STUDIES ON SINGULAR PERTURBATION MODELLING AND CONTROL OF LARGE-SCALE SYSTEMS

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ABSTRACT

In this thesis, modelling and control problems of large-scale continuous-time and discrete-time systems are studied within a framework of a decomposition-coordination principle from a temporal viewpoint.

The thesis is divided into six chapters.

In Chapter 1, a background and a basic concept of studies on these problems is described and the outline of the subsequent chapters is given.

In Chapter 2, the problem of time-scale modelling is dealt with. A brief survey of time-scale decomposition methods and singular perturbation methods is given. After formulations of singularly perturbed discrete-time versions are discussed, singular perturbation modelling concepts and algorithms based on structural properties are developed.

In Chapter 3, the boundary layer method for discrete-time versions is studied, which provides us a useful tool for controller design.

In Chapter 4, controller design problems are considered in the discrete-time domain. After the recent progress in this field is briefly reviewed, various design problems for slow-time-scale and fast-time-scale versions are studied. Furthermore, on the basis of these results, a multirate controller design is developed.

In Chapter 5, a variety of linear quadratic problems with multi-time-scale property are treated in the continuous-time domain. The mode-decoupled near optimum controller designs are given and some existence conditions are investigated.

In the last chapter, Chapter 6, the results obtained are summarized along with some additional remarks and discussions.

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CHAPTER 1. INTRODUCTION

In recent years, the problems associated with modelling, analysis and control of large-scale dynamic systems have aroused interests of numerous researchers from different points of view. The typical references to these problems may be found in recent surveys (Sandell et al. [San.78], Nishikawa [Ns.78]) and the books (Singh [Sin.79], Mahmoud et al. [Mam.81], Bernussou et al. [Be.82], Jamshidi [Jam.83]). In order to overcome various difficulties due to the dimensionality, the information structure constraints and the uncertainty etc., which appear in the large-scale and complex systems, the model reduction methods (Lamba et al. [Lam.82], Hickin et al. [Hi.80]), the hierarchical control methods (Singh et al. [Sin.80],[Sin.81.1], Mahmoud [Mam.77]), the decentralized control methods (Singh [Sin.81.2]), the stability analysis of interconnected systems (Šiljak [Sil.78], Michel [Mi.83]) etc. have been extensively developed. In order to gain a conceptual insight and a numerical simplification, a variety of decomposition concepts and algorithms have been performed in these investigations : e.g., the ε-coupling method (Kokotović [Ko.72.1], Mahmoud [Mam.78]) and the chained aggregation (Tse et al. [Ts.78], [Ts.80]) among model order-reduction methods, the spatial decomposition based on a physical intuition (Singh [Sin.80],[Sin.81.1], Mahmoud [Mam.77], Šiljak [Sil.78], Michel [Mi.83]), the overlapping decomposition (Ikeda [Ik.80], Šiljak [Sil.79]), the area decomposition (Avramovic et al. [Av.80], Kokotović et al. [Ko.82]), the graph-theoretic decomposition from the viewpoint of stability and input reachability (Araki et al. [Ar.81], Sezer et al. [Se.81], Šiljak [Sil.83]).

In addition to these spatial and structural decompositions, a temporal decomposition has been also considered. Many physical large-scale models often involve interacting dynamic phenomena of widely different speeds. Since these models have high dimensionality and numerical stiffness in general, concepts and algorithms of order reduction and separation of time-scales play an important role to alleviate these difficulties. For this reason, Singular Perturbations and Time-Scale Decomposition Methods have been applied in broad categories of modelling, analysis and control (Kokotović et al. [Ko.76]), and these investigations are still in progress (Saksena et al. [Sak.84]).

In view of this situation, an attempt is made to consider primarily modelling and control of large-scale systems on the basis of the temporal decomposition-coordination principle in this thesis. Most of the present discussions are focused on linear, continuous- and discrete-time, deterministic systems with multi-time-scales.

The rest of this thesis is organized into five chapters according to their contents, and furthermore each chapter is divided into some sections.

In Chapter 2, after a brief survey of singular perturbations and time-scale decomposition methods is given, the problem of singular perturbation modelling is investigated. As a primary factor that obstructs an application of the singular perturbation methods, a lack of modelling procedures and computer-oriented decomposition techniques for model reduction design of control systems has been so far indicated in recent surveys of large-scale systems (Sandell et al. [San.78], Šiljak [Sil.83]). In view of this situation, the problems of such a singular perturbation modelling are studied, i.e., the modelling concepts and algorithms which are based on structural properties are developed, along with discussions for the canonical forms of the singularly perturbed discrete-time systems.

