\delta(n_\delta))$ a *critical point* of C_δ with respect to D for each $\delta \in Y^*(D)$. Hence we are looking at $\langle N_i \mid i_\delta \leq i \in C_\delta \cap D \rangle$.

2.4 Theorem. Let $\mathcal{F}^V \subseteq \text{Seq}^\omega(\omega_1)$ be defined by $\langle C_\delta \mid \delta \in A \rangle$. Then $\mathcal{F}^{V[G_\alpha]}$ -generic-limsup-semiproper combined with semiproper iterates under the simple iteration. (The $\mathcal{F}^{V[G_\alpha]}$ are uniformly defined from the ladder system $\langle C_\delta \mid \delta \in A \rangle$ in each intermediate universe $V[G_\alpha]$. The exact value of $\mathcal{F}^{V[G_\alpha]}$ increasingly changes as α gets bigger.)

□

2.5 Corollary. Let $\langle C_\delta \mid \delta \in A \rangle$ be weak club guessing so that $\mathcal{F}$ defined from $\langle C_\delta \mid \delta \in A \rangle$ is ω -stationary. Let us recall $\mathcal{F} = \{\langle C_\delta(k(m)) \mid m < \omega \rangle \mid \delta \in A, \langle k(m) \mid m < \omega \rangle \text{ is a sequence of strictly increasing natural numbers}\}$. We may force the following, if there exists a supercompact cardinal.

- (1) If $\langle C_\delta \mid \delta \in A \rangle$ is weak club guessing in the ground model, then it remains to be so in the extensions.
- (2) Forcing axiom⁺ holds for the class of partially ordered sets P such that P is semiproper and that P is $\mathcal{F}$ -generic-limsup-semiproper.

2.6 Lemma. Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system. Let $\mathcal{F} \subseteq \text{Seq}^\omega(\omega_1)$ be defined from the system. Let $\mathcal{F}$ -Reflection Principle hold. Let us consider $(WCG)^+$ with respect to the system. Let P be a preorder. Then the following are equivalent on P .

- (1) P is $(WCG)^+$ -preserving.
- (2) P is $\mathcal{F}$ -generic-limsup-semiproper.

□

The $\mathcal{F}$ -RP gets forced by a little better notion of forcing than semiproper + $\mathcal{F}$ -generic-limsup-semiproper partially ordered set.

2.7 Lemma. Let $\mathcal{F} \subseteq \text{Seq}^\omega(\omega_1)$ be as above. Let forcing axiom hold for the class of preordered sets P such that P is proper and that P is $\mathcal{F}$ -limsup-semiproper. Then $\mathcal{F}$ -Reflection Principle holds.

□

2.8 Corollary. The following is consistent, if there exists a supercompact cardinal.

- (1) $\langle C_\delta \mid \delta \in A \rangle$ is weak club guessing.
- (2) The forcing axiom⁺ holds for the class of preordered sets P such that P preserves every stationary subset of ω_1 and that P preserves every member of $(WCG)^+$ with respect to $\langle C_\delta \mid \delta \in A \rangle$.

Proof of lemma 2.6. (2) implies (1): No use of $\mathcal{F}$ -RP made in this direction. Let $X \subseteq \omega_1$ such that $\langle C_\delta \mid \delta \in X \cap A \rangle$ is weak club guessing. Suppose $p \Vdash_P \text{"}\dot{C} \subseteq \omega_1 \text{ is a club"}$. Want $q \leq p$ and $\delta \in X \cap A$ such that $q \Vdash_P \text{"}\dot{C}_\delta \cap \dot{C} = \omega\text{"}$. To this end, let θ be a sufficiently large regular cardinal. Since $\langle C_\delta \mid \delta \in X \cap A \rangle$ is weak club guessing, we may take an $\in$ -chain $\langle N_n \mid n < \omega \rangle$ in H_θ such that $\{N_n \cap \omega_1 \mid n < \omega\}$ is a cofinal subset of C_δ , where $\delta = N_\omega \cap \omega_1$ and $N_\omega = \bigcup \{N_n \mid n < \omega\}$. We may assume $P, \langle C_\delta \mid \delta \in A \rangle, p, \dot{C} \in N_0$. Since P is $\mathcal{F}$ -generic-limsup-semiproper and $\langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}$, we have q such that $q \leq p$ and $q \Vdash_P \text{"}N_n[G] \cap \omega_1 = N_n \cap \omega_1 \text{ for infinitely many } n < \omega\text{"}$. Hence $q \Vdash_P \text{"}N_n[G] \cap \omega_1 = N_n \cap \omega_1 \in \dot{C} \cap C_\delta \text{ for infinitely many } n < \omega\text{"}$ and so $q \Vdash_P \text{"}\dot{C}_\delta \cap \dot{C} = \omega\text{"}$.

(1) implies (2): Suppose P is $(WCG)^+$ -preserving. Want to show P is $\mathcal{F}$ -generic-limsup-semiproper. It suffices to show the following. There exists a club $C \subseteq [H_{\theta_0}]^\omega$, where $\theta_0 = |\text{TC}(P)|^+ (\geq \omega_2)$, such that for any $\in$ -chain $\langle N_n \mid n < \omega \rangle$ in H_{θ_0} through C (i.e. for all $n < \omega$, $N_n \in C$), if $\langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}$, then for any $p \in N_0 \cap P$, there exists q such that $q \leq p$ and $q \Vdash_P \text{"}N_n[G] \cap \omega_1 = N_n \cap \omega_1 \text{ for infinitely many } n < \omega\text{"}$. We show this by contradiction. Suppose not. Let $S = \{\langle N_n \mid n < \omega \rangle \mid \langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F} \text{ and there is } p \in N_0 \cap P \text{ such that for all } q \text{ with } q \leq p, \neg(q \Vdash_P \text{"}N_n[G] \cap \omega_1 = N_n \cap \omega_1 \text{ for infinitely many } n < \omega\text{"})\}$. Then this S is ω -stationary. Namely, for any club C in $[H_{\theta_0}]^\omega$, there exists $\langle N_n \mid n < \omega \rangle \in S$ through C . By Fodor's Lemma for ω -stationary sets (may see [M]), we may assume that there exists $p_0 \in P$ such that $S \subseteq \{\langle N_n \mid n < \omega \rangle \mid \langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}, p_0 \in N_0 \cap P \text{ and for all } q \text{ with } q \leq p_0, \neg(q \Vdash_P \text{"}N_n[G] \cap \omega_1 = N_n \cap \omega_1 \text{ for infinitely many } n < \omega\text{"})\}$.

Let S^* denote the set of the end-segments of the elements of S . Apply $\mathcal{F}$ -RP to $(H_{\theta_0}, S^*, \lambda, (P, p_0))$, where λ is sufficiently large. Get a club D in ω_1 and an $\in$ -chain $\langle M_i \mid i < \omega_1 \rangle$ in H_λ as in $\mathcal{F}$ -RF.

Claim. $B = \{\delta \in Y^*(D) \mid \delta \models \text{"(1) in } \mathcal{F}\text{-RP"}\} \in (WCG)^+$.

