

A finite Toda representation of the box-ball system with box capacity

Kazuki Maeda

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

Abstract. A connection between the finite ultradiscrete Toda lattice and the box-ball system is extended to the case where each box has own capacity and a carrier has a capacity parameter depending on time. In order to consider this connection, new carrier rules “size limit for solitons” and “recovery of balls”, and a concept “expansion map” are introduced. A particular solution to the extended system of a special case is also presented.

E-mail: kmaeda@amp.i.kyoto-u.ac.jp

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1. Introduction

The box-ball system (BBS), proposed by Takahashi and Satsuma [9], is one of the most important cellular automata obtained from discrete integrable systems through a limiting procedure called ultradiscretization [12]. It is well known that the time evolution of the original BBS is determined by the ultradiscrete KdV (u-KdV) equation:

$$U_n^{(t+1)} = \min \left(1 - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) \right), \quad n, t \in \mathbb{Z}, \quad (1)$$

where $U_n^{(t)} \in \{0, 1\}$ denotes the number of balls in the n th box at time t . The equation (1) is an ultradiscrete analogue of the discrete KdV (d-KdV) equation, and consequently the original BBS has ultradiscrete soliton solutions. It is also known that the nonautonomous discrete KP (nd-KP) equation with 2-reduction condition yields a time evolution equation of the BBS with three extensions [1]: box capacity [10], carrier capacity [8], and kind of balls [11].

In this paper, we consider another type of time evolution equations of the BBS first presented by Nagai *et al* [5]. They discovered that the ultradiscrete Toda (u-Toda) equation on a (non-periodic) finite lattice

$$Q_n^{(t+1)} = \min \left(E_{n+1}^{(t)}, \sum_{j=0}^n Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)} \right), \quad (2a)$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad (2b)$$

$$E_0^{(t)} = E_N^{(t)} = +\infty, \quad (2c)$$

which we refer to as the finite u-Toda lattice simply, determines the time evolution of the original BBS. In this equation, the dependent variables $Q_n^{(t)}$ and $E_n^{(t)}$ denote the size of the n th soliton at time t and the size of the n th empty block at time t , respectively. Figure 1 shows an example of the connection between the finite u-Toda lattice and the BBS. We call this representation *finite Toda representation* of the BBS.

	$Q_0^{(0)}$	$E_1^{(0)}$	$Q_1^{(0)}$	$E_2^{(0)}$	$Q_2^{(0)}$		$Q_0^{(0)}E_1^{(0)}$	$Q_1^{(0)}E_2^{(0)}$	$Q_2^{(0)}$
t=0:	.11111		.1111	...11			5	5	4
1:		.11111		.111	...111		5	4	3
2:		.1111	...11		.11111		4	3	2
3:		.111	...111		.11111		3	2	3
4:		.11	...1111		.11111		2	3	4
5:		.11	...1111		.11111		2	5	4
6:		.11	...1111		.11111		2	7	4

Figure 1. Example of the connection between the finite u-Toda lattice and the original BBS. The variables $Q_n^{(t)}$ and $E_n^{(t)}$ denote the size of the n th soliton and of the n th empty block at time t , respectively.

The correspondence between the u-KdV equation and the finite u-Toda equation via the BBS is similar to the Euler-Lagrange correspondence of cellular automaton [3]. This terminology comes from hydrodynamics; the dependent variables of the Euler representation denote the number of particles at each point and the ones of the Lagrange representation denote the position of each particle. According to these definitions, we use the following terms in this paper:

- Euler representation of BBS: the equation of the BBS with the variables which denote the number of balls in each box.
- Lagrange representation of BBS: the equation of the BBS with the variables which denote the start position of each soliton and each empty block.
- Finite Toda representation of BBS: the equation of the BBS with the variables which denote the size of each soliton and each empty block.

The u-KdV equation (1) and the finite u-Toda lattice (2) gives the Euler representation and the Lagrange representation of the original BBS, respectively. Additionally, if we know the start position of the first soliton, we can calculate the start positions of all solitons and empty blocks from the values of the variables of the finite Toda representation. In other words, the finite Toda representation and the Lagrange representation can be transformed to each other.

It has not been clarified by now why these two different ultradiscrete equations with different type boundary conditions describe the same original BBS. Moreover, the finite Toda representation of the extended BBSs are not studied sufficiently. Among the three extensions for the BBS, two types of the finite Toda representations have been clarified: Tokihiro *et al* [11] discussed the case of with several kinds of balls using the finite ultradiscrete hungry Toda lattice; Tsujimoto and the author [2] showed that the finite nonautonomous ultradiscrete Toda (nu-Toda) lattice determines the time evolution of the BBS with a carrier. The main purpose of this paper is to discuss the

remaining case; we derive a finite Toda representation of the BBS with box capacity. For this purpose, we use a map from a state of the BBS to a binary sequence, which we call “expansion map”. By using the expansion map, we can define the size of solitons in the BBS with box capacity for all time t , especially for interacting solitons. We also consider the finite Toda representation of the BBS with both box capacity and carrier capacity using new carrier rules “size limit for solitons” and “recovery of balls”. Furthermore, we present a particular solution for the fixed box capacity case.

The outline of this paper is the following. In section 2, we recall the derivation of the Euler representation of the BBS with a carrier from the 2-reduced nd-KP equation. In addition, we introduce new carrier rules “size limit for solitons” and “recovery of balls” which are used to derive the finite Toda representation of the BBS with a carrier [2] and play an important role in section 4. In section 3, we recall the finite Toda representation of the original BBS and give some remarks. In section 4, we discuss the finite Toda representation of the BBS with variable box capacity. First we discuss the case of variable box capacity Δ_n and no restricted carrier capacity. After that, we discuss the case of both box capacity and carrier capacity are variable. In section 5, we give a particular solution to the finite Toda representation of the BBS with fixed box capacity. In section 6, we give concluding remarks.

2. Euler representation of the BBS with a carrier

The nd-KP equation is given by [13]

$$(a_n - b_t)f_{n+1}^{k,t+1}f_n^{k+1,t} + (b_t - c_k)f_{n+1}^{k,t}f_n^{k+1,t+1} + (c_k - a_n)f_n^{k,t+1}f_{n+1}^{k+1,t} = 0, \quad k, n, t \in \mathbb{Z}. \quad (3)$$

It is shown that the N -soliton solution to the nd-KP equation (3) is presented by

$$f_n^{k,t} = 1 + \sum_{\substack{J \subset \{0,1,\dots,N-1\} \\ J \neq \emptyset}} \left(\prod_{\substack{i,j \in J \\ i \neq j}} w_{i,j} \prod_{i \in J} h_{i,n}^{k,t} \right),$$

$$h_{i,n}^{k,t} := \xi_i \prod_{j=0}^{n-1} \frac{a_j - p_i}{a_j - q_i} \prod_{j=0}^{t-1} \frac{b_j - p_i}{b_j - q_i} \prod_{j=0}^{k-1} \frac{c_j - p_i}{c_j - q_i}, \quad w_{i,j} := \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)},$$

where ξ_i , p_i and q_i , $i = 0, 1, \dots, N-1$, are some constants. Now we impose the 2-reduction condition with respect to the variable k , that is $f_n^{k+2,t} = f_n^{k,t}$ and $c_{k+2} = c_k$ for all $k \in \mathbb{Z}$, and set $a_n = 1 + \delta_n$, $b_t = -\mu_t$, $c_0 = 1$ and $c_1 = 0$. Then the nd-KP equation (3) reduces to the forms

$$(1 + \delta_n + \mu_t)f_{n+1}^{0,t+1}f_n^{1,t} = (1 + \mu_t)f_{n+1}^{0,t}f_n^{1,t+1} + \delta_n f_n^{0,t+1}f_{n+1}^{1,t}, \quad (4a)$$

$$(1 + \delta_n + \mu_t)f_n^{0,t}f_{n+1}^{1,t+1} = (1 + \delta_n)f_{n+1}^{0,t}f_n^{1,t+1} + \mu_t f_n^{0,t+1}f_{n+1}^{1,t}, \quad (4b)$$

and an N -soliton solution to the reduced equations is given by

$$f_n^{k,t} = 1 + \sum_{\substack{J \subset \{0,1,\dots,N-1\} \\ J \neq \emptyset}} \left(\prod_{\substack{i,j \in J \\ i \neq j}} w_{i,j} \prod_{i \in J} h_{i,n}^{k,t} \right), \quad k = 0, 1, \quad (5a)$$

$$h_{i,n}^{0,t} := \xi_i \prod_{j=0}^{n-1} \frac{1 + \delta_j - p_i}{p_i + \delta_j} \prod_{j=0}^{t-1} \frac{p_i + \mu_j}{1 + \mu_j - p_i}, \quad h_{i,n}^{1,t} := \frac{1 - p_i}{p_i} h_{i,n}^{0,t}, \quad (5b)$$

$$w_{i,j} := \left(\frac{p_i - p_j}{1 - p_i - p_j} \right)^2. \quad (5c)$$

Let us define the dependent variables as

$$u_n^{(t)} = \frac{f_{n+1}^{0,t+1} f_n^{1,t+1}}{f_n^{0,t+1} f_{n+1}^{1,t+1}}, \quad \bar{u}_n^{(t)} = (1 + \delta_n + \mu_t) \frac{f_n^{0,t} f_{n+1}^{0,t+1}}{f_{n+1}^{0,t} f_n^{0,t+1}}, \quad \bar{z}_n^{(t)} = \frac{f_n^{0,t} f_n^{1,t+1}}{f_n^{0,t+1} f_n^{1,t}}. \quad (6)$$