In Chapter 3, initial value problems of discrete-time systems are mainly considered. Based on the singularly perturbed canonical forms discussed in Chapter 2, the boundary layer method for discrete-time versions is formulated. Its method provides us a powerful tool for controller design in the discrete-time domain, which are studied in the following chapter.

In Chapter 4, at the beginning, recent progress in the studies of discrete-time versions via singular perturbations and time-scale decomposition methods is briefly reviewed. After two different discrete-time versions, i.e., a slow-time-scale version and a fast-time-scale version, are formulated and their properties are discussed, various controller design problems are investigated for two versions, e.g., a stabilizing problem, an observer problem and a regulator problem. The results of these two versions lead to a multirate controller design naturally. In a last section, such the multirate controller design is developed on the basis of the decomposition-coordination principle with respect to time-scales.

In Chapter 5, a variety of linear quadratic problems are dealed with in the continuous-time domain, i.e., a problem having a terminal condition with zero slope, a fixed-endpoint problem and a regulator problem. Near optimum controller designs are developed and some existence conditions are studied. After two-time-scale cases are investigated, these results are extended to multi-time-scale cases based on multi-time-scale modelling concepts.

In these chapters, some numerical examples are given in order to verify the validity of results.

Finally, in Chapter 6, the results obtained are summarized along with some additional remarks and discussions.

CHAPTER 2. TIME-SCALE DECOMPOSITION METHODS AND SINGULAR PERTURBATION MODELLING

2.1 Introduction

In order to cope with the high dimensionality and the ill-conditioning resulting from the interaction of slow and fast behaviors, which are present in large-scale systems, Singular Perturbations and Time-Scale Decomposition Methods have been watched with keenest interest (Kokotović et al. [Ko.76], Saksena et al. [Sak.84]). These methods belong to one class among model reduction methods of large-scale systems (Genesio et al. [Gen.76], Decoster et al. [De.76], Kando et al. [Ka.81.1], [Ka.81.2], Sandell et al. [San.78], Jamshidi [Jam.83], Lamba et al. [Lam.82]). But in recent years, it has been recognized that they provide a powerful tool in broad categories of modelling, analysis and control of large-scale systems because of characteristics of order-reduction and separation of time-scales (Saksena et al. [Sak.84]).

In the large-scale systems, the presence of small parastic parameters (Kokotović [Ko.72.2]), high gain feedback (Suzuki [Su.80]) and weak coupling (Peponides et al. [Pe.83]) etc. often gives rise to multi-time-scale behaviors. Many models with these properties have been shown in a recent survey (Saksena et al. [Sak.84]). They can be decomposed into slow and fast subsystems via a variety of time-scale decomposition methods, and as the result, reduction of computation and simplification of structure can be achieved. These characteristics are useful for analysis and design of large-scale systems.

In Section 2.2, these time-scale decomposition methods, i.e., the Riccati-type iteration method, the subspace iteration method and the quasi-steady-state (q.s.s.) method are briefly reviewd, and then the relationships among them are discussed. An alternative scheme to tackle these multi-time-scale systems is one via the singular perturbation method. However, it requires a special form, so-called the singularly perturbed form. Actually, how to identify a small parameter ε, how to group the state variables into slow and fast states, and how to verify the validity of modelling, they are nontrivial tasks (Sandell et al. [San.78], Šiljak [Sil.83]).

In Section 2.3, at the beginning, some former modelling approaches from the viewpoint of model reduction methods are briefly described. After discussions for canonical forms of singularly perturbed discrete-time systems, modelling concepts and algoritms based on norm conditions are developed for both discrete- and continuous-time systems.

2.2 Time-Scale Decomposition Methods

We consider the following linear time-invariant system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{2.1}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$.

If the above coefficient matrix $\tilde{A}$ is dichotomically separable (Medanic [Me.82]), i.e., the eigenvalues $\lambda(\tilde{A})$ can be expressed as a disjoint union of slow eigenvalues $\lambda(\tilde{A}_s)$ and fast eigenvalues $\lambda(\tilde{A}_f)$, where

$$\max_i |\lambda_i(\tilde{A}_s)| \ll \min_j |\lambda_j(\tilde{A}_f)| \tag{2.2}$$

it is said that (2.2) exhibits a two-time-scale property (Anderson [An.78.2], Chow et al. [Cho.76.1]).