Proof. Let E be a club in ω_1 . Want $\delta \in B$ with $|C_\delta \cap E| = \omega$. Since S is ω -stationary in $\text{Seq}^\omega(H_{\theta_0})$, we may take an $\in$ -chain $\langle N_n^* \mid n < \omega \rangle$ in H_λ such that $D, \langle M_i \mid i < \omega_1 \rangle, E \in N_0^*$ and that $\langle N_n^* \cap H_{\theta_0} \mid n < \omega \rangle \in S$. Let $N_\omega^* = \bigcup \{N_n^* \mid n < \omega\}$ and $\delta = N_\omega^* \cap \omega_1$. Then $\langle N_n^* \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}$ and is through $D \cap E$. In particular, $\delta \in Y^*(D)$ and $|C_\delta \cap E| = \omega$. Let $\{N_n^* \cap \omega_1 \mid n < \omega\} \subseteq C_\delta \cap D = \{C_\delta(k_\delta(m)) \mid m < \omega\}$. By considering an end-segment, we may assume $\langle N_n^* \cap H_{\theta_0} \mid n < \omega \rangle \in S^*$ with $C_\delta(k_\delta(n_\delta)) < N_0^* \cap \omega_1$.

Want (1) holds at this δ so that $\delta \in B$. Since $N_n^* \supseteq_{\omega_1} M_{N_n^* \cap \omega_1}$ for all $n < \omega$ and $\langle N_n^* \cap H_{\theta_0} \mid n < \omega \rangle \in S^*$, (2) in $\mathcal{F}$ -RP fails. Hence (1) must hold in $\mathcal{F}$ -RP. $\square$

Since $p_0 \Vdash_P \text{"}\langle C_\delta \mid \delta \in B \rangle \text{ remains to be weak club guessing"}$, $p_0 \Vdash_P \text{"there exists } \delta \in B \text{ such that } |C_\delta \cap \dot{D}| = \omega \text{, where } \dot{D} = \{i < \omega_1 \mid M_i[G] \cap \omega_1 = M_i \cap \omega_1 = i \in D\} \text{ a club in } \omega_1\text{"}$. Take q and $\delta \in B$ such that $q \leq p_0$ and $q \Vdash_P \text{"}\dot{C}_\delta \cap \dot{D} = \omega\text{"}$. Since $\delta \in B$, we have (1) in $\mathcal{F}$ -RP. Hence $\langle M_{C_\delta(k_\delta(m))} \cap H_{\theta_0} \mid n_\delta \leq m < \omega \rangle \in S^*$, where $\langle C_\delta(k) \mid k < \omega \rangle$ enumerates C_δ and $\langle C_\delta(k_\delta(m)) \mid m < \omega \rangle$ enumerates $C_\delta \cap D$.

Since $\langle M_{C_\delta(k_\delta(m))} \cap H_{\theta_0} \mid n_\delta \leq m < \omega \rangle \in S^*$, there exists $\langle N_n \mid n < \omega \rangle \in S$ such that $\langle M_{C_\delta(k_\delta(m))} \cap H_{\theta_0} \mid n_\delta \leq m < \omega \rangle$ is an end-segment of $\langle N_n \mid n < \omega \rangle$. Since $\langle N_n \mid n < \omega \rangle \in S$, we have $\neg(q \Vdash_P \text{"}N_n[G] \cap \omega_1 = N_n \cap \omega_1 \text{ for infinitely many } n < \omega\text{"})$. However, $q \Vdash_P \text{"if } C_\delta(k_\delta(m)) \in C_\delta \cap \dot{D} \text{ and } N_n = M_{C_\delta(k_\delta(m))} \cap H_{\theta_0}, \text{ then } N_n[G] \cap \omega_1 = (M_{C_\delta(k_\delta(m))} \cap H_{\theta_0})[G] \cap \omega_1 = M_{C_\delta(k_\delta(m))}[G] \cap \omega_1 = M_{C_\delta(k_\delta(m))} \cap \omega_1 = N_n \cap \omega_1 = C_\delta(k_\delta(m))\text{"}$. Hence $q \Vdash_P \text{"}N_n[G] \cap \omega_1 = N_n \cap \omega_1 \text{ for infinitely many } n < \omega\text{"}$. This is a contradiction. $\square$

Proof of lemma 2.7. Let (K, S, θ, a) be as in the hypothesis of $\mathcal{F}$ -RF. We force $\dot{D}$, $\langle \dot{N}_i \mid i < \omega_1 \rangle$ and $\langle \dot{n}_\delta \mid \delta \in A \rangle$ by initial segments. Let $p \in P$, if $p = (\alpha^p, D^p, \langle N_i^p \mid i \leq \alpha^p \rangle, \langle n_\delta^p \mid \delta \in A \cap (\alpha^p + 1) \rangle)$ satisfies the following:

- (1) $\alpha^p < \omega_1$.
- (2) $D^p \subseteq \alpha^p + 1$, $\alpha^p \in D^p$ and D^p is closed.
- (3) $\langle N_i^p \mid i \leq \alpha^p \rangle$ is an $\in$ -chain in H_θ with $a \in N_0^p$.
- (4) For each $\delta \in A \cap (\alpha^p + 1)$, $n_\delta^p < \omega$. If $|C_\delta \cap D^p| = \omega$, then let $\langle C_\delta(k) \mid k < \omega \rangle$ enumerate C_δ and let $\langle C_\delta(k_\delta^p(m)) \mid m < \omega \rangle$ enumerate $C_\delta \cap D^p$. And we demand either (i) or (ii);

(i) $\langle N_{C_\delta(k_\delta^p(m))}^p \cap K \mid n_\delta^p \leq m < \omega \rangle \in S$.

(ii) For any strictly increasing sequence $\langle m_l \mid l < \omega \rangle$ of natural numbers with $n_\delta^p \leq m_0$ and any $\in$ -chain $\langle M_l \mid l < \omega \rangle$ in H_θ such that $\langle M_l \mid l < \omega \rangle \supseteq_{\omega_1} \langle N_{C_\delta(k_\delta^p(m_l))}^p \mid l < \omega \rangle$, we have $\langle M_l \cap K \mid l < \omega \rangle \notin S$.

For $p, q \in P$, we set $q \leq p$, if

- (5) $\alpha^p \leq \alpha^q$.
- (6) $D^p = D^q \cap (\alpha^p + 1)$.
- (7) For all $i \leq \alpha^p$, $N_i^p = N_i^q$.
- (8) For all $\delta \in A \cap (\alpha^p + 1)$, $n_\delta^p = n_\delta^q$.

Claim. (Dense, Extension by Escape) For any (p, η, x) such that $p \in P$, $\alpha^p < \eta < \omega_1$ and $x \in H_\theta$, there exists $q \in P$ such that $q \leq p$, $\alpha^q = \eta$ and $x \in N_\eta^q$.

Proof. Let $\alpha^q = \eta$, $D^q = D^p \cup \{\eta\}$ and $\langle N_i^q \mid i \leq \eta \rangle$ be any $\in$ -chain which end-extends $\in$ -chain $\langle N_i^p \mid i \leq \alpha^p \rangle$ and $x \in N_\eta^q$. Let $\langle n_\delta^q \mid \delta \in A \cap (\eta + 1) \rangle$ be any sequence of natural numbers which end-extends $\langle n_\delta^p \mid \delta \in A \cap (\alpha^p + 1) \rangle$. Since for any $\delta \in A \cap (\eta + 1)$, $|C_\delta \cap D^q| = \omega$ iff $(\delta \leq \alpha^p \text{ and } |C_\delta \cap D^p| = \omega)$, this q works.

Claim. (Targeted-Extension) Let λ be a sufficiently large regular cardinal and M be a countable elementary substructure of H_λ with $P \in M$. Let $\delta_M = M \cap \omega_1$. Then for any (p, ξ, D) such that $p \in M \cap P$, $\xi < \delta_M$ and $D \in M$ is a dense subset of P , there exists $r \in M \cap D$ such that $r \leq p$, $\xi < \alpha^r$ and $C_{\delta_M} \cap D^r = C_{\delta_M} \cap D^p$.