Then the 2-reduced nd-KP equation (4) yields the equations

$$\bar{u}_n^{(t+1)} = \delta_n \frac{1}{u_n^{(t)}} + (1 + \mu_{t+1}) \bar{z}_n^{(t+1)}, \quad (7a)$$

$$\bar{z}_n^{(t+1)} = \frac{(1 + \delta_n) \bar{z}_{n-1}^{(t+1)} u_{n-1}^{(t)} + \mu_{t+1}}{\bar{u}_{n-1}^{(t+1)}}, \quad (7b)$$

and the identity

$$u_n^{(t+1)} = u_n^{(t)} \frac{\bar{z}_n^{(t+1)}}{\bar{z}_{n+1}^{(t+1)}} \quad (7c)$$

holds. For positivity, we choose the parameters as $0 \leq \delta_n \leq 1$ and $0 \leq \mu_t \leq 1$ for all $n, t \in \mathbb{Z}$. When the values of the dependent variables are all positive for all $n, t \in \mathbb{Z}$, we can ultradiscretize the equations (7): putting $u_n^{(t)} = e^{-U_n^{(t)}/\epsilon}$, $\bar{u}_n^{(t)} = e^{-\bar{U}_n^{(t)}/\epsilon}$, $\bar{z}_n^{(t)} = e^{-\bar{Z}_n^{(t)}/\epsilon}$, $\delta_n = e^{-\Delta_n/\epsilon}$, $\mu_t = e^{-M_t/\epsilon}$ into (7) and taking a limit $\epsilon \rightarrow +0$, we obtain the ultradiscrete system

$$\bar{U}_n^{(t+1)} = \min(\Delta_n - U_n^{(t)}, \bar{Z}_n^{(t+1)}), \quad (8a)$$

$$\bar{Z}_n^{(t+1)} = \min(\bar{Z}_{n-1}^{(t+1)} + U_{n-1}^{(t)}, M_{t+1}) - \bar{U}_{n-1}^{(t+1)}, \quad (8b)$$

$$U_n^{(t+1)} = U_n^{(t)} + \bar{Z}_n^{(t+1)} - \bar{Z}_{n+1}^{(t+1)}, \quad (8c)$$

where $\Delta_n, M_t \geq 0$ for all $n, t \in \mathbb{Z}$. Note that we have used the fundamental formula for the ultradiscretization:

$$\lim_{\epsilon \rightarrow +0} -\epsilon \log(e^{-A/\epsilon} + e^{-B/\epsilon}) = \min(A, B).$$

An N -soliton solution to the ultradiscrete system (8) is obtained as follows. Let us take the constants p_i , $i = 0, 1, \dots, N-1$, to satisfy the condition $0 < p_i < 1$. Putting $f_n^{k,t} = e^{-F_n^{k,t}/\epsilon}$, $h_{i,n}^{k,t} = e^{-H_{i,n}^{k,t}/\epsilon}$, $p_i = e^{-P_i/\epsilon}$, $\xi_i = e^{-\Xi_i/\epsilon}$, $w_{i,j} = e^{-W_{i,j}/\epsilon}$ into (5) and (6), and taking a limit $\epsilon \rightarrow +0$, we obtain

$$\begin{aligned} U_n^{(t)} &= F_{n+1}^{0,t+1} - F_n^{0,t+1} + F_n^{1,t+1} - F_{n+1}^{1,t+1}, \\ \bar{U}_n^{(t)} &= F_n^{0,t} - F_{n+1}^{0,t} + F_{n+1}^{0,t+1} - F_n^{0,t+1}, \\ \bar{Z}_n^{(t)} &= F_n^{0,t} - F_n^{0,t+1} + F_n^{1,t+1} - F_n^{1,t}, \\ F_n^{k,t} &= \min \left(0, \min_{\substack{J \subset \{0,1,\dots,N-1\} \\ J \neq \emptyset}} \left(\sum_{\substack{i,j \in J \\ i \neq j}} W_{i,j} + \sum_{i \in J} H_{i,n}^{k,t} \right) \right), \quad k = 0, 1, \\ H_{i,n}^{0,t} &= \Xi_i - \sum_{j=0}^{n-1} \min(P_i, \Delta_j) + \sum_{j=0}^{t-1} \min(P_i, M_j), \quad H_{i,n}^{1,t} = H_{i,n}^{0,t} - P_i, \\ W_{i,j} &= 2 \min(P_i, P_j), \end{aligned}$$

and $P_i \geq 0$, $i = 0, 1, \dots, N-1$.

Let us introduce the time evolution rule of the BBS with the n th box capacity Δ_n and the carrier capacity M_{t+1} from time t to $t+1$. We consider the time evolution rule from time t to $t+1$ as the composition of *size limit process* and *recovery process*.

- (i) Size limit process: the carrier of balls moves from left ($n = -\infty$) to right ($n = +\infty$). When the carrier passes each box, the carrier gets all balls in the box; and if the number of balls exceeds the carrier capacity M_{t+1} , the excess balls are removed from the system. At the same time, the carrier puts the balls which the carrier holds into the box as many as possible.
- (ii) Recovery process: after the size limit process, all the removed balls are recovered to the boxes in which the balls were.

Figure 2 illustrates these rules.

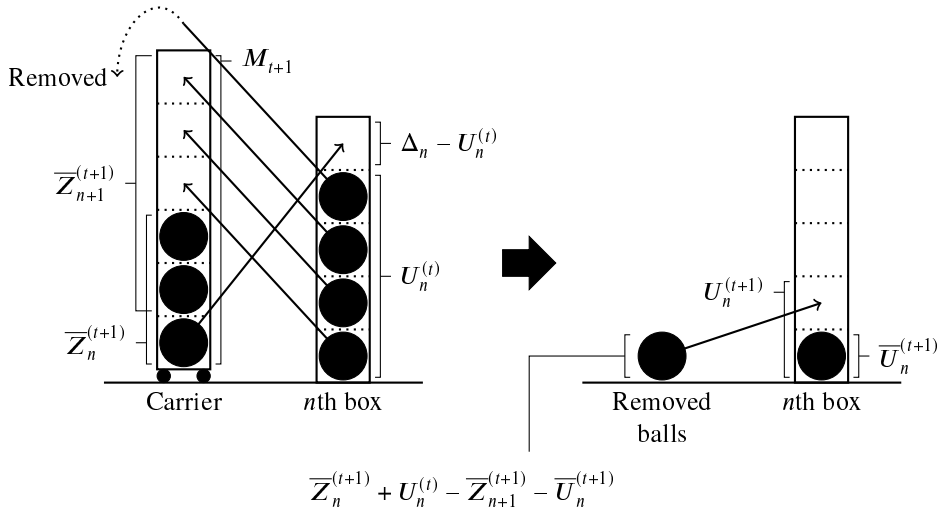


Figure 2. Illustration of the time evolution rule of the BBS with a carrier. The left figure illustrates the size limit process (8a) and (8b), and the right one illustrates the recovery process (8c).

Suppose that the dependent variables $U_n^{(t)}$, $\bar{U}_n^{(t+1)}$ and $\bar{Z}_n^{(t+1)}$ denote the following quantities:

- $U_n^{(t)} \in \{0, 1, \dots, \Delta_n\}$: the number of balls in the n th box at time t ;
- $\bar{U}_n^{(t+1)} \in \{0, 1, \dots, \Delta_n\}$: the number of balls in the n th box after the size limit process from time t to $t+1$;
- $\bar{Z}_n^{(t+1)} \in \{0, 1, \dots, M_{t+1}\}$: the number of balls in the carrier arriving at the n th box in the size limit process from time t to $t+1$.