By applying the following transformation in (2.1)

$$x_f = \tilde{L}x_1 + x_2 \tag{2.3}$$

we have the block triangular form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} \tilde{A}_s & \tilde{A}_{12} \\ R(\tilde{L}) & \tilde{A}_f \end{bmatrix} \begin{bmatrix} x_1 \\ x_f \end{bmatrix} \tag{2.4}$$

where $\tilde{A}_s = \tilde{A}_{11} - \tilde{A}_{12}\tilde{L}$ (2.5.a)

$\tilde{A}_f = \tilde{A}_{22} + \tilde{L}\tilde{A}_{12}$ (2.5.b)

$R(\tilde{L}) = \tilde{A}_{21} + \tilde{L}\tilde{A}_{11} - \tilde{A}_{22}\tilde{L} - \tilde{L}\tilde{A}_{12}\tilde{L} = 0.$ (2.6)

The matrix $\tilde{L} \in R^{n_2 \times n_1}$ which satisfies both a nonsymmetric algebraic Riccati equation (NARE) (2.6) and the two-time-scale property (2.2) is called a dichotomic solution (Medanic [Me.82]) and is unique (Anderson [An.78.2]), although there are generally many real solutions of NARE corresponding to each possible partitioning of the eigenvalues of $\tilde{A}$.

Furthermore, the substitution of the transformation

$$x_s = x_1 + \tilde{M}x_f \tag{2.7}$$

into (2.4) yields the following block diagonal form:

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} \tilde{A}_s & R(\tilde{M}) \\ 0 & \tilde{A}_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{2.8.a}$$

where $R(\tilde{M}) = -\tilde{A}_s\tilde{M} + \tilde{M}\tilde{A}_f + \tilde{A}_{12} = 0.$ (2.8.b)

The fast transformation is defined from (2.3) and (2.7) as follows:

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = \begin{bmatrix} I+\tilde{M}\tilde{L} & \tilde{M} \\ \tilde{L} & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & -\tilde{M} \\ -\tilde{L} & I+\tilde{L}\tilde{M} \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}. \tag{2.9}$$

In this way, the two-time-scale system (2.1) can be decomposed into lower order slow and fast subsystems via the fast transformation.

Alternatively, by applying a slow transformation:

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = \begin{bmatrix} I & -\tilde{L}^s \\ -\tilde{M}^s & I+\tilde{M}^s\tilde{L}^s \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I+\tilde{L}^s\tilde{M}^s & \tilde{L}^s \\ \tilde{M}^s & I \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{2.10}$$

which is dual to (2.9), the following block diagonal form is obtained:

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} \tilde{A}_s^s & 0 \\ 0 & \tilde{A}_f^s \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{2.11}$$

where

$$\begin{cases} \tilde{A}_s^s = \tilde{A}_{11} - \tilde{L}^s \tilde{A}_{21} & (2.12.a) \\ \tilde{A}_f^s = \tilde{A}_{22} + \tilde{A}_{21} \tilde{L}^s & (2.12.b) \end{cases}$$

$$R(\tilde{L}^s) = \tilde{A}_{12} + \tilde{A}_{11}\tilde{L}^s - \tilde{L}^s\tilde{A}_{21}\tilde{L}^s - \tilde{L}^s\tilde{A}_{22} = 0 \tag{2.13}$$

$$R(\tilde{M}^s) = -\tilde{M}^s\tilde{A}_s^s + \tilde{A}_f^s\tilde{M}^s + \tilde{A}_{21} = 0. \tag{2.14}$$

From the above results, a combined transformation is defined by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & \tilde{L}^s \\ -\tilde{L} & I \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{2.15}$$

where $\tilde{L}$ and $\tilde{L}^s$ satisfy (2.6) and (2.13) respectively, which transforms (2.1) into the following:

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} \tilde{A}_s & 0 \\ 0 & \tilde{A}_f^s \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{2.16}$$

where $\tilde{A}_s$ and $\tilde{A}_f^s$ are given in (2.5.a) and (2.12.b).