Proof. We consider a family of maps indexed by $p \in P$. Let $p \in P$. Let η be such that $\alpha^p < \eta < \omega_1$. Let $r = f_p(\eta)$, where $r \in D$, $r \leq p$, $\eta < \alpha^r$ and $D^r \cap (\alpha^p, \eta] = \{\eta\}$. We may assume $\langle f_p \mid p \in P \rangle \in M$. Since $p \in M$, $f_p \in M$. Let $C(f_p) = \{\beta < \omega_1 \mid \text{for all } \eta \text{ with } \alpha^p < \eta < \beta, \alpha^{f_p(\eta)} < \beta\}$. Then $C(f_p)$ is a club in ω_1 and $C(f_p) \in M$. Take $\beta \in C(f_p) \cap M$ such that $\alpha^p, \xi < \beta$. Let η be such that $\alpha^p, \xi, \max(C_{\delta_M} \cap \beta) < \eta < \beta$. Then $f_p(\eta) \in M$ and so there exists $r \in M \cap D$ such that $r \leq p$, $\xi < \alpha^r$ and $C_{\delta_M} \cap D^r = C_{\delta_M} \cap D^p$.

Claim. (Proper) P is proper and σ -Baire.

Proof. Let λ be a sufficiently large regular cardinal and M be a countable elementary substructure of H_λ with $P \in M$. Let $p \in M \cap P$. Let $\delta_M = M \cap \omega_1$. Then by repeating above claim, we may construct a (P, M) -generic sequence $\langle p_n \mid n < \omega \rangle$ such that $C_{\delta_M} \cap D^{p_n} = C_{\delta_M} \cap D^p$ for all $n < \omega$. Let $\alpha^q = \delta_M$, $D^q = \bigcup \{D^{p_n} \mid n < \omega\} \cup \{\delta_M\}$, $\langle N_i^q \mid i \leq \delta_M \rangle = \bigcup \{ \langle N_i^{p_n} \mid i \leq \alpha^{p_n} \rangle \mid n < \omega \} \cup \{(\delta_M, M \cap H_\theta)\}$ and $\langle n_\delta^q \mid \delta \in A \cap (\delta_M + 1) \rangle$ be any end-extension of all $\langle n_\delta^{p_n} \mid \delta \in A \cap (\alpha^{p_n} + 1) \rangle$. Notice that $C_{\delta_M} \cap D^q = C_{\delta_M} \cap D^p$ which is finite. Hence this q is a lower bound of the p_n 's.

Claim. P is $\mathcal{F}$ -limsup-semiproper.

Proof. Let λ be a sufficiently large regular cardinal. Let $\langle N_n \mid n < \omega \rangle$ be an $\in$ -chain in H_λ such that $K, \theta, P, \langle C_\delta \mid \delta \in A \rangle \in N_0$ and $\langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}$. Let $N_\omega = \bigcup \{N_n \mid n < \omega\}$ and let $\delta^* = N_\omega \cap \omega_1$. Let $\langle C_{\delta^*}(k) \mid k < \omega \rangle$ enumerate C_{δ^*} and $\langle C_{\delta^*}(k_\delta^*(n)) \mid n < \omega \rangle$ enumerate $\{N_n \cap \omega_1 \mid n < \omega\}$.

Let $p \in N_0 \cap P$. We want q such that $q \leq p$ and q is (P, N_n) -semi-generic for infinitely many $n < \omega$. To this end, we argue in two cases.

Case 1. There exists a sequence $\langle n_l \mid l < \omega \rangle$ of strictly increasing natural numbers and an $\in$ -chain $\langle M_l \mid l < \omega \rangle$ in H_θ such that $\langle M_l \mid l < \omega \rangle \supseteq_{\omega_1} \langle N_{n_l} \cap H_\theta \mid l < \omega \rangle$ and $\langle M_l \cap K \mid l < \omega \rangle \in S$:

Apply the Sequential 3 H Lemma (see [M]) to get an $\in$ -chain $\langle M_l^* \mid l < \omega \rangle$ in H_λ such that

- $\langle M_l^* \mid l < \omega \rangle \supseteq_{\omega_1} \langle N_{n_l} \mid l < \omega \rangle$.

- For all $l < \omega$, $M_l^* \cap H_{|TC(K)|+} = M_l \cap H_{|TC(K)|+}$.

And so

- $\langle M_l^* \cap K \mid l < \omega \rangle = \langle M_l \cap K \mid l < \omega \rangle \in S$.

Notice $p \in N_0 \subseteq N_{n_0} \subseteq M_0^*$. It suffices to get q such that $q \leq p$ and for all $l < \omega$, q is (P, M_l^*) -generic and so (P, N_{n_l}) -semi-generic. Construct a descending sequence $\langle q_l \mid l < \omega \rangle$ of conditions such that

- $p \geq q_l \in M_{l+1}^*$ and q_l is (P, M_l^*) -generic.
- $D^{q_l} \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{M_0^* \cap \omega_1, \dots, M_l^* \cap \omega_1\}$.
- $\alpha^{q_l} = M_l^* \cap \omega_1$ and $N_{\alpha^{q_l}} = M_l^* \cap H_\theta$.

Let q be defined by

- $\alpha^q = \delta^* = M_\omega^* \cap \omega_1$, where $M_\omega^* = \bigcup \{M_l^* \mid l < \omega\}$.
- $D^q = \bigcup \{D^{q_l} \mid l < \omega\} \cup \{\delta^*\}$.
- $\langle N_i^q \mid i \leq \alpha^q \rangle = \bigcup \{ \langle N_i^{q_l} \mid i \leq \alpha^{q_l} \rangle \mid l < \omega \} \cup \{(\delta^*, M_\omega^* \cap H_\theta)\}$.
- $\langle n_\delta^q \mid \delta \in A \cap (\alpha^q + 1) \rangle = \bigcup \{ \langle n_\delta^{q_l} \mid \delta \in A \cap (\alpha^{q_l} + 1) \rangle \mid l < \omega \} \cup \{(\delta^*, n_{\delta^*}^q)\}$,
where $n_{\delta^*}^q < \omega$ gets specified as follows;

Let $C_{\delta^*} \cap D^q = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid m < \omega\}$. Since $D^q \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{M_l^* \cap \omega_1 \mid l < \omega\}$, there exists $n_{\delta^*}^q < \omega$ such that

$$\{M_l^* \cap \omega_1 \mid l < \omega\} = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid n_{\delta^*}^q \leq m < \omega\}.$$

And so

$$\langle N_{C_{\delta^*}(k_{\delta^*}^q(m))}^q \cap K \mid n_{\delta^*}^q \leq m < \omega \rangle = \langle (M_l^* \cap H_\theta) \cap K \mid l < \omega \rangle = \langle M_l \cap K \mid l < \omega \rangle \in S.$$

Hence this q works. □

Case 2. For all sequences $\langle n_l \mid l < \omega \rangle$ of strictly increasing natural numbers and all $\in$ -chains $\langle M_l \mid l < \omega \rangle$ in H_θ with $\langle M_l \mid l < \omega \rangle \supseteq_{\omega_1} \langle N_{n_l} \cap H_\theta \mid l < \omega \rangle$, we have $\langle M_l \cap K \mid l < \omega \rangle \notin S$:

It suffices to get $q \in P$ such that $q \leq p$ and for all $n < \omega$, q is (P, N_n) -generic. To this end, we may construct a descending sequence $\langle q_n \mid n < \omega \rangle$ of conditions such that

- $p \geq q_n \in N_{n+1}$ and q_n is (P, N_n) -generic.
- $D^{q_n} \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{N_0 \cap \omega_1, \dots, N_n \cap \omega_1\}$.
- $\alpha^{q_n} = N_n \cap \omega_1$ and $N_{\alpha^{q_n}} = N_n \cap H_\theta$.