Then the equations (8) give the time evolution rule: the equations (8a) and (8b) define the size limit process

$$\{U_n^{(t)}\}_{n=-\infty}^{+\infty} \mapsto (\{\bar{U}_n^{(t+1)}\}_{n=-\infty}^{+\infty}, \{\bar{Z}_n^{(t+1)}\}_{n=-\infty}^{+\infty})$$

and, since

$$\begin{aligned} & \bar{U}_n^{(t+1)} + \left((\bar{Z}_n^{(t+1)} + U_n^{(t)}) - \min(\bar{Z}_n^{(t+1)} + U_n^{(t)}, M_{t+1}) \right) \\ &= \bar{U}_n^{(t+1)} + \bar{Z}_n^{(t+1)} + U_n^{(t)} - \bar{Z}_{n+1}^{(t+1)} - \bar{U}_n^{(t+1)} \\ &= U_n^{(t)} + \bar{Z}_n^{(t+1)} - \bar{Z}_{n+1}^{(t+1)} \end{aligned}$$

gives the number of removed balls by the size limit at the n th box, the equation (8c) defines the recovery process

$$(\{\bar{U}_n^{(t+1)}\}_{n=-\infty}^{+\infty}, \{\bar{Z}_n^{(t+1)}\}_{n=-\infty}^{+\infty}) \mapsto \{U_n^{(t+1)}\}_{n=-\infty}^{+\infty}.$$

$\bar{U}_n^{(t)}$:	$U_n^{(t)}$:
Δ : 353535353535353535	Δ : 353535353535353535
t=0: 15213..2.....	t=0: 15213..2.....
1: ..14.5..2.....	1: .. 34 .5..2.....
2: ...13.33.2.....	2: ... 23 133.2.....
3:4.2312.....	3:4. 4 312.....
4:31.413.....	4:31 15 13.....
5:31.2231.....	5:31. 33 31.....
6:22.2.42.....	6:22.2 15 2.....
7:22.2.15.....	7:22.2. 35
8:13.2..33..	8:13.2. 23 3..
9:13.2..231	9:13.2.. 4 31

Figure 3. Example of a 3-soliton solution to the time evolution equation (8). The leftmost box is the 0th box in the both figures. The carrier capacity $M_t = 6$ for all $t \geq 1$. Each number denotes the number of balls in a box and ‘.’ denotes an empty box. In the right figure, boxes containing recovered balls are shown in boldface (compare to the left figure).

Figure 3 shows an example of a 3-soliton solution to the time evolution equation (8). The parameters are chosen as

$$\Delta_n = \begin{cases} 3 & \text{if } n \text{ is even,} \\ 5 & \text{if } n \text{ is odd,} \end{cases} \quad M_t = \begin{cases} +\infty & \text{if } t \leq 0, \\ 6 & \text{if } t > 0. \end{cases}$$

Remark 1. Eliminating the variable $\bar{U}_n^{(t+1)}$ from the equations (8a) and (8b), we have the relation

$$-\bar{Z}_{n+1}^{(t+1)} = \min(\Delta_n - U_n^{(t)}, \bar{Z}_n^{(t+1)}) - \min(\bar{Z}_n^{(t+1)} + U_n^{(t)}, M_{t+1}). \quad (9)$$

From the equation (8c), the relation

$$\bar{Z}_n^{(t+1)} = \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) \quad (10)$$

also holds. Substituting (10) into (9), we obtain the equation

$$\begin{aligned} U_n^{(t+1)} = \min & \left(\Delta_n - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) \right) \\ & + \max \left(0, \sum_{j=-\infty}^n U_j^{(t)} - \sum_{j=-\infty}^{n-1} U_j^{(t+1)} - M_{t+1} \right), \end{aligned} \quad (11)$$

where we have used the formula

$$-\min(-A, -B) = \max(A, B).$$

The equation (11) has the same form as of the time evolution equation of the “BBS with a carrier” presented by Takahashi and Matsukidaira [8].

If we choose $M_{t+1} = +\infty$, then (11) yields

$$U_n^{(t+1)} = \min \left(\Delta_n - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) \right), \quad (12)$$

which is the nonautonomous u-KdV equation. In addition, (8b) yields

$$\bar{Z}_n^{(t+1)} = \bar{Z}_{n-1}^{(t+1)} + U_{n-1}^{(t)} - \bar{U}_{n-1}^{(t+1)},$$

and comparing this relation with (8c), we have the relation $U_n^{(t+1)} = \bar{U}_n^{(t+1)}$.

3. Finite Toda representation of the original BBS

We recall the relation between the finite u-Toda lattice and the original BBS. The bilinear form of the discrete Toda lattice is given by

$$\tau_n^{(t-1)} \tau_n^{(t+1)} = \tau_{n+1}^{(t-1)} \tau_{n-1}^{(t+1)} + \tau_n^{(t)} \tau_n^{(t)}, \quad n, t \in \mathbb{Z}. \quad (13)$$

Let us introduce the dependent variables

$$q_n^{(t)} = \frac{\tau_n^{(t)} \tau_{n+1}^{(t+1)}}{\tau_{n+1}^{(t)} \tau_n^{(t+1)}}, \quad e_n^{(t)} = \frac{\tau_{n+1}^{(t)} \tau_{n-1}^{(t+1)}}{\tau_n^{(t)} \tau_n^{(t+1)}}, \quad d_n^{(t)} = \frac{\tau_n^{(t)} \tau_{n+1}^{(t)}}{\tau_{n+1}^{(t-1)} \tau_n^{(t+1)}}.$$

Then (13) yields the equation

$$q_n^{(t+1)} = e_{n+1}^{(t)} + d_n^{(t+1)}, \quad (14a)$$

and the identities

$$e_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \quad d_n^{(t+1)} = d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} \quad (14b)$$

hold. Putting $q_n^{(t)} = e^{-Q_n^{(t)}/\epsilon}$, $e_n^{(t)} = e^{-E_n^{(t)}/\epsilon}$, $d_n^{(t)} = e^{-D_n^{(t)}/\epsilon}$, and taking a limit $\epsilon \rightarrow +0$, we obtain the u-Toda lattice

$$Q_n^{(t+1)} = \min(E_{n+1}^{(t)}, D_n^{(t+1)}), \quad (15a)$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad (15b)$$

$$D_n^{(t+1)} = D_{n-1}^{(t+1)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}. \quad (15c)$$

Furthermore, we impose the terminating condition for discussing the finite Toda representation:

$$E_0^{(t)} = E_N^{(t)} = +\infty, \quad D_0^{(t+1)} = Q_0^{(t)}, \quad (15d)$$

where N is a positive integer, which denotes the number of solitons in the original BBS.

Let the variables $Q_n^{(t)}$, $E_n^{(t)}$ and $D_n^{(t+1)}$, respectively, denote the following quantities of the original BBS:

- $Q_n^{(t)}$: the size of the n th soliton at time t ($n = 0, 1, \dots, N-1$);

- $E_n^{(t)}$: the size of the n th empty block, namely, the distance between the $(n-1)$ th soliton and the n th one at time t ($n = 1, 2, \dots, N-1$);
- $D_n^{(t+1)}$: the number of balls in the carrier after getting $Q_n^{(t)}$ balls ($n = 0, 1, \dots, N-1$).

Then the next theorem gives a fundamental result on the connection between the finite u-Toda lattice and the BBS.

Theorem 1 (Nagai *et al* [5]). *The finite u-Toda lattice (15) determines the time evolution of the original BBS.*

Remark 2. Conventionally, the d-Toda lattice (qd-type) has been written in the form

$$q_n^{(t+1)} + e_n^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)}, \quad q_{n-1}^{(t+1)} e_n^{(t+1)} = q_n^{(t)} e_n^{(t)},$$

or

$$q_n^{(t+1)} = q_n^{(t)} - e_n^{(t+1)} + e_n^{(t)}, \quad (16a)$$

$$e_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \quad (16b)$$

which we cannot ultradiscretize directly due to “negative problem”. The equations (16) are called Rutishauser’s qd algorithm in numerical algorithms [6]. On the other hand, the equations (14) are called dqd algorithm. The dqd algorithm is the subtraction-free form of the qd algorithm and computes matrix eigenvalues or singular values more accurately than the qd algorithm.