The studies on time-scale decomposition methods described above have been begun by Chang [Cha.72],[Cha.74.1],[Cha.74.2], and have been extensively developed by numerous researchers. Among them, Kokotović [Ko.75], Anderson et al. [An.81], O'Malley et al. [OM.79] have investigated mainly the problems of computational algorithms for a dichotomic solution of NARE. Various properties and existence conditions of the dichotomic solution have been discussed in detail by Medanic [Me.82], Anderson [An.78.2], Narasimhamurthi et al. [Nr.77] and Avramovic [Av.79]. These studies have been extended to time-varying systems by Anderson [An.80] and O'Malley et al. [OM.82].

In the next section, a variety of computational algorithms, i.e., the Riccati iteration method, the subspace iteration method and the q.s.s. method are briefly described and the relationships among them are discussed. For brevity, the case of the fast transformation for the continuous-time system is mainly treated.

2.2.1 Riccati iteration method

An iterative algorithm to compute solutions of the NARE (2.6) and the Sylvester equation (2.8.b) has been studied by Anderson et al. [An.78.2],[An.81],[OM.79] as follows:

$$\tilde{L}_{k+1} = \tilde{L}_k + (\tilde{A}_{22}+\tilde{L}_k\tilde{A}_{12})^{-1}R(\tilde{L}_k) \tag{2.17}$$

where $R(\tilde{L}_k) = \tilde{A}_{21} + \tilde{L}_k\tilde{A}_{11} - \tilde{A}_{22}\tilde{L}_k - \tilde{L}_k\tilde{A}_{12}\tilde{L}_k$

$$(\tilde{L}_{k+1} = \tilde{L}_k + R(\tilde{L}_k)(\tilde{A}_{11}-\tilde{A}_{12}\tilde{L}_k)^{-1} \tag{2.17)'$$

where $R(\tilde{L}_k) = \tilde{A}_{21} + \tilde{L}_k\tilde{A}_{11} - \tilde{A}_{22}\tilde{L}_k - \tilde{L}_k\tilde{A}_{12}\tilde{L}_k$)

and

$$\tilde{M}_{k+1} = \tilde{M}_k - R(\tilde{M}_k)(\tilde{A}_{22}+\tilde{L}\tilde{A}_{12})^{-1} \tag{2.18}$$

where $R(\tilde{M}_k) = -\tilde{A}_s\tilde{M}_k + \tilde{M}_k\tilde{A}_f + \tilde{A}_{12}$

$$(\tilde{M}_{k+1} = \tilde{M}_k + (\tilde{A}_{11}-\tilde{A}_{12}\tilde{L})^{-1}R(\tilde{M}_k) \tag{2.18)'$$

where $R(\tilde{M}_k) = -\tilde{A}_s\tilde{M}_k + \tilde{M}_k\tilde{A}_f + \tilde{A}_{12}$).

The parentheses (2.17)' and (2.18)' represent the discrete-time case.

When we consider a singularly perturbed system, the above iterations lead to Kokotović's iteration [Ko.75]:

$$\tilde{L}_{k+1} = \tilde{A}_{22}^{-1}[\tilde{L}_k(\tilde{A}_{11}-\tilde{A}_{12}\tilde{L}_k) + \tilde{A}_{21}] \;;\quad \tilde{L}_0 = \tilde{A}_{22}^{-1}\tilde{A}_{21} \tag{2.19}$$

$$(\tilde{L}_{k+1} = [\tilde{A}_{22}\tilde{L}_k + \tilde{L}_k\tilde{A}_{12}\tilde{L}_k - \tilde{A}_{21}]\tilde{A}_{11}^{-1} \;;\quad \tilde{L}_0 = -\tilde{A}_{21}\tilde{A}_{11}^{-1}) \tag{2.19)'$$

$$\tilde{M}_{k+1} = [(\tilde{A}_{11}-\tilde{A}_{12}\tilde{L})\tilde{M}_k - \tilde{M}_k\tilde{L}\tilde{A}_{12} - \tilde{A}_{12}]\tilde{A}_{22}^{-1} \;;\quad \tilde{M}_0 = -\tilde{A}_{12}\tilde{A}_{22}^{-1} \tag{2.20}$$

$$(\tilde{M}_{k+1} = \tilde{A}_{11}^{-1}[\tilde{M}_k(\tilde{A}_{22}+\tilde{L}\tilde{A}_{12}) + \tilde{A}_{12}\tilde{L}\tilde{M}_k + \tilde{A}_{12}] \;;\quad \tilde{M}_0 = \tilde{A}_{11}^{-1}\tilde{A}_{12}). \tag{2.20)'$$

It is noted that (2.19) is locally convergent, but it does not require an inverse computation of $(\tilde{A}_{22}+\tilde{L}_k\tilde{A}_{12})$ at each iteration.