Let q be defined by

- $\alpha^q = \delta^* = N_\omega \cap \omega_1$.
- $D^q = \bigcup \{D^{q_n} \mid n < \omega\} \cup \{\delta^*\}$.
- $\langle N_i^q \mid i \leq \alpha^q \rangle = \bigcup \{ \langle N_i^{q_n} \mid i \leq \alpha^{q_n} \rangle \mid n < \omega \} \cup \{(\delta^*, N_\omega \cap H_\theta)\}$.
- $\langle n_\delta^q \mid \delta \in A \cap (\alpha^q + 1) \rangle = \bigcup \{ \langle n_\delta^{q_n} \mid \delta \in A \cap (\alpha^{q_n} + 1) \rangle \mid n < \omega \} \cup \{(\delta^*, n_{\delta^*}^q)\}$,
where $n_{\delta^*}^q < \omega$ gets specified as follows;

Let $C_{\delta^*} \cap D^q = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid m < \omega\}$. Since $D^q \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{N_n \cap \omega_1 \mid n < \omega\}$, there exists $n_{\delta^*}^q < \omega$ such that

$$\{N_n \cap \omega_1 \mid n < \omega\} = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid n_{\delta^*}^q \leq m < \omega\}.$$

And so

$$\langle N_{C_{\delta^*}(k_{\delta^*}^q(m))}^q \mid n_{\delta^*}^q \leq m < \omega \rangle = \langle N_{N_n \cap \omega_1}^q \mid n < \omega \rangle = \langle N_n \cap H_\theta \mid n < \omega \rangle.$$

For all sequences $\langle m_l \mid l < \omega \rangle$ of strictly increasing natural numbers with $n_{\delta^*}^q \leq m_0$ and all $\in$ -chains $\langle M_l \mid l < \omega \rangle$ in H_θ with $\langle M_l \mid l < \omega \rangle \supseteq_{\omega_1} \langle N_{C_{\delta^*}^q(k_{\delta^*}^q(m_l))}^q \mid l < \omega \rangle$ which is a subsequence of $\langle N_n \cap H_\theta \mid n < \omega \rangle$, we have $\langle M_l \cap K \mid l < \omega \rangle \notin S$.

Hence this q works.

Now apply forcing axiom to this P . We have a club D in ω_1 and an $\in$ -chain $\langle N_i \mid i < \omega_1 \rangle$ which works for (K, S, θ, a) in $\mathcal{F}$ -RP.

§3. Iteration theorem

We show $\mathcal{F}^{V[G_\alpha]}$ -generic-limsup-semiproper combined with semiproper iterates under the simple iterations. For an account on the simple iterations, see [M].

3.1 Theorem. Let $I = (\langle P_\alpha \mid \alpha \leq \nu \rangle, \langle \dot{Q}_\alpha \mid \alpha < \nu \rangle)$ be a simple iteration such that

- For all $\alpha < \nu$, $\Vdash_{P_\alpha} \text{"}\dot{Q}_\alpha \text{ is semiproper"}$.
- For all $\alpha < \nu$, $\Vdash_{P_\alpha} \text{"}\dot{Q}_\alpha \text{ is } \mathcal{F}^{V[G_\alpha]} \text{-generic-limsup-semiproper"}$,

where $\mathcal{F}^{V[G_\alpha]}$ is formed in $V[G_\alpha]$ as the set of cofinal subsequences of the $\langle C_\delta(n) \mid n < \omega \rangle$'s for all $\delta \in A$. Hence $\mathcal{F}^{V[G_\alpha]}$ may contain new cofinal subsequences than the original $\mathcal{F} = \mathcal{F}^V$.

Then for all $\alpha \leq \nu$, we have $\Vdash_{P_\alpha} \text{"the tails } P_{\alpha\nu} \text{ are semiproper and } \mathcal{F}^{V[G_\alpha]} \text{-generic-limsup-semiproper"}$.

In particular, P_ν is semiproper and $\mathcal{F}^V$ -generic-limsup-semiproper.

3.2 Iteration Lemma. Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $I, \langle C_\delta \mid \delta \in A \rangle \in N$.

Let $(\alpha, \alpha^*, a, p, \langle \dot{M}_n \mid n < \omega \rangle)$ be such that

- $\alpha < \alpha^* \leq \nu$.
- $a \in P_\alpha, p \in P_{\alpha^*}$ and $a \leq p \restriction \alpha$.
- $a \Vdash_{P_\alpha} \text{"}\langle \dot{M}_n \mid n < \omega \rangle \text{ is an } \in\text{-chain in } H_\theta^{V[G_\alpha]} \text{ such that } N \cup \{G_\alpha, p\} \subseteq \dot{M}_0 \text{ and that } \langle \dot{M}_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}^{V[G_\alpha]}\text{"}$.

Then there exists $q \in P_{\alpha^*}$ such that

- $q \restriction \alpha = a$ and $q \leq p$.
- $a \Vdash_{P_\alpha} \text{"} q \restriction [\alpha, \alpha^*) \Vdash_{P_{\alpha\alpha^*}^{V[G_\alpha]}} \text{"}\dot{M}_n[G_{\alpha\alpha^*}] \cap \omega_1 = \dot{M}_n \cap \omega_1 \text{ for infinitely many } n < \omega\text{"}$.

□

We extract sort of typical constructions involved as (Technical construction 1-3).

Lemma. (Technical construction 1) Let $(\alpha, \alpha^*, a, p, \langle \dot{\delta}_k^p \mid k < \omega \rangle, \dot{x})$ be such that

- $\alpha < \alpha^* \leq \nu$ and α^* is limit.
- $a \in P_\alpha, p \in P_{\alpha^*}$ and $a \leq p \restriction \alpha$.
- $\langle \dot{\delta}_k^p \mid k < \omega \rangle$ are stages for p .

- $a \cap 1 \Vdash_{P_{\alpha^*}} \dot{\delta}_0^p = \alpha$.
- $a \Vdash_{P_\alpha} \dot{x} \leq p$ and $\dot{x} \restriction \alpha \in G_\alpha$.

Then there exists $(\beta, b, x, \langle \dot{\delta}_k^x \mid k < \omega \rangle)$ such that

- $\alpha < \beta < \alpha^*$.
- $b \in P_\beta$, $b \restriction \alpha \leq a$, $x \in P_{\alpha^*}$ and $b \leq x \restriction \beta$.
- $b \restriction \alpha \Vdash_{P_\alpha} \dot{x} = x$.
- $\langle \dot{\delta}_k^x \mid k < \omega \rangle$ are stages for x and for all $k < \omega$, $\Vdash_{P_{\alpha^*}} \dot{\delta}_{k+1}^p \leq \dot{\delta}_k^x$ (a step ahead).
- $b \cap 1 \Vdash_{P_{\alpha^*}} \dot{\delta}_0^x = \beta$.