Suppose the finite lattice condition $e_0^{(t)} = e_N^{(t)} = 0$. Nagai *et al* [5] rewrote (16a) using (16b) as

$$\begin{aligned} q_n^{(t+1)} &= e_{n+1}^{(t)} + q_n^{(t)} - e_n^{(t+1)} \\ &= e_{n+1}^{(t)} + \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} (q_{n-1}^{(t+1)} - e_n^{(t)}) \\ &= e_{n+1}^{(t)} + \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} (q_{n-1}^{(t)} - e_{n-1}^{(t+1)}) \\ &= \dots \\ &= e_{n+1}^{(t)} + \frac{\prod_{j=0}^n q_j^{(t)}}{\prod_{j=0}^{n-1} q_j^{(t+1)}}. \end{aligned}$$

Then they could ultradiscretize the finite Toda lattice:

$$Q_n^{(t+1)} = \min \left(E_{n+1}^{(t)}, \sum_{j=0}^n Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)} \right), \quad (17a)$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad (17b)$$

$$E_0^{(t)} = E_N^{(t)} = +\infty. \quad (17c)$$

On the other hand, by introducing an auxiliary variable

$$d_n^{(t+1)} := q_n^{(t)} - e_n^{(t+1)} = q_n^{(t+1)} - e_{n+1}^{(t)},$$

we can ultradiscretize the finite d-Toda lattice directly without the negative problem and obtain the finite u-Toda lattice of the dqd form (15). From the viewpoint of the

BBS, the variable $D_n^{(t+1)}$ denotes the number of balls in the carrier. Therefore, the finite u-Toda lattice of the dqd form (15) is important to consider the finite Toda representation of the BBS with a carrier.

Remark 3. Here we remark on the Lagrange representation of the BBS, which is also a terminology from hydrodynamics as the Euler representation; the dependent variables of the Lagrange representation denote the position of solitons. Let the variables $X_n^{(t)}$ and $Y_n^{(t)}$ denote the start position of the n th soliton and the one of the n th empty block at time t , respectively (see Figure 4). Then the Lagrange representation of the BBS is given by [4]

$$X_n^{(t+1)} = Y_{n+1}^{(t)}, \quad (18a)$$

$$Y_n^{(t+1)} = Y_n^{(t)} + \min \left(X_n^{(t)} - Y_n^{(t)}, \sum_{j=1}^n (Y_j^{(t)} - X_{j-1}^{(t)}) - \sum_{j=1}^{n-1} (Y_j^{(t+1)} - X_{j-1}^{(t+1)}) \right), \quad (18b)$$

$$Y_0^{(t)} = -\infty, \quad X_N^{(t)} = +\infty. \quad (18c)$$

Relations between these variables and the variables of the finite u-Toda lattice (17) are given by

$$X_n^{(t)} = Y_n^{(t)} + E_n^{(t)}, \quad Y_n^{(t)} = X_{n-1}^{(t)} + Q_{n-1}^{(t)}. \quad (19)$$

We can readily show that (18) and (19) yield the finite u-Toda lattice (17). Conversely, we can calculate the values of $\{X_n^{(t)}\}_{n=1}^{N-1}$ and $\{Y_n^{(t)}\}_{n=1}^N$ from the values of $X_0^{(t)}$, $\{Q_n^{(t)}\}_{n=0}^{N-1}$ and $\{E_n^{(t)}\}_{n=1}^{N-1}$ using the relations (19). In other words, the finite u-Toda lattice (17) and the equation $X_0^{(t+1)} = X_0^{(t)} + Q_0^{(t)}$, which is obtained from (18a) and (19), uniquely determine the time evolution of the BBS.

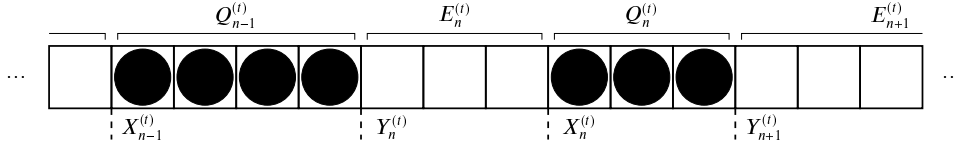


Figure 4. Lagrange representation and finite Toda representation of the BBS.

4. Extension of the finite Toda representation to the case of variable box capacity Δ_n and variable carrier capacity M_t

In previous studies, the finite Toda representation is considered only for the BBS with box capacity 1. In this section, we extend the finite Toda representation to the case in which each box has own capacity Δ_n . First we consider the case of carrier capacity $M_t = +\infty$. The Euler representation of this case is given by (12).

We first define the size of solitons and the one of empty blocks for the BBS with variable box capacity Δ_n at any time t . For this purpose, we refer to the work of Takahashi and Satsuma [10]. They analyzed the BBS with the fixed box capacity Δ using a map from a state of box capacity Δ to a binary sequence. We generalize this map for the case of variable box capacity Δ_n .

- (1) $V_n^{(t)} = 0$ for $n < 0$.
- (2) Let $s_0 = 0$ and $s_n = \sum_{j=0}^{n-1} \Delta_j$ for $n = 1, 2, \dots$. From $n = 0$ to $+\infty$, if $V_{s_{n-1}}^{(t)} = 1$, then

$$\begin{aligned} V_{s_n}^{(t)} &= V_{s_n+1}^{(t)} = \dots = V_{s_n-U_n^{(t)}-1+\Delta_n}^{(t)} = 0, \\ V_{s_n-U_n^{(t)}+\Delta_n}^{(t)} &= V_{s_n-U_n^{(t)}+\Delta_n+1}^{(t)} = \dots = V_{s_n+\Delta_n-1}^{(t)} = 1. \end{aligned}$$

Figures 5 and 6 show examples of the expansion map. As shown in Figure 6, the expansion map enables us to define the size of the n th soliton $Q_n^{(t)}$ and the one of the n th empty block $E_n^{(t)}$ for the BBS with box capacity Δ_n at any time t in the same way as for the BBS with box capacity 1. Let $D_n^{(t+1)}$ denote the number of balls which the

carrier holds after getting $Q_n^{(t)}$ balls, and $\Lambda_n^{(t)}$ denote the capacity of the box which contains the beginning (leftmost) segment of the n th empty block. Then we arrive at the following theorem.

Theorem 2. *Let the variables $Q_n^{(t)}$, $E_n^{(t)}$ and $D_n^{(t+1)}$ denote the quantities of the BBS as explained in the previous section. Then the time evolution of the BBS with variable box capacity $\Lambda_n^{(t)}$ is given by*

$$Q_n^{(t+1)} = \min \left(E_{n+1}^{(t)} - \max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)}), D_n^{(t+1)} \right), \quad (20a)$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)} - \max(0, \Lambda_n^{(t)} - D_{n-1}^{(t+1)}) + \max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)}), \quad (20b)$$

$$D_n^{(t+1)} = D_{n-1}^{(t+1)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad (20c)$$

$$E_0^{(t)} = E_N^{(t)} = +\infty, \quad D_0^{(t+1)} = Q_0^{(t)}. \quad (20d)$$

We note that, from (20a), $Q_n^{(t+1)} \leq D_n^{(t+1)}$ holds for all n and t . Since the size of the n th soliton $Q_n^{(t)}$ should be positive for all n and t , from (20c), the inequality $D_n^{(t+1)} \geq 1$ holds for all n and t . Thus, all $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ are equal to zero when $\Lambda_n^{(t)} = 1$ for all n and t , the case of the original BBS. In this case, the equations (20) reduce to the finite Toda representation of the original BBS (15). Hence we can say that the ultradiscrete system (20) is a generalization of the finite u-Toda lattice (15).

Proof. Let us consider the general $\Lambda_n^{(t)} \geq 1$ case. As we mentioned above, (20) has additional terms $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ which do not appear in the case of box capacity 1 (15). Hence, we shall investigate the role of the terms $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$.

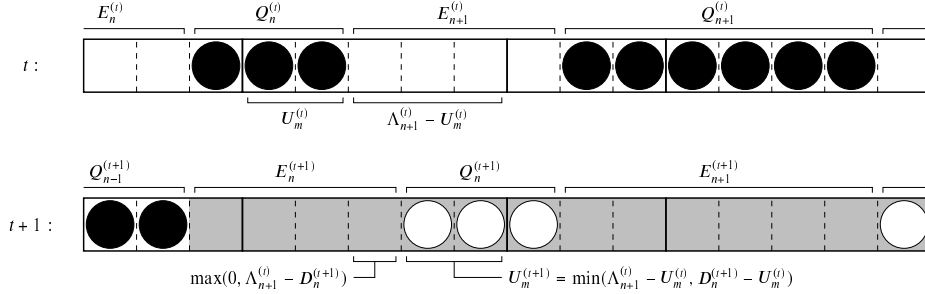


Figure 7. Illustration of the quantity $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$. We now need to determine the quantities on the area filled with gray: $E_n^{(t+1)}$, $Q_n^{(t+1)}$, $E_{n+1}^{(t+1)}$, $\dots$, $Q_{N-1}^{(t+1)}$. In this figure, $D_{n-1}^{(t+1)} - Q_{n-1}^{(t+1)} = 1$ is assumed; the carrier has one ball just before getting $Q_n^{(t)}$ balls.