On the other hand, since (2.17) is equivalent to the subspace iteration method (Anderson et al. [An.81], Avramovic [Av.79]), (2.17) is globally convergent. As an eigensolver to compute slow eigenvalues and eigenvectors (Anderson et al. [An.81]) and as a computational method of solving a conventional symmetric Riccati algebraic equation (Anderson et al. [An.83]), the Riccati iteration method may be also applied.

2.2.2 Subspace iteration method

It has been shown (Medanic [Me.82], Anderson [An.78.2], Avramovic [Av.79]) that the dichotomic solution $\tilde{L}$ of the NARE can be expressed as

$$\tilde{L} = \tilde{V}_2^{-1}\tilde{V}_1 \quad ; \quad \tilde{L} \in R^{n_2 \times n_1}, \quad \tilde{V}_1 \in R^{n_2 \times n_1}, \quad \tilde{V}_2 \in R^{n_2 \times n_2}$$

where we take the rows of $[\tilde{V}_1, \tilde{V}_2] \in R^{n_2 \times n}$ as a basis for left eigenspace corresponding to n_2 fast (dominant) eigenvalues.

The subspace iteration method provides us these bases for dominant (left) eigenspace by using a simple iteration scheme (Stewart [St.76.1], [St.76.2]):

$$\tilde{V}^{k+1} = \tilde{V}^k\tilde{A} \quad (n_1 \geq n_2) \quad ; \quad \tilde{V}^k = [\tilde{V}_1^k, \tilde{V}_2^k]. \tag{2.21}$$

After various computational devices for rounding errors and an improvement of covergence speed are employed, generalized eigenvectors or Shur vectors of matrix $\tilde{A}^T$ can be obtained under mild conditions, where no row of an initial value $\tilde{V}^0$ is orthogonal to any basis of dominant eigenspace. This iteration method is globally convergent and has capability of detecting the dimension of slow and fast states and of redefining it by observing the convergence speed of rows (Avramovic [Av.79], Stewart [St.76.1]). By using $\tilde{V}^k$, whose rows approach to generalized eigenvectors (Anderson [An.78.2],[An.81]) or Shur vectors (Laub [Lau.79]) of $\tilde{A}^T$, and by setting

$$\tilde{L}_k = (\tilde{V}_2^k)^{-1}(\tilde{V}_1^k) \tag{2.22}$$

where $\tilde{L}_k \in R^{n_2 \times n_1}$, $\tilde{V}_1^k \in R^{n_2 \times n_1}$, $\tilde{V}_2^k \in R^{n_2 \times n_2}$

it can be shown that the subspace iteration (2.21) leads to the Riccati iteration (2.17) (Anderson et al. [An.81], Avramovic [Av.79]).

[Remarks]

By using the eigenvectors, the dichotomic solution $\tilde{L} \in R^{n_2 \times n_1}$ is given by

$$\tilde{L} = \tilde{Q}_{22}^{-1}\tilde{Q}_{21} = -\tilde{M}_{21}\tilde{M}_{11}^{-1} \tag{2.23}$$

where $[\tilde{M}_{11}^T, \tilde{M}_{21}^T]^T$ represents eigenvectors corresponding to the slow eigenvalues $\lambda(\tilde{A}_s)$, and $[\tilde{Q}_{21}, \tilde{Q}_{22}]$ represents eigenrows corresponding to the fast eigenvalues $\lambda(\tilde{A}_f)$. The relationship between the fast transformation (2.9) and the modal matrices $[\tilde{M}_{ij}]$ is given by (Anderson [An.78.2])

$$\begin{bmatrix} I+\tilde{M}\tilde{L} & \tilde{M} \\ \tilde{L} & I \end{bmatrix} = \begin{bmatrix} \tilde{M}_{11}\tilde{Q}_{11} & \tilde{M}_{11}\tilde{Q}_{12} \\ \tilde{Q}_{22}^{-1}\tilde{Q}_{21} & I \end{bmatrix} \tag{2.24}$$

where $[\tilde{Q}_{ij}] = [\tilde{M}_{ij}]^{-1}$.