□

Lemma. (Technical construction 2) Let $(\alpha, \alpha^*, a, p, \langle \dot{\delta}_k^p \mid k < \omega \rangle, \dot{x})$ be as in technical construction 1. Then there exists $(\langle \beta_i, b_i, x_i, \langle \dot{\delta}_k^{x_i} \mid k < \omega \rangle \mid i < \mu \rangle)$ such that

- $\beta = \beta_i, b = b_i, x = x_i, \langle \dot{\delta}_k^x \mid k < \omega \rangle = \langle \dot{\delta}_k^{x_i} \mid k < \omega \rangle$ serve exactly as in technical construction 1.
- For $i, j < \mu$ such that $i \neq j$, we have that $b_i \restriction \alpha$ and $b_j \restriction \alpha$ are incompatible in P_α .
- $\{b_i \restriction \alpha \mid i < \mu\}$ forms a maximal antichain below a in P_α .

□

Lemma. (Technical construction 3) Let $(\alpha, \alpha^*, a, p, \langle \dot{\delta}_k^p \mid k < \omega \rangle, \langle \dot{M}_n \mid n < \omega \rangle)$ be such that

- $\alpha < \alpha^* \leq \nu$ and α^* is limit.
- $a \in P_\alpha, p \in P_{\alpha^*}$ and $a \leq p \restriction \alpha$.
- $\langle \dot{\delta}_k^p \mid k < \omega \rangle$ are stages for p in P_{α^*} .
- $a \cap 1 \Vdash_{P_{\alpha^*}} \dot{\delta}_0^p = \alpha$.
- $a \Vdash_{P_\alpha} \langle \dot{M}_n \mid n < \omega \rangle$ is an $\in$ -chain in $H_\theta^{V[G_\alpha]}$ with $N \cup \{G_\alpha, p, \langle \dot{\delta}_k^p \mid k < \omega \rangle\} \subseteq \dot{M}_0$.

Let T be a tree such that $T \subset {}^{<\omega}ON$ with $\{\emptyset\} = T_0$. For $\sigma = \emptyset$, let

- $\alpha^\emptyset = \alpha, a^\emptyset = a, p^\emptyset = p, \langle \dot{\delta}_k^\emptyset \mid k < \omega \rangle = \langle \dot{\delta}_k^p \mid k < \omega \rangle$ and $\langle \dot{M}_n^\emptyset \mid n < \omega \rangle = \langle \dot{M}_n \mid n < \omega \rangle$.

For all $\sigma \in T$, we have $(\alpha^\sigma, a^\sigma, p^\sigma, \langle \dot{\delta}_k^\sigma \mid k < \omega \rangle, \langle \dot{M}_n^\sigma \mid n < \omega \rangle)$ such that

- $\alpha \leq \alpha^\sigma < \alpha^*$.
- $a^\sigma \in P_{\alpha^\sigma}, p^\sigma \in P_{\alpha^*}, a^\sigma \restriction \alpha \leq a, p^\sigma \leq p$ and $a^\sigma \leq p^\sigma \restriction \alpha^\sigma$.
- $\langle \dot{\delta}_k^\sigma \mid k < \omega \rangle$ are stages for p^σ in P_{α^*} .
- $a^\sigma \cap 1 \Vdash_{P_{\alpha^*}} \dot{\delta}_0^{p^\sigma} = \alpha^\sigma$.
- $a^\sigma \Vdash_{P_{\alpha^\sigma}} \langle \dot{M}_n^\sigma \mid n < \omega \rangle$ is an $\in$ -chain in $H_\theta^{V[G_{\alpha^\sigma}]}$ with $N \cup \{G_{\alpha^\sigma}, p^\sigma, \langle \dot{\delta}_k^\sigma \mid k < \omega \rangle\} \subseteq \dot{M}_0^\sigma$.

For all $\tau \in \text{succ}_T(\sigma)$, we have $\langle \dot{m}(\tau, n) \mid n < \omega \rangle$ such that

- $\alpha^\sigma \leq \alpha^\tau$.
- $a^\tau \restriction \alpha^\sigma \leq a^\sigma$.
- $p^\tau \leq p^\sigma$ in P_{α^*} .
- $\langle \dot{\delta}_k^\tau \mid k < \omega \rangle$ are a step ahead of $\langle \dot{\delta}_k^\sigma \mid k < \omega \rangle$.
- $a^\tau \restriction \alpha^\sigma \Vdash_{P_{\alpha^\sigma}} p^\tau \restriction [\alpha^\sigma, \alpha^*) \Vdash_{P_{\alpha^\sigma \alpha^*}} \dot{M}_0^\sigma[G_{\alpha^\sigma \alpha^*}] \cap \omega_1 = \dot{M}_0^\sigma \cap \omega_1$.
- $a^\tau \Vdash_{P_{\alpha^\tau}} \langle \dot{m}(\tau, n) \mid n < \omega \rangle$ is a sequence of strictly increasing natural numbers.
- $a^\tau \Vdash_{P_{\alpha^\tau}} \dot{M}_n^\tau = \dot{M}_{\dot{m}(\tau, n)}^\sigma[G_{\alpha^\sigma \alpha^\tau}]$.

- $a^\tau \Vdash_{P_{\alpha^\tau}} \text{“for all } n < \omega, \dot{M}_n^\tau \cap \omega_1 = \dot{M}_{m(\tau, n)}^\tau \cap \omega_1 \text{”}$.

Let $q \leq p$ be a fusion of a^σ 's in P_{α^σ} .

Then there exists a sequence $\langle \dot{n}_k \mid k < \omega \rangle$ of P_{α^σ} -names such that

- $q \Vdash_{P_{\alpha^\sigma}} \text{“}\langle \dot{n}_k \mid k < \omega \rangle \text{ is a sequence of strictly increasing natural numbers”}$.
- For all $k < \omega$, we have $q \Vdash_{P_{\alpha^\sigma}} \text{“}\dot{M}_{\dot{n}_k} \cap \omega_1 = \dot{M}_{\dot{n}_k}[G_{\alpha\alpha^\sigma}] \cap \omega_1 \text{”}$.

More specifically, we may calculate $\langle i_k \mid k < \omega \rangle$, $\langle a_k \mid k < \omega \rangle$, $\langle \alpha_k \mid k < \omega \rangle$, $\langle p_k \mid k < \omega \rangle$ and $\langle m(k, n) \mid k, n < \omega \rangle$ in $V[G_{\alpha^\sigma}]$ such that, where $M =_{\omega_1} N$ abbreviates $M \cap \omega_1 = N \cap \omega_1$,

- $M_0[G_{\alpha_0\alpha^\sigma}] \subseteq M_0^\emptyset[G_{\alpha_0\alpha^\sigma}] =_{\omega_1} M_0^\emptyset = M_0$.
- $M_{m(0,0)}[G_{\alpha_0\alpha^\sigma}] \subseteq M_{m(0,0)}^\emptyset[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha^\sigma}] = M_0^{\langle i_0 \rangle}[G_{\alpha_1\alpha^\sigma}] =_{\omega_1} M_0^{\langle i_0 \rangle} =_{\omega_1} M_{m(0,0)}^\emptyset = M_{m(0,0)}$.
- $M_{m(0,m(1,0))}[G_{\alpha\alpha^\sigma}] \subseteq M_{m(0,m(1,0))}^\emptyset[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha_2}][G_{\alpha_2\alpha^\sigma}] = M_{m(1,0)}^{\langle i_0 \rangle}[G_{\alpha_1\alpha_2}][G_{\alpha_2\alpha^\sigma}] = M_0^{\langle i_0, i_1 \rangle}[G_{\alpha_2\alpha^\sigma}] =_{\omega_1} M_0^{\langle i_0, i_1 \rangle} =_{\omega_1} M_{m(1,0)}^{\langle i_0 \rangle} =_{\omega_1} M_{m(0,m(1,0))}^\emptyset = M_{m(0,m(1,0))}$.