Let us consider the time evolution of the BBS with box capacity Δ_n from time t to $t+1$. Assume that $Q_j^{(t+1)}$, $j = 0, 1, \dots, n-1$, and $E_j^{(t+1)}$, $j = 1, 2, \dots, n-1$, are given (see Figure 7). Let m be the index of the box which contains the leftmost

segment of the $(n+1)$ th empty block at time t . Then the capacity of the m th box Δ_m is equal to $\Lambda_{n+1}^{(t)}$ by definition. Moreover, the relation

$$D_n^{(t+1)} = \sum_{j=0}^n Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)} = \sum_{j=-\infty}^{m-1} (U_j^{(t)} - U_j^{(t+1)}) + U_m^{(t)},$$

where $U_k^{(t)}$ denotes the number of balls in the k th box at time t , also holds by definition. Hence, we can calculate the quantity $U_m^{(t+1)}$ by the nu-KdV equation (12):

$$\begin{aligned} U_m^{(t+1)} &= \min \left(\Delta_m - U_m^{(t)}, \sum_{j=-\infty}^{m-1} (U_j^{(t)} - U_j^{(t+1)}) \right) \\ &= \min(\Lambda_{n+1}^{(t)} - U_m^{(t)}, D_n^{(t+1)} - U_m^{(t)}). \end{aligned}$$

Then we obtain the relation

$$\begin{aligned} \Delta_m - U_m^{(t)} - U_m^{(t+1)} &= \Lambda_{n+1}^{(t)} - U_m^{(t)} - \min(\Lambda_{n+1}^{(t)} - U_m^{(t)}, D_n^{(t+1)} - U_m^{(t)}) \\ &= -\min(0, D_n^{(t+1)} - \Lambda_{n+1}^{(t)}) \\ &= \max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)}), \end{aligned}$$

where we have used the identity $-\min(-A, -B) = \max(A, B)$. This relation implies that the term $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ denotes the size of interspace inserted between the n th soliton at time t and the n th one at time $t+1$.

Once we notice the role of the terms $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$, we can now clarify the meaning of the equations (20a) and (20b). Since the term $E_{n+1}^{(t)} - \max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ in (20a) denotes the difference between the size of the inserted space and the one of the $(n+1)$ th empty block, $Q_n^{(t+1)}$ should be determined by (20a). Similarly, $E_n^{(t+1)}$ should be determined by (20b). It is also true for $n=0$ and 1 , then the proof is completed by induction. $\square$

Next, we construct the finite Toda representation of the BBS with both box capacity and carrier capacity from two time evolution maps: the size limit map and the recovery map. This is similar as for the construction of the Euler representation explained in section 2. Figure 8 shows an example.

In the next theorem, we use the following notations:

- $Q_n^{(t)}, E_n^{(t)}$: the size of the n th soliton and the one of the n th empty block at time t , respectively;
- $\overline{Q}_n^{(t+1)}, \overline{E}_n^{(t+1)}$: the size of the n th soliton and the one of the n th empty block after the size limit process from time t to $t+1$;
- $\overline{C}_n^{(t+1)}, \overline{D}_n^{(t+1)}$: some quantities which will be explained in the proof of the next theorem in detail;
- $K_n^{(t)}, \Lambda_n^{(t)}$: the capacity of the box which contains the leftmost segment of the n th soliton and the one of the n th empty block at time t , respectively.

Figure 8. Example of the expansion map for the BBS with box capacity Δ_n and carrier capacity $M_t = 6$ for $t > 0$. This is obtained from the example in Figure 3 via the expansion map.

If $M_{t+1} = +\infty$, then (21c) and (21d) reduce to $\overline{C}_n^{(t+1)} = \overline{D}_{n-1}^{(t+1)} - \overline{Q}_{n-1}^{(t+1)} + K_n^{(t)}$ and $\overline{D}_n^{(t+1)} = \overline{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}$, respectively. Thus we have the equation $\overline{D}_n^{(t+1)} = \overline{D}_{n-1}^{(t+1)} - \overline{Q}_{n-1}^{(t+1)} + Q_n^{(t)}$ and, substituting them into (21e) and (21f), we obtain $Q_n^{(t+1)} = \overline{Q}_n^{(t+1)}$ and $E_n^{(t+1)} = \overline{E}_n^{(t+1)}$. Hence, in this case, the ultradiscrete system (21) reduces to the system (20). We can therefore say that the system (21) is

a generalization of the system (20).

Proof. Let us show that the equations (21a)–(21d) describe the size limit process and (21e)–(21f) describe the recovery process.

First, we consider the size limit process. The equations (21a) and (21b) have the same forms as of (20a) and (20b). Thus we shall investigate the variables $\bar{C}_n^{(t+1)}$ and $\bar{D}_n^{(t+1)}$ which are defined by (21c) and (21d). Suppose that the carrier capacity is chosen as $K_n^{(t)} \leq M_{t+1} < +\infty$, $\bar{Q}_j^{(t+1)}$, $j = 0, 1, \dots, n-1$, and $\bar{E}_j^{(t+1)}$, $j = 1, 2, \dots, n-1$, are given, and the quantity $\bar{D}_{n-1}^{(t+1)}$ denotes the number of balls which the carrier holds after getting $Q_{n-1}^{(t)}$ balls from boxes and restricting the number of the holding balls to M_{t+1} balls. Since the inequality $\bar{Q}_{n-1}^{(t+1)} \leq \bar{D}_{n-1}^{(t+1)}$ holds from (21a), it is sufficient to consider the following two cases: the case of which the carrier drops off all balls temporarily ($\bar{D}_{n-1}^{(t+1)} - \bar{Q}_{n-1}^{(t+1)} = 0$) and the case of which the carrier has balls just before getting $Q_n^{(t)}$ balls ($\bar{D}_{n-1}^{(t+1)} - \bar{Q}_{n-1}^{(t+1)} > 0$).

- (i) If $\bar{D}_{n-1}^{(t+1)} - \bar{Q}_{n-1}^{(t+1)} = 0$, then $\bar{C}_n^{(t+1)} = \min(\bar{D}_{n-1}^{(t+1)} - \bar{Q}_{n-1}^{(t+1)} + K_n^{(t)}, M_{t+1}) = K_n^{(t)}$ holds from the assumption. We should note that, in this case, the number of balls which the carrier holds is zero temporarily before getting $Q_n^{(t)}$ balls. Thus, from (21d), we have $\bar{D}_n^{(t+1)} = \min(Q_n^{(t)}, M_{t+1})$, which indicates that the quantity $\bar{D}_n^{(t+1)}$ is again the number of balls which the carrier holds after getting $Q_n^{(t)}$ balls and restricting the number of the holding balls to M_{t+1} balls.
- (ii) The case of $\bar{D}_{n-1}^{(t+1)} - \bar{Q}_{n-1}^{(t+1)} > 0$. Let m be the index of the box which contains the leftmost segment of the n th soliton at time t . Under the assumption, in the terms of the variables of the Euler representation (8), $\bar{D}_{n-1}^{(t+1)} - \bar{Q}_{n-1}^{(t+1)} > 0$ implies that $\bar{U}_m^{(t+1)} = \Delta_m - U_m^{(t)}$ should hold (see Figure 9). Now $\Delta_m = K_n^{(t)}$ by definition.