2.2.3 Quasi-steady-state (q.s.s) method

By using q.s.s. concepts, an alternative method has been proposed by Kokotović and his co-workers (Kokotović et al. [Ko.80], Winkelman et al. [Win.80]). It consists of the following iterative scheme:

$$\eta_k = \eta_{k-1} + (\tilde{A}_{22}^{k-1})^{-1}\tilde{A}_{21}^{k-1}x_1 \tag{2.25}$$

where $\eta_0 = x_2$

$$\tilde{A}_{11}^k = \tilde{A}_{11}^{k-1} - \tilde{A}_{12}(\tilde{A}_{22}^{k-1})^{-1}\tilde{A}_{21}^{k-1} \quad ; \quad \tilde{A}_{11}^0 = \tilde{A}_{11} \tag{2.26.a}$$

$$\tilde{A}_{21}^k = (\tilde{A}_{22}^{k-1})^{-1}\tilde{A}_{21}^{k-1}\tilde{A}_{11}^k \quad ; \quad \tilde{A}_{21}^0 = \tilde{A}_{21} \tag{2.26.b}$$

$$\tilde{A}_{22}^k = \tilde{A}_{22}^{k-1} + (\tilde{A}_{22}^{k-1})^{-1}\tilde{A}_{21}^{k-1}\tilde{A}_{12} \quad ; \quad \tilde{A}_{22}^0 = \tilde{A}_{22}. \tag{2.26.c}$$

Then, applying the above iteration to (2.1) yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\eta}_k \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11}^k & \tilde{A}_{12} \\ \tilde{A}_{21}^k & \tilde{A}_{22}^k \end{bmatrix} \begin{bmatrix} x_1 \\ \eta_k \end{bmatrix}. \tag{2.27}$$

On the other hand, applying the fast transformation (2.3) to (2.1), i.e., $x_f = \tilde{L}_{k-1}x_1 + x_2$, where $\tilde{L}_{k-1}$ is a (k-1)th iterative solution of the Riccati iteration (2.17) ((2.19) in case of the singularly perturbed system) leads to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_f^k \end{bmatrix} = \begin{bmatrix} \tilde{A}_s^k & \tilde{A}_{12} \\ R(\tilde{L}_{k-1}) & \tilde{A}_f^k \end{bmatrix} \begin{bmatrix} x_1 \\ x_f^k \end{bmatrix} \tag{2.28}$$

where
$$\tilde{A}_s^k = \tilde{A}_{11} - \tilde{A}_{12}\tilde{L}_{k-1} \quad ; \quad \tilde{A}_s^0 = \tilde{A}_{11}$$
$$\tilde{A}_f^k = \tilde{A}_{22} + \tilde{L}_{k-1}\tilde{A}_{12} \quad ; \quad \tilde{A}_f^0 = \tilde{A}_{22}$$
$$R(\tilde{L}_{k-1}) = \tilde{A}_{21} + \tilde{L}_{k-1}\tilde{A}_{11} - \tilde{A}_{22}\tilde{L}_{k-1} - \tilde{L}_{k-1}\tilde{A}_{12}\tilde{L}_{k-1} \; ; \; \tilde{L}_{-1} = 0.$$

After algebraic manipulation, it can be easily shown that the mode-decoupled system (2.27) via the q.s.s. method is equivalent to the system (2.28) via the Riccati iteration method. In this way, the q.s.s. method is globally convergent too.

Similar discussions on computational algorithms can be developed for the case of the discrete-time system or the slow transformation.

2.3 Singular Perturbation Modelling

2.3.1 Singular perturbation modelling via other model reduction methods

As the systems become larger and more complex, it becomes more difficult to obtain a singular perturbed form via only physical intuitions and experiences. Much attention to this problem has been paid so far and many approaches have been proposed mainly from the viewpoint of order reduction methods.

In the category of the dominant mode method, Litz et al. [Liz.81.1], [Liz.81.2] have defined the indices to select the dominant eigenvalues and to measure dependence of the state variables on these dominant eigenvalues, and by means of these indices, have transformed the original system into the singularly perturbed form.

Georgakis et al. [Geo.77],[Cr.80] have treated this problem via the perturbation and the quasi-modal transformation in order to alleviate disadvantage of the modal transformation.