In general,

$$\begin{aligned}
 a^\emptyset &= a = a_0. \\
 a^{\langle i_0, \dots, i_k \rangle} &= a_{k+1}. \\
 \alpha_k &= l(a_k). \\
 M_n^\emptyset &= M_n. \\
 M_n^{\langle i_0 \rangle} &= M_{m(0,n)}[G_{\alpha_0\alpha_1}]. \\
 M_n^{\langle i_0 \rangle} &=_{\omega_1} M_{m(0,n)}. \\
 M_n^{\langle i_0, \dots, i_k, i_{k+1} \rangle} &= M_{m(k+1,n)}^{\langle i_0, \dots, i_k \rangle}[G_{\alpha_{k+1}\alpha_{k+2}}]. \\
 M_n^{\langle i_0, \dots, i_k, i_{k+1} \rangle} &=_{\omega_1} M_{m(k+1,n)}^{\langle i_0, \dots, i_k \rangle}. \\
 M_0^\emptyset[G_{\alpha_0\alpha^\sigma}] &=_{\omega_1} M_0^\emptyset. \\
 M_0^{\langle i_0, \dots, i_k \rangle}[G_{\alpha_{k+1}\alpha^\sigma}] &=_{\omega_1} M_0^{\langle i_0, \dots, i_k \rangle}.
 \end{aligned}$$

And so

$$\begin{aligned}
 M_0[G_{\alpha_0\alpha^\sigma}] &=_{\omega_1} M_0. \\
 M_{m(0,m(1,\dots,m(k,0)\dots))}[G_{\alpha_0\alpha^\sigma}] &=_{\omega_1} M_{m(0,m(1,\dots,m(k,0)\dots))}.
 \end{aligned}$$

Where $m(k, n)$ abbreviates $m(\langle i_0, \dots, i_k \rangle, n)$.

We give a proof of iteration lemma 3.2.

3.2 Lemma. (Iteration Lemma) Let θ be a sufficiently large regular cardinal. Let N be a countable elementary substructure of H_θ with $I, \langle C_\delta \mid \delta \in A \rangle \in N$.

Let $(\alpha, \alpha^*, a, p, \langle \dot{M}_n \mid n < \omega \rangle)$ satisfy

- $\alpha < \alpha^* \leq \nu$.
- $a \in P_\alpha$, $p \in P_{\alpha^\sigma}$ and $a \leq p \restriction \alpha$.
- $a \Vdash_{P_\alpha} \text{“} N \cup \{\dot{G}_\alpha, p\} \subseteq \dot{M}_0$, $\langle \dot{M}_n \mid n < \omega \rangle$ is an $\in$ -chain in $H_\theta^{V[G_\alpha]}$ and $\langle \dot{M}_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}^{V[G_\alpha]} \text{”}$.

Then there exists $(q, \langle \dot{m}(n) \mid n < \omega \rangle)$ such that

- $q \in P_{\alpha^*}$ and $q \leq p$.
- $q \restriction \alpha = a$.
- $q \restriction_{P_{\alpha^*}} \dot{m}(n)$ are strictly increasing natural numbers and $\dot{M}_{\dot{m}(n)}[\dot{G}_{\alpha^*} \restriction [\alpha, \alpha^*]] \cap \omega_1 = \dot{M}_{\dot{m}(n)} \cap \omega_1$.

Notational Remark. Let $\alpha < \beta$ and G_β be P_β -generic over V . Then

- G_α denotes $G_\beta \restriction \alpha = \{r \restriction \alpha \mid r \in G_\beta\}$ which is P_α -generic over V .
- H_α denotes $\{r(\alpha)[G_\alpha] \mid r \in G_\beta\}$ which is $Q_\alpha = \dot{Q}_\alpha[G_\alpha]$ -generic over $V[G_\alpha]$.

If $\dot{x}$ is a P_α -name, then we may view $\dot{x}[G_\alpha]$ as a term $\check{x}[\dot{G}_\beta \restriction \check{\alpha}]$ being interpreted by G_β in $V[G_\beta]$. We simply denote this by x for easier notation. For sequences $s = \langle \dot{x}_n \mid n < \omega \rangle$ of P_α -names, we abbreviate as follows:

- $x_n = \dot{x}_n[G_\alpha]$ (the n -th value of the interpretation of a term $\langle \check{s}(n)[\dot{G}_\alpha] \mid n < \check{\omega} \rangle$ by G_α in $V[G_\alpha]$)
 $= \dot{x}_n[G_\beta \restriction \alpha]$ (the n -th value of the interpretation of a term $\langle \check{s}(n)[\dot{G}_\beta \restriction \check{\alpha}] \mid n < \check{\omega} \rangle$ by G_β in $V[G_\beta]$) for each $n < \omega$.

Proof. By induction on α^* for all $(\alpha, a, p, \langle \dot{M}_n \mid n < \omega \rangle)$.

Case 1. (Successor Steps Essential) Let $(\alpha, \alpha + 1, a, p, \langle \dot{M}_n \mid n < \omega \rangle)$ be as in the hypothesis. Since $\restriction_{P_\alpha} \dot{Q}_\alpha$ is $\mathcal{F}^{V[\dot{G}_\alpha]}$ -generic-limsup-semiproper" and $a \restriction_{P_\alpha} "p(\alpha) \in \dot{Q}_\alpha \cap \dot{M}_0, \langle \dot{M}_n \mid n < \omega \rangle$ is an $\in$ -chain in $H_\beta^{V[\dot{G}_\alpha]}$ and $\langle \dot{M}_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}^{V[\dot{G}_\alpha]}$ ", we have $a \restriction_{P_\alpha}$ "there exists $\pi \leq p(\alpha)$ in $\dot{Q}_\alpha$ such that $\pi \restriction_{\dot{Q}_\alpha} " \{n < \omega \mid M_n[H_\alpha] \cap \omega_1 = M_n \cap \omega_1\}$ is infinite"". Let $q \in P_{\alpha+1}$ such that $q \restriction \alpha = a$ and $a \restriction_{P_\alpha} "q(\alpha) = \pi"$. Then $q \leq p$ in $P_{\alpha+1}$ and $q \restriction_{P_{\alpha+1}} \dot{M}_n[G_{\alpha\alpha+1}] \cap \omega_1 = \dot{M}_n[H_\alpha] \cap \omega_1 = \dot{M}_n \cap \omega_1$ for infinitely many $n < \omega$.

Case 2. (Successor Steps General) Let $(\alpha, \beta + 1, a, p, \langle \dot{M}_n \mid n < \omega \rangle)$ be as in the hypothesis. We may assume $\alpha < \beta$. Apply the hypothesis of induction to $(\alpha, \beta, a, p \restriction \beta, \langle \dot{M}_n \mid n < \omega \rangle)$. We have $(q', \langle \dot{m}(n) \mid n < \omega \rangle)$ such that

- $q' \in P_\beta$, $a = q' \restriction \alpha$ and $q' \leq p \restriction \beta$.
 - $q' \restriction_{P_\beta} \dot{m}(n)$ are strictly increasing natural numbers and $M_{\dot{m}(n)}[G_{\alpha\beta}] \cap \omega_1 = M_{\dot{m}(n)} \cap \omega_1$ for all $n < \omega$.
- Hence $q' \restriction_{P_\beta} " \langle M_{\dot{m}(n)}[G_{\alpha\beta}] \mid n < \omega \rangle$ is an $\in$ -chain in $H_\beta^{V[\dot{G}_\beta]}$ such that $N \cup \{\dot{G}_\beta, p\} \subseteq M_{\dot{m}(0)}[G_{\alpha\beta}]$ and $\langle M_{\dot{m}(n)}[G_{\alpha\beta}] \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}^{V[\dot{G}_\beta]}$ ". Now we are in case 1 with $(\beta, \beta + 1, q', p, \langle M_{\dot{m}(n)}[G_{\alpha\beta}] \mid n < \omega \rangle)$. Hence we have $q \in P_{\beta+1}$ such that $q \restriction \beta = q'$, $q \leq p$ and $q \restriction_{P_{\beta+1}} "M_{\dot{m}(n)}[G_{\alpha\beta}][G_{\beta\beta+1}] \cap \omega_1 = M_{\dot{m}(n)}[G_{\alpha\beta}] \cap \omega_1$ for infinitely many $n < \omega$ ". And so $q \restriction \alpha = a$ and $q \restriction_{P_{\beta+1}} "M_m[G_{\alpha\beta+1}] \cap \omega_1 = M_m \cap \omega_1$ for infinitely many $m < \omega$ ".