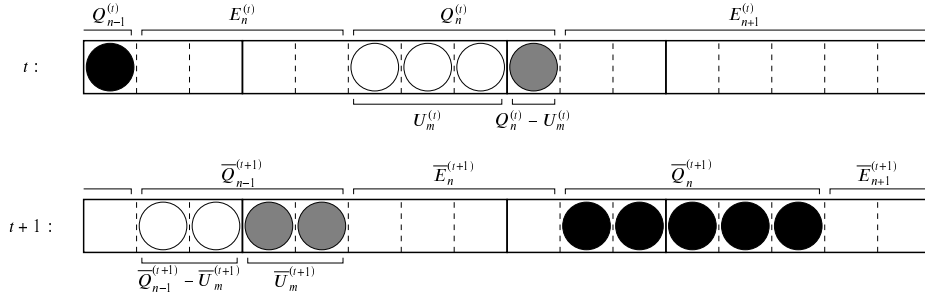


Figure 9. Illustration of the size limit process when the carrier parameter $M_{t+1} = 6$ and $\bar{D}_{n-1}^{(t+1)} - \bar{Q}_{n-1}^{(t+1)} > 0$. We can see that $\bar{C}_n^{(t+1)}$ denotes the number of balls which the carrier holds after getting $U_m^{(t)}$ balls (white balls) and restricting the number of the holding balls to M_{t+1} , and $\bar{D}_n^{(t+1)}$ denotes the number of balls which the carrier holds after getting $Q_n^{(t)} - U_m^{(t)}$ balls (gray balls) and restricting the number of the holding balls to M_{t+1} balls.

Hence we can write (21c) and (21d) as $\bar{C}_n^{(t+1)} = \min(\bar{D}_{n-1}^{(t+1)} - (\bar{Q}_{n-1}^{(t+1)} - \bar{U}_m^{(t+1)}) + U_m^{(t)}, M_{t+1})$ and $\bar{D}_n^{(t+1)} = \min(\bar{C}_n^{(t+1)} + (Q_n^{(t)} - U_m^{(t)}) - \bar{U}_m^{(t+1)}, M_{t+1})$, respectively.

Table 1. Change of the number of balls which the carrier holds.

State of the carrier	The number of balls which the carrier holds
$\vdots$	$\vdots$
Getting $Q_{n-1}^{(t)}$ balls	$\overline{D}_{n-1}^{(t+1)}$
Putting $\overline{Q}_{n-1}^{(t+1)} - \overline{U}_m^{(t+1)}$ balls	$\overline{D}_{n-1}^{(t+1)} - (\overline{Q}_{n-1}^{(t+1)} - \overline{U}_m^{(t+1)})$
Getting $U_m^{(t)}$ balls	$\overline{D}_{n-1}^{(t+1)} - \overline{Q}_{n-1}^{(t+1)} + K_n^{(t)}$
Size limit to M_{t+1} balls	$\overline{C}_n^{(t+1)}$
Putting $\overline{U}_m^{(t+1)}$ balls	$\overline{C}_n^{(t+1)} - \overline{U}_m^{(t+1)}$
Getting $Q_n^{(t)} - U_m^{(t)}$ balls	$\overline{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}$
Size limit to M_{t+1} balls	$\overline{D}_n^{(t+1)}$
$\vdots$	$\vdots$

Therefore, the quantity $\overline{D}_n^{(t+1)}$ is again the number of balls which the carrier holds after getting $Q_n^{(t)}$ balls and restricting the number of the holding balls to M_{t+1} balls. We can summarize the change of the number of balls which the carrier holds as Table 1.

Thus, together with the proof of Theorem 3, it is proved that (21a)–(21d) describe the size limit process by induction.

Furthermore, since the number of balls removed by the size limit process are given by $(\overline{D}_{n-1}^{(t+1)} - \overline{Q}_{n-1}^{(t+1)} + K_n^{(t)}) - \overline{C}_n^{(t+1)}$ and $(\overline{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}) - \overline{D}_n^{(t+1)}$, we obtain the equations of the recovery process

$$\begin{aligned}
Q_n^{(t+1)} &= \overline{Q}_n^{(t+1)} + (\overline{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)} - \overline{D}_n^{(t+1)}) \\
&\quad + (\overline{D}_n^{(t+1)} - \overline{Q}_n^{(t+1)} + K_{n+1}^{(t)} - \overline{C}_{n+1}^{(t+1)}) \\
&= Q_n^{(t)} + \overline{C}_n^{(t+1)} - \overline{C}_{n+1}^{(t+1)} - K_n^{(t)} + K_{n+1}^{(t)}, \\
E_n^{(t+1)} &= \overline{E}_n^{(t+1)} - (\overline{D}_{n-1}^{(t+1)} - \overline{Q}_{n-1}^{(t+1)} + K_n^{(t)} - \overline{C}_n^{(t+1)}) \\
&\quad - (\overline{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)} - \overline{D}_n^{(t+1)}) \\
&= \overline{E}_n^{(t+1)} + \overline{Q}_{n-1}^{(t+1)} - Q_n^{(t)} - \overline{D}_{n-1}^{(t+1)} + \overline{D}_n^{(t+1)},
\end{aligned}$$

which lead to the equations (21e) and (21f), and the proof is completed. $\square$

Remark 4. Furthermore, the variables $X_0^{(t)}$ and $\overline{X}_0^{(t)}$, which denote the index of the leftmost segment of the 0th soliton at time t satisfy the equations

$$\begin{aligned}
\overline{X}_0^{(t+1)} &= X_0^{(t)} + Q_0^{(t)} + \max(0, \Lambda_1^{(t)} - \overline{D}_0^{(t+1)}), \\
X_0^{(t+1)} &= \overline{X}_0^{(t+1)} - Q_0^{(t)} + \overline{D}_0^{(t+1)} = X_0^{(t)} + \max(\overline{D}_0^{(t+1)}, \Lambda_1^{(t)}).
\end{aligned}$$

5. Particular solution for the fixed box capacity case

In this section, we discuss a particular solution to the ultradiscrete system (21) with a special condition: all boxes have constant capacity Δ .

Let us consider the bilinear equations

$$\bar{\tau}_n^{0,t+1} \tau_n^{1,t-1} = \delta \tau_{n+1}^{0,t-1} \bar{\tau}_{n-1}^{1,t+1} + \tau_n^{0,t} \bar{\tau}_n^{1,t}, \quad (23a)$$

$$\delta \bar{\tau}_n^{0,t} \tau_n^{1,t} = (\delta - \mu_t) \tau_n^{0,t} \bar{\tau}_n^{1,t} + \mu_t \bar{\tau}_n^{0,t+1} \tau_n^{1,t-1}, \quad (23b)$$

$$\tau_{n+1}^{0,t} \bar{\tau}_n^{1,t} = \bar{\tau}_{n+1}^{0,t} \tau_n^{1,t} + \mu_t \bar{\tau}_n^{0,t+1} \tau_{n+1}^{1,t-1}, \quad (23c)$$

$$(\delta - \mu_t) \bar{\tau}_n^{0,t+1} \tau_{n+1}^{1,t-1} + \tau_{n+1}^{0,t} \bar{\tau}_n^{1,t} = \tau_{n+1}^{0,t-1} \bar{\tau}_n^{1,t+1}, \quad (23d)$$

where δ is a constant and μ_t is a parameter depending on t . We introduce the dependent variables

$$\begin{aligned} q_n^{(t)} &= \frac{\bar{\tau}_{n+1}^{0,t+1} \tau_n^{1,t}}{\bar{\tau}_n^{0,t+1} \tau_{n+1}^{1,t}}, & \bar{q}_n^{(t)} &= \frac{\delta}{\delta - \mu_t} \frac{\bar{\tau}_{n+1}^{0,t+1} \bar{\tau}_n^{1,t}}{\bar{\tau}_n^{0,t+1} \bar{\tau}_{n+1}^{1,t}}, \\ e_n^{(t)} &= \delta^2 \frac{\tau_{n+1}^{0,t} \bar{\tau}_{n-1}^{1,t+1}}{\tau_n^{0,t} \bar{\tau}_{n-1}^{1,t+1}}, & \bar{e}_n^{(t)} &= \delta(\delta - \mu_t) \frac{\bar{\tau}_{n+1}^{0,t} \bar{\tau}_{n-1}^{1,t+1}}{\bar{\tau}_n^{0,t} \bar{\tau}_{n-1}^{1,t+1}}, \\ \bar{c}_n^{(t)} &= \delta \frac{\bar{\tau}_n^{0,t} \tau_n^{1,t}}{\bar{\tau}_n^{0,t+1} \tau_n^{1,t-1}}, & \bar{d}_n^{(t)} &= \frac{\delta}{\delta - \mu_t} \frac{\tau_{n+1}^{0,t} \bar{\tau}_n^{1,t}}{\bar{\tau}_n^{0,t+1} \tau_{n+1}^{1,t-1}}. \end{aligned}$$

Then (23d) yields the relation

$$1 + \delta^{-1} d_n^{(t)} = (\delta - \mu_t)^{-1} \frac{\tau_{n+1}^{0,t-1} \bar{\tau}_n^{1,t+1}}{\bar{\tau}_n^{0,t+1} \tau_{n+1}^{1,t-1}}.$$