On the other hand, from the viewpoint of the Routh approximation method, Hutton [Hu.75],[Hu.77] has transformed the original system into the Routh canonical state-space form via a certain nonsingular transformation and a reciprocal transformation, and by observing the gap of alpha-coefficients, has obtained the singularly perturbed form. Furthermore, he has shown that the Routh approximation model is equivalent to the reduced system of the singularly perturbed form.

An approach via an aggregation method has been proposed by Hickin [Hi.77].

Fernando et al. [Fe.82] have studied this problem from the viewpoint of a recent balanced approach. By using the balancing transformation (Laub [Lau.80]), an internally balanced system (Iri et al. [Ir.82]) has been obtained, and a strong and a weak subsystem has been identified. Then, by observing the eigenvalue gap of these subsystems, a singular perturbation modelling has been employed.

On the other hand, by using time-scale decomposition concepts, a two-stage modelling approach, which consists of modelling stage and validation stage, has been investigated by Winkelman et al. [Win.80]. In the first stage, by physical intuition, slow and fast states are identified roughly, and in the second stage, validity of the grouping is verified by observing variation of eigenvalues and eigenvectors via the q.s.s. iteration.

In the next section, from a viewpoint different from those described above, an approach based on norm conditions, under which the structural properties of singularly perturbed systems are satisfied, is developed. After a discussion for the formulation of singularly perturbed discrete-time versions is given, the problem of singular perturbation modelling is investigated.

2.3.2 Singular perturbation modelling

(a) Formulation of singularly perturbed discrete-time models

It is well known that the singularly perturbed continuous-time system is represented by

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon\, \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u. \tag{2.29}$$

In this section, firstly, we will consider the discrete-time version of the above system under a sample-and-hold device. Applying the mode-decoupling transformation (Saksena et al. [Sak.84], Phillips [Ph.80])

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I, & -\varepsilon M \\ -L, & I+\varepsilon LM \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix}, \quad \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = \begin{bmatrix} I+\varepsilon ML, & \varepsilon M \\ L, & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2.30}$$

where $R(L) = A_{21} + \varepsilon LA_{11} - A_{22}L - \varepsilon LA_{12}L = 0$ (2.31.a)

$$R(M) = -\varepsilon A_s M + MA_f + A_{12} = 0 \tag{2.31.b}$$

$$A_s = A_{11} - A_{12}L, \quad A_f = A_{22} + \varepsilon LA_{12} \tag{2.32.a,b}$$

and discretizing with a sampling rate T leads to

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} A_{11}^d(T,\varepsilon), A_{12}^d(T,\varepsilon) \\ A_{21}^d(T,\varepsilon), A_{22}^d(T,\varepsilon) \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} + \begin{bmatrix} B_1^d(T,\varepsilon) \\ B_2^d(T,\varepsilon) \end{bmatrix} u_i \tag{2.33}$$

where $x_{1,i} = x_1(iT,\varepsilon), \quad x_{2,i} = x_2(iT,\varepsilon), \quad u_i = u(iT,\varepsilon)$

$$A_{11}^d(T,\varepsilon) = \exp(A_sT)(I+\varepsilon ML) - \varepsilon M\exp(\tfrac{A}{\varepsilon}fT)L$$

$$A_{12}^d(T,\varepsilon) = \varepsilon(\exp(A_sT)M - M\exp(\tfrac{A}{\varepsilon}fT))$$

$$A_{21}^d(T,\varepsilon) = -L\exp(A_sT)(I+\varepsilon ML) + (I+\varepsilon LM)\exp(\tfrac{A}{\varepsilon}fT)L$$

$$A_{22}^d(T,\varepsilon) = -\varepsilon L\exp(A_sT)M + (I+\varepsilon LM)\exp(\tfrac{A}{\varepsilon}fT)$$

$$B_1^d(T,\varepsilon) = (\exp(A_sT)-I)A_s^{-1}B_s - \varepsilon M(\exp(\tfrac{A}{\varepsilon}fT)-I)A_f^{-1}B_f$$

$$B_2^d(T,\varepsilon) = -L(\exp(A_sT)-I)A_s^{-1}B_s + (I+\varepsilon LM)(\exp(\tfrac{A}{\varepsilon}fT)-I)A_f^{-1}B_f$$

$$B_s = B_1 + MB_f, \quad B_f = B_2 + \varepsilon LB_1.$$

In the above, we assume that A_s is a nonsingular matrix and A_f is a Hurwitz matrix. Since the above model depends on the sampling rate T critically, the selection of the fast sampling $T_f = \alpha_1\varepsilon$ or the slow sampling $T_s = \alpha_2[1/\varepsilon]T_f$ (where $[1/\varepsilon]$ is the largest integer $\leq 1/\varepsilon$) leads to two different formulas, respectively.