Case 3. (Limit) Let $(\alpha, \alpha^*, a, p, \langle \dot{M}_n \mid n < \omega \rangle)$ be as in the hypothesis. We assume α^* is limit. Construct a tree representation T and a map $\langle \sigma \mapsto (\alpha^\sigma, a^\sigma, p^\sigma, \langle \dot{\delta}_n^\sigma \mid n < \omega \rangle, \langle \dot{M}_n^\sigma \mid n < \omega \rangle) \mid \sigma \in T \rangle$ such that

For $\langle \rangle = \emptyset \in T_0$, we set

- (0) $\alpha^{(\cdot)} = \alpha, a^{(\cdot)} = a, p^{(\cdot)} = p, \langle \dot{\delta}_k^{(\cdot)} \mid k < \omega \rangle$ be any stages for p such that $\dot{\delta}_0^{(\cdot)} = \check{\alpha}$ and that $a \restriction_{P_\alpha} " \langle \dot{\delta}_k^{(\cdot)} \mid k < \omega \rangle \in \dot{M}_0"$ and $\langle \dot{M}_n^{(\cdot)} \mid n < \omega \rangle = \langle \dot{M}_n \mid n < \omega \rangle$.

In general, for $\sigma = \langle i_0, \dots, i_{k-1} \rangle \in T_k$, we demand

- (1) $\alpha \leq \alpha^\sigma < \alpha^*$.
- (2) $a^\sigma \in P_{\alpha^\sigma}$ and $a^\sigma \restriction \alpha \leq a$.
- (3) $p^\sigma \in P_{\alpha^*}$ and $p^\sigma \leq p$.
- (4) $a^\sigma \leq p^\sigma \restriction \alpha^\sigma$.
- (5) $\langle \dot{\delta}_n^\sigma \mid n < \omega \rangle$ are stages for p^σ .
- (6) $p^\sigma \restriction \alpha^\sigma \frown 1 \restriction_{P_{\alpha^*}} " \dot{\delta}_0^\sigma = \alpha^\sigma"$ (0th-stage self-decisive condition).

- (7) $\langle \dot{M}_n^\sigma \mid n < \omega \rangle$ is a sequence of P_{α^σ} -names such that $a^\sigma \Vdash_{P_{\alpha^\sigma}} "N \cup \{\dot{G}_{\alpha^\sigma}, p^\sigma, \langle \dot{\delta}_k^\sigma \mid k < \omega \rangle\} \subseteq \dot{M}_0^\sigma$, $\langle \dot{M}_n^\sigma \mid n < \omega \rangle$ is an $\in$ -chain in $H_\theta^{V[\dot{G}_{\alpha^\sigma}]}$ and $\langle \dot{M}_n^\sigma \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}^{V[\dot{G}_{\alpha^\sigma}]}$.

For $\tau = \sigma \smallfrown \langle i \rangle = \langle i_0, \dots, i_{k-1}, i_k \rangle \in T_{k+1}$, there exists a sequence $\langle \dot{m}(\tau, n) \mid n < \omega \rangle$ of P_{α^τ} -names and we demand

- (8) $\alpha^\sigma \leq \alpha^\tau$.
- (9) $a^\tau \restriction \alpha^\sigma \leq a^\sigma$.
- (10) $p^\tau \leq p^\sigma$.
- (11) $a^\tau \restriction \alpha^\sigma \Vdash_{P_{\alpha^\sigma}} "p^\tau, \langle \dot{\delta}_n^\tau \mid n < \omega \rangle \in \dot{M}_1^\sigma$ and $p^\tau \restriction [\alpha^\sigma, \alpha^*]$ is $(P_{\alpha^\sigma \alpha^*}, \dot{M}_0^\sigma)$ -semi-generic".
- (12) For all $n < \omega$, $\Vdash_{P_{\alpha^\sigma}} "\dot{\delta}_{n+1}^\sigma \leq \dot{\delta}_n^\sigma"$ (a step ahead).
- (13) $p^\tau \restriction \alpha^\tau \smallfrown 1 \Vdash_{P_{\alpha^\sigma}} "\dot{\delta}_0^\tau = \alpha^\tau"$ (0th-stage self-decisive condition).
- (14) $a^\tau \Vdash_{P_{\alpha^\tau}} "\langle \dot{m}(\tau, n) \mid n < \omega \rangle$ is a sequence of strictly increasing natural numbers and for all $n < \omega$, $\dot{m}(\tau, n) \geq 1$, $\dot{M}_{\dot{m}(\tau, n)}^\sigma[G_{\alpha^\sigma \alpha^\tau}] = \dot{M}_n^\tau$ and $\dot{M}_{\dot{m}(\tau, n)}^\sigma \cap \omega_1 = \dot{M}_n^\tau \cap \omega_1"$.

The construction is by recursion on $k < \omega$. For $k = 0$, we set $T_0 = \{\emptyset\}$ and set $\alpha^\emptyset, a^\emptyset, p^\emptyset, \langle \dot{\delta}_n^\emptyset \mid n < \omega \rangle$ and $\langle \dot{M}_n^\emptyset \mid n < \omega \rangle$ as specified. This is possible as $a \Vdash_{P_\alpha} "N \cup \{\dot{G}_\alpha, p\} \subseteq \dot{M}_0"$. Then it is easy to see that all the assumptions (1) through (7) for $\sigma = \emptyset$ are satisfied.

Suppose we have constructed T_k and $\alpha^\sigma, a^\sigma, p^\sigma, \langle \dot{\delta}_n^\sigma \mid n < \omega \rangle$ and $\langle \dot{M}_n^\sigma \mid n < \omega \rangle$ for each $\sigma \in T_k$ such that (1) through (7) are satisfied. Let $\gamma = \alpha^\sigma$, $w = a^\sigma$, $x = p^\sigma$, $\langle \dot{\delta}_n \mid n < \omega \rangle = \langle \dot{\delta}_n^\sigma \mid n < \omega \rangle$ and $\langle \dot{N}_n \mid n < \omega \rangle = \langle \dot{M}_n^\sigma \mid n < \omega \rangle$ for shorter notation. Then $w \in P_\gamma$ forces that

- $N \cup \{\dot{G}_\gamma, x, \langle \dot{\delta}_n \mid n < \omega \rangle\} \subseteq \dot{N}_0$ and $x \restriction \gamma \in \dot{G}_\gamma$.