Further, (23a)–(23c) yield the equations

$$\bar{q}_n^{(t+1)} = e_{n+1}^{(t)} (1 + \delta^{-1} \bar{d}_n^{(t+1)}) + \bar{d}_n^{(t+1)}, \quad (24a)$$

$$\bar{c}_n^{(t+1)} = (\delta - \mu_{t+1}) \frac{\bar{d}_{n-1}^{(t+1)}}{\bar{q}_{n-1}^{(t+1)}} + \mu_{t+1}, \quad (24b)$$

$$\bar{d}_n^{(t+1)} = (\delta - \mu_{t+1})^{-1} \bar{c}_n^{(t+1)} q_n^{(t)} + \frac{\mu_{t+1}}{1 - \delta^{-1} \mu_{t+1}}, \quad (24c)$$

and the identities

$$\bar{e}_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{\bar{q}_{n-1}^{(t+1)}} \frac{1 + \delta^{-1} \bar{d}_{n-1}^{(t+1)}}{1 + \delta^{-1} \bar{d}_n^{(t+1)}}, \quad (24d)$$

$$q_n^{(t+1)} = q_n^{(t)} \frac{\bar{c}_n^{(t+1)}}{\bar{c}_{n+1}^{(t+1)}}, \quad (24e)$$

$$e_n^{(t+1)} = \bar{e}_n^{(t+1)} \frac{\bar{q}_{n-1}^{(t+1)} \bar{d}_n^{(t+1)}}{q_n^{(t)} \bar{d}_{n-1}^{(t+1)}}, \quad (24f)$$

hold. In addition, we impose the finite lattice condition

$$e_0^{(t)} = e_N^{(t)} = \bar{e}_0^{(t)} = \bar{e}_N^{(t)} = 0. \quad (24g)$$

In the bilinear equations (23), this condition implies

$$\tau_{-1}^{k,t} = \tau_{N+1}^{k,t} = \bar{\tau}_{-1}^{k,t} = \bar{\tau}_{N+1}^{k,t} = 0.$$

We assume that the constant δ and the parameter μ_t satisfy the condition $0 < \mu_t < \delta$ for all $t \in \mathbb{Z}$. Then, putting $q_n^{(t)} = e^{-Q_n^{(t)}/\epsilon}$, $e_n^{(t)} = e^{-E_n^{(t)}/\epsilon}$, $\bar{q}_n^{(t)} = e^{-\bar{Q}_n^{(t)}/\epsilon}$, $\bar{e}_n^{(t)} = e^{-\bar{E}_n^{(t)}/\epsilon}$, $\bar{c}_n^{(t)} = e^{-\bar{C}_n^{(t)}/\epsilon}$, $\bar{d}_n^{(t)} = e^{-\bar{D}_n^{(t)}/\epsilon}$, $\delta = e^{-\Delta/\epsilon}$ into (24)

and taking a limit $\epsilon \rightarrow +0$, we obtain the ultradiscrete system (21) with the condition $K_n^{(t)} = \Lambda_n^{(t)} = \Delta \leq M_{t+1}$ for all $n, t \in \mathbb{Z}$.

The following theorem is proved by using a determinant identity called the Plücker relation.

Theorem 4. *A particular solution to the bilinear equations (23) with the semi-infinite lattice condition $\tau_{-1}^{k,t} = \bar{\tau}_{-1}^{k,t} = 0$ for all $k, t \in \mathbb{Z}$ is given by the Hankel determinants*

$$\tau_n^{k,t} = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ |\xi_{k+i+j}^{(t)}|_{0 \leq i,j \leq n-1} & \text{if } n > 0, \end{cases} \quad (25a)$$

$$\bar{\tau}_n^{k,t} = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ |\bar{\xi}_{k+i+j}^{(t)}|_{0 \leq i,j \leq n-1} & \text{if } n > 0, \end{cases} \quad (25b)$$

where $\xi_n^{(t)}$ and $\bar{\xi}_n^{(t)}$ are arbitrary functions satisfying the dispersion relation

$$\bar{\xi}_n^{(t+1)} = -\delta \xi_{n+1}^{(t)} + \xi_n^{(t)} = (\mu_t - \delta) \bar{\xi}_{n+1}^{(t)} + \bar{\xi}_n^{(t)}, \quad n = 0, 1, \dots \quad (26)$$

Hereafter, we choose the arbitrary functions as

$$\xi_n^{(t)} = \sum_{i=0}^{N-1} \frac{\eta_i^{(t)}}{p_i(p_i + \delta)^n}, \quad \bar{\xi}_n^{(t)} = \sum_{i=0}^{N-1} \frac{\eta_i^{(t-1)}}{(p_i + \delta)^{n+1}}, \quad \eta_i^{(t)} := \frac{w_i \prod_{j=0}^t (p_i + \mu_j)}{(p_i + \delta)^t}, \quad (27)$$

where p_i and w_i , $i = 0, 1, \dots, N-1$, are some constants. Then the dispersion relation (26) is satisfied and the finite lattice condition $\tau_{-1}^{k,t} = \tau_{N+1}^{k,t} = \bar{\tau}_{-1}^{k,t} = \bar{\tau}_{N+1}^{k,t} = 0$ holds for all $k, t \in \mathbb{Z}$. Substituting (27) to (25) and expanding the Hankel determinants using the Cauchy-Binet formula, we obtain

$$\tau_n^{k,t} = \sum_{0 \leq r_0 < r_1 < \dots < r_{n-1} \leq N-1} \left(\left(\prod_{0 \leq i < j \leq n-1} \frac{p_{r_i} - p_{r_j}}{(p_{r_i} + \delta)(p_{r_j} + \delta)} \right)^2 \prod_{i=0}^{n-1} \frac{w_{r_i} \prod_{j=0}^t (p_{r_i} + \mu_j)}{p_{r_i} (p_{r_i} + \delta)^{t+k}} \right),$$

$$\bar{\tau}_n^{k,t} = \sum_{0 \leq r_0 < r_1 < \dots < r_{n-1} \leq N-1} \left(\left(\prod_{0 \leq i < j \leq n-1} \frac{p_{r_i} - p_{r_j}}{(p_{r_i} + \delta)(p_{r_j} + \delta)} \right)^2 \prod_{i=0}^{n-1} \frac{w_{r_i} \prod_{j=0}^{t-1} (p_{r_i} + \mu_j)}{(p_{r_i} + \delta)^{t+k}} \right),$$

for $n = 1, 2, \dots, N$. These expressions can be ultradiscretized directly: putting $p_n = e^{-P_n/\epsilon}$, $w_n = e^{-W_n/\epsilon}$, $\tau_n^{k,t} = e^{-T_n^{k,t}/\epsilon}$, $\bar{\tau}_n^{k,t} = e^{-\bar{T}_n^{k,t}/\epsilon}$, and taking a limit $\epsilon \rightarrow +0$, we obtain the next theorem.

Theorem 5. *A particular solution to the ultradiscrete system (21) with the condition $K_n^{(t)} = \Lambda_n^{(t)} = \Delta \leq M_{t+1}$ for all $n, t \in \mathbb{Z}$ is given by*

$$\begin{aligned} Q_n^{(t)} &= \bar{T}_{n+1}^{0,t+1} - \bar{T}_n^{0,t+1} + T_n^{1,t} - T_{n+1}^{1,t}, & \bar{Q}_n^{(t)} &= \bar{T}_{n+1}^{0,t+1} - \bar{T}_n^{0,t+1} + \bar{T}_n^{1,t} - \bar{T}_{n+1}^{1,t}, \\ E_n^{(t)} &= T_{n+1}^{0,t} - T_n^{0,t} + \bar{T}_{n-1}^{1,t+1} - \bar{T}_n^{1,t+1} + 2\Delta, & \bar{E}_n^{(t)} &= \bar{T}_{n+1}^{0,t} - \bar{T}_n^{0,t} + \bar{T}_{n-1}^{1,t+1} - \bar{T}_n^{1,t+1} + 2\Delta, \\ \bar{C}_n^{(t)} &= \bar{T}_n^{0,t} - \bar{T}_n^{0,t+1} + T_n^{1,t} - T_n^{1,t-1} + \Delta, & \bar{D}_n^{(t)} &= T_{n+1}^{0,t} - \bar{T}_n^{0,t+1} + \bar{T}_n^{1,t} - T_{n+1}^{1,t-1}, \end{aligned}$$

$$T_n^{k,t} = \min_{0 \leq r_0 < r_1 < \dots < r_{n-1} \leq N-1} \left(\sum_{i=0}^{n-1} \left(W_{r_i} + (2(n-1-i)-1)P_{r_i} - (2(n-1) + t + k) \min(P_{r_i}, \Delta) + \sum_{j=0}^t \min(P_{r_i}, M_j) \right) \right), \quad n = 1, 2, \dots, N,$$

$$\bar{T}_n^{k,t} = \min_{0 \leq r_0 < r_1 < \dots < r_{n-1} \leq N-1} \left(\sum_{i=0}^{n-1} \left(W_{r_i} + 2(n-1-i)P_{r_i} - (2(n-1) + t + k) \min(P_{r_i}, \Delta) + \sum_{j=0}^{t-1} \min(P_{r_i}, M_j) \right) \right), \quad n = 1, 2, \dots, N,$$

$$T_{-1}^{k,t} = T_{N+1}^{k,t} = \bar{T}_{-1}^{k,t} = \bar{T}_{N+1}^{k,t} = +\infty, \quad T_0^{k,t} = \bar{T}_0^{k,t} = 0,$$

where P_i and W_i , $i = 0, 1, \dots, N-1$, are some constants satisfying $P_0 \leq P_1 \leq \dots \leq P_{N-1}$.