Hence by the iteration lemma for semiproperness and lemmas on stages (please see [M] for an account), there exists $(\beta, y, \langle \dot{\delta}_n^y \mid n < \omega \rangle)$ in $V[\dot{G}_\gamma]$ such that

- $\gamma < \beta < \alpha^*$.
- $y \leq x$ in P_{α^*} .
- $\langle \dot{\delta}_n^y \mid n < \omega \rangle$ are stages for y .
- $y \restriction \beta \smallfrown 1 \Vdash_{P_{\alpha^*}} "\dot{\delta}_0^y = \beta"$.
- For all $n < \omega$, $\Vdash_{P_{\alpha^*}} "\dot{\delta}_{n+1}^y \leq \dot{\delta}_n^y"$ (a step ahead).
- $y \restriction \gamma \in \dot{G}_\gamma$.
- $y \restriction [\gamma, \alpha^*]$ is $(P_{\gamma \alpha^*}, \dot{N}_0)$ -semi-generic
- $(\beta, y, \langle \dot{\delta}_n^y \mid n < \omega \rangle) \in \dot{N}_1$.

Then for any $d \leq w$ in P_γ such that d decides the values of β, y and $\langle \dot{\delta}_n^y \mid n < \omega \rangle$, we may consider $(\gamma, \beta, d, y \restriction \beta, \langle \dot{N}_n \mid 1 \leq n < \omega \rangle)$ satisfying

- $\gamma < \beta < \alpha^*$.
- $d \in P_\gamma$, $y \restriction \beta \in P_\beta$ and $d \leq (y \restriction \beta) \restriction \gamma$.
- $d \Vdash_{P_\gamma} "N \cup \{\dot{G}_\gamma, y \restriction \beta\} \subseteq \dot{N}_1$ and $\langle \dot{N}_n \mid 1 \leq n < \omega \rangle$ is an $\in$ -chain in $H_\theta^{V[\dot{G}_\gamma]}$ and $\langle \dot{N}_n \cap \omega_1 \mid 1 \leq n < \omega \rangle \in \mathcal{F}^{V[\dot{G}_\gamma]}$.

Now we apply the hypothesis of induction at β . Hence there exists $(b, \langle \dot{m}(n) \mid n < \omega \rangle)$ such that

- $b \in P_\beta$, $b \restriction \gamma = d$ and $b \leq y \restriction \beta$.
- $b \Vdash_{P_\beta} "\langle \dot{m}(n) \mid n < \omega \rangle$ is a sequence of strictly increasing natural numbers such that $1 \leq \dot{m}(n)$ and $\dot{N}_{\dot{m}(n)}[G_{\gamma \beta}] \cap \omega_1 = \dot{N}_{\dot{m}(n)} \cap \omega_1"$.

And so

- $b \Vdash_{P_\beta} "N \cup \{\dot{G}_\beta, y, \langle \dot{\delta}_n^y \mid n < \omega \rangle\} \subseteq \dot{N}_{\dot{m}(0)}[G_{\gamma \beta}]$, $\langle \dot{N}_{\dot{m}(n)}[G_{\gamma \beta}] \mid n < \omega \rangle$ is an $\in$ -chain in $H_\theta^{V[\dot{G}_\beta]}$ and $\langle \dot{N}_{\dot{m}(n)}[G_{\gamma \beta}] \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}^{V[\dot{G}_\beta]}$.

Since there exists d as above predense many below w , we may construct T_{k+1} and

$$\langle \tau \mapsto (\alpha^\tau, a^\tau, p^\tau, \langle \dot{\delta}_n^\tau \mid n < \omega \rangle, \langle \dot{M}_n^\tau \mid n < \omega \rangle, \langle \dot{m}(\tau, n) \mid n < \omega \rangle) \mid \tau \in T_{k+1} \rangle,$$

where the correspondences are $\alpha^\tau = \beta, a^\tau = b, p^\tau = y, \langle \dot{\delta}_n^\tau \mid n < \omega \rangle = \langle \dot{\delta}_n^y \mid n < \omega \rangle, \langle \dot{m}(\tau, n) \mid n < \omega \rangle = \langle \dot{m}(n) \mid n < \omega \rangle$ and $\langle \dot{M}_n^\tau \mid n < \omega \rangle = \langle \dot{N}_{\dot{m}(n)}[G_{\gamma\beta}] \mid n < \omega \rangle$. This completes the construction.

Let q be a fusion of the tree representation T . Let G_α^* be P_{α^*} -generic over V with $q \in G_{\alpha^*}$. Let us calculate $\langle i_n \mid n < \omega \rangle$ from the generic cofinal path through T so that for all $k < \omega$, $\langle i_n \mid n < k \rangle \in T_k$ and $a^{\langle i_n \mid n < k \rangle} \in G_{\alpha^{\langle i_n \mid n < k \rangle}}$.

Let

$$\begin{aligned} M_n &= \dot{M}_n[G_\alpha], \\ \alpha^k &= \alpha^{\langle i_n \mid n < k \rangle}, \quad a^k = a^{\langle i_n \mid n < k \rangle}, \quad p^k = p^{\langle i_n \mid n < k \rangle}, \\ \dot{\delta}_m^k &= \dot{\delta}_m^{\langle i_n \mid n < k \rangle}, \quad \dot{M}_m^k = \dot{M}_m^{\langle i_n \mid n < k \rangle}, \quad M_m^k = \dot{M}_m^k[G_{\alpha^k}], \quad m(k, n) = \dot{m}(\langle i_0, \dots, i_k \rangle, n)[G_{\alpha^{k+1}}]. \end{aligned}$$

Then

$$a^k \in G_{\alpha^k}, \quad p^k \in G_{\alpha^*}.$$

$$M_0 =_{\omega_1} M_0[G_{\alpha^*}],$$

$$M_{m(0, m(1, \dots, m(k, 0) \dots))}[G_{\alpha^0 \alpha^1}] \cdots [G_{\alpha^k \alpha^{k+1}}] = M_0^{k+1},$$

$$M_{m(0, m(1, \dots, m(k, 0) \dots))}[G_{\alpha^0 \alpha^1}] \cdots [G_{\alpha^k \alpha^{k+1}}][G_{\alpha^{k+1} \alpha^*}] = M_0^{k+1}[G_{\alpha^{k+1} \alpha^*}],$$

$$M_{m(0, m(1, \dots, m(k, 0) \dots))}[G_{\alpha^0 \alpha^*}] \subseteq M_{m(0, m(1, \dots, m(k, 0) \dots))}[G_{\alpha^0 \alpha^1}] \cdots [G_{\alpha^k \alpha^{k+1}}][G_{\alpha^{k+1} \alpha^*}].$$

Hence

$$M_{m(0, m(1, \dots, m(k, 0) \dots))}[G_{\alpha^0 \alpha^*}] \subseteq M_0^{k+1}[G_{\alpha^{k+1} \alpha^*}] \supseteq_{\omega_1} M_0^{k+1} \supseteq_{\omega_1} M_{m(0, m(1, \dots, m(k, 0) \dots))}.$$

So

$$M_{m(0, m(1, \dots, m(k, 0) \dots))}[G_{\alpha^0 \alpha^*}] \cap \omega_1 \leq M_0^{k+1}[G_{\alpha^{k+1} \alpha^*}] \cap \omega_1 = M_0^{k+1} \cap \omega_1 = M_{m(0, m(1, \dots, m(k, 0) \dots))} \cap \omega_1.$$

So

$$M_{m(0, m(1, \dots, m(k, 0) \dots))} \subseteq_{\omega_1} M_{m(0, m(1, \dots, m(k, 0) \dots))}[G_{\alpha^0 \alpha^*}].$$

Note that $m(0, m(1, \dots, m(k, 0) \dots))$ strictly increase.

□

References

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