Remark 5. There exists a Bäcklund transformation from the discrete system (24) to the nonautonomous discrete Toda (nd-Toda) lattice:

$$\begin{aligned} \mathbf{q}_n^{(t)} &= \frac{\delta^{-1} q_n^{(t)}}{\delta(1 + \delta^{-1} q_n^{(t)})(1 + \delta^{-1} e_n^{(t)})}, & \mathbf{e}_n^{(t)} &= \frac{\delta^{-1} e_n^{(t)}}{\delta(1 + \delta^{-1} q_{n-1}^{(t)})(1 + \delta^{-1} e_n^{(t)})}, \\ \bar{\mathbf{q}}_n^{(t)} &= \frac{\delta^{-1} \bar{q}_n^{(t)}}{(\delta - \mu_t)(1 + \delta^{-1} \bar{q}_n^{(t)})(1 + \delta^{-1} \bar{e}_n^{(t)})}, & \bar{\mathbf{e}}_n^{(t)} &= \frac{\delta^{-1} \bar{e}_n^{(t)}}{(\delta - \mu_t)(1 + \delta^{-1} \bar{q}_{n-1}^{(t)})(1 + \delta^{-1} \bar{e}_n^{(t)})}. \end{aligned}$$

In fact, these variables have τ -function expressions

$$\begin{aligned} \mathbf{q}_n^{(t)} &= \delta^{-1} \frac{\tau_n^{0,t} \bar{\tau}_{n+1}^{0,t+1}}{\tau_{n+1}^{0,t} \bar{\tau}_n^{0,t+1}}, & \bar{\mathbf{q}}_n^{(t)} &= (\delta - \mu_t)^{-1} \frac{\bar{\tau}_n^{0,t} \tau_{n+1}^{0,t+1}}{\bar{\tau}_{n+1}^{0,t} \tau_n^{0,t+1}}, \\ \mathbf{e}_n^{(t)} &= \delta \frac{\tau_{n+1}^{0,t} \bar{\tau}_{n-1}^{0,t+1}}{\tau_n^{0,t} \bar{\tau}_n^{0,t+1}}, & \bar{\mathbf{e}}_n^{(t)} &= (\delta - \mu_t) \frac{\bar{\tau}_{n+1}^{0,t} \tau_{n-1}^{0,t+1}}{\bar{\tau}_n^{0,t} \tau_n^{0,t+1}}. \end{aligned}$$

Since the bilinear equations

$$\begin{aligned} \tau_n^{0,t-1} \bar{\tau}_n^{0,t+1} &= \delta(\delta - \mu_t) \tau_{n+1}^{0,t-1} \bar{\tau}_{n-1}^{0,t+1} + \tau_n^{0,t} \bar{\tau}_n^{0,t}, \\ \delta \tau_n^{0,t} \tau_{n+1}^{0,t} &= (\delta - \mu_t) \tau_n^{0,t} \bar{\tau}_{n+1}^{0,t} + \mu_t \tau_{n+1}^{0,t-1} \bar{\tau}_n^{0,t+1} \end{aligned}$$

hold (these are proved by using the Plücker relation), we have the equations

$$\bar{\mathbf{q}}_n^{(t+1)} = \mathbf{e}_{n+1}^{(t)} + \bar{\mathbf{d}}_n^{(t+1)}, \quad \bar{\mathbf{d}}_n^{(t+1)} = \bar{\mathbf{d}}_{n-1}^{(t+1)} \frac{\mathbf{q}_n^{(t)}}{\bar{\mathbf{q}}_{n-1}^{(t+1)}} + \sigma_{t+1},$$

where

$$\bar{\mathbf{d}}_n^{(t)} := (\delta - \mu_t)^{-1} \frac{\bar{\tau}_n^{0,t} \tau_{n+1}^{0,t}}{\tau_{n+1}^{0,t-1} \bar{\tau}_n^{0,t+1}}, \quad \sigma_t := \frac{\delta^{-1} \mu_t}{\delta - \mu_t}.$$

Additionally, we have the identities

$$\bar{\mathbf{e}}_n^{(t+1)} = \mathbf{e}_n^{(t)} \frac{\mathbf{q}_n^{(t)}}{\bar{\mathbf{q}}_{n-1}^{(t+1)}}, \quad \mathbf{q}_n^{(t+1)} = \bar{\mathbf{q}}_n^{(t+1)} \frac{\bar{\mathbf{d}}_{n-1}^{(t+1)} \mathbf{q}_n^{(t)}}{\bar{\mathbf{d}}_n^{(t+1)} \bar{\mathbf{q}}_{n-1}^{(t+1)}}, \quad \mathbf{e}_n^{(t+1)} = \bar{\mathbf{e}}_n^{(t+1)} \frac{\bar{\mathbf{d}}_n^{(t+1)} \bar{\mathbf{q}}_{n-1}^{(t+1)}}{\bar{\mathbf{d}}_{n-1}^{(t+1)} \mathbf{q}_n^{(t)}}.$$

Eliminating $\bar{d}_n^{(t+1)}$ from these equations, we obtain the modified nd-Toda lattice [2]

$$\begin{aligned}\bar{q}_n^{(t+1)} + \bar{e}_n^{(t+1)} &= q_n^{(t)} + e_{n+1}^{(t)} + \sigma_{t+1}, & \bar{q}_{n-1}^{(t+1)} \bar{e}_n^{(t+1)} &= q_n^{(t)} e_n^{(t)}, \\ q_n^{(t+1)} + e_{n+1}^{(t+1)} &= \bar{q}_n^{(t+1)} + \bar{e}_{n+1}^{(t+1)} - \sigma_{t+1}, & q_n^{(t+1)} e_n^{(t+1)} &= \bar{q}_n^{(t)} \bar{e}_n^{(t)},\end{aligned}$$

and the finite lattice condition is given by

$$e_0^{(t)} = e_N^{(t)} = \bar{e}_0^{(t)} = \bar{e}_N^{(t)} = 0.$$

6. Concluding remarks

In this paper, we have derived the finite Toda representation of the BBS with box capacity by introducing the expansion map from a state of the BBS to a binary sequence. Furthermore, we have given a particular solution for the fixed box capacity case. Hence we can say that the ultradiscrete system (21) is integrable if the parameters $K_n^{(t)}$ and $\Lambda_n^{(t)}$ are chosen as constants. Since there is a connection between the ultradiscrete system (21) and the BBS with variable box capacity which is integrable, we expect that the ultradiscrete system (21) of the variable box capacity case is also integrable and a discrete system derived through the inverse-ultradiscretization has determinant solutions. This problem is left for future research.

In the proof of Theorem 3, the variables $\bar{C}_n^{(t+1)}$ and $\bar{D}_n^{(t+1)}$ have played important roles; these variables denote the number of balls which the carrier has. Moreover, these variables correspond to the variables which are introduced to remove subtractions in the discrete equations (see Remark 2). This result gives us a guideline for ultradiscretization of Toda-type integrable systems and making connections between these systems and BBSs.

In 2000, Spiridonov and Zhedanov [7] proposed a Toda-type nonautonomous integrable system called R_{II} chain, which is derived as the compatibility conditions of spectral transformations for some biorthogonal rational functions. By using techniques developed in this paper, we will be able to ultradiscretize the R_{II} chain and consider a corresponding BBS.

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