By assuming $\alpha_1 = \alpha_2 = 1$, i.e., $T_f = \varepsilon$, $T_s = [1/\varepsilon]T_f \approx 1$ and by neglecting $O(\varepsilon)$ errors, we obtain the following discrete-time models, where n represents the fast sampling point and k the slow sampling point, and $n = k[1/\varepsilon]$.

[I] Fast sampling model

$$\begin{bmatrix} x_{1,n+1} \\ x_{2,n+1} \end{bmatrix} = \begin{bmatrix} I+\varepsilon A_{11}^f, & \varepsilon A_{12}^f \\ A_{21}^f, & A_{22}^f \end{bmatrix} \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix} + \begin{bmatrix} \varepsilon B_1^f \\ B_2^f \end{bmatrix} u_n \tag{2.34}$$

where $x_{i,n} = x_i(iT_f,\varepsilon)$, $u_n = u(nT_f,\varepsilon)$

$$\begin{cases} A_{11}^f = A_0 + M_0(I-\exp(A_{22}))L_0, & A_{12}^f = M_0(I-\exp(A_{22})) \\ A_{21}^f = (\exp(A_{22})-I)L_0, & A_{22}^f = \exp(A_{22}) \\ B_1^f = B_0 - M_0(\exp(A_{22})-I)A_{22}^{-1}B_2, & B_2^f = (\exp(A_{22})-I)A_{22}^{-1}B_2 \end{cases} \tag{2.35}$$

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_0 = B_1 - A_{12}A_{22}^{-1}B_2.$$

In the above, the notations A_{ij}, B_i etc. represent the continuous-time version and A_{ij}^f, B_i^f etc. the discrete-time version. Furthermore, the approximate relations

$$\begin{cases} L(\varepsilon) = L_0 + O(\varepsilon) : L_0 = A_{22}^{-1}A_{21} & \text{for } \forall\varepsilon \in [0,\varepsilon^*) \\ M(\varepsilon) = M_0 + O(\varepsilon) : M_0 = -A_{12}A_{22}^{-1} & \end{cases}$$

have been used in the derivation of the above system.

[II] Slow sampling model

We will discretize the continuous-time system (2.29) with the slow sampling rate $T_s = [1/\varepsilon]T_f \approx 1$. Then, under the assumption 「A_f : Hurwitz matrix」, there exist $k_f>0$ and $\alpha_f>0$ such that

$$\| \exp(\tfrac{A}{\varepsilon}{}_f T_s) \| = k_f \exp(-\tfrac{\alpha}{\varepsilon}{}_f T_s) = O(\varepsilon).$$

Setting $\exp(\tfrac{A}{\varepsilon}{}_f T_s) = \varepsilon \bar{A}_f$

and neglecting $O(\varepsilon)$ errors yields

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} A_{11}^s, \varepsilon A_{12}^s \\ A_{21}^s, \varepsilon A_{22}^s \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} B_1^s \\ B_2^s \end{bmatrix} u_k \tag{2.36}$$

where
$$\begin{cases} A_{11}^s = \exp(A_0), \quad A_{12}^s = \exp(A_0)M_0 \\ A_{21}^s = -L_0\exp(A_0), \quad A_{22}^s = -L_0\exp(A_0)M_0+\bar{A}_f \\ B_1^s = (\exp(A_0)-I)A_0^{-1}B_0, \quad B_2^s = -L_0(\exp(A_0)-I)A_0^{-1}B_0 - A_{22}^{-1}B_2. \end{cases} \tag{2.37}$$

The above slow sampling model (2.36) coincides with the state space form (Naidu et al. [Na.81],[Na.82]) of the singularly perturbed difference equations (Comstock et al. [Co.76], Hsio et al. [Hs.79]).

Thus, by discretizing the singularly perturbed continuous-time systems with slow and fast sampling rates, two different discrete-time models have been formulated. The detailed discussions for these models will be given in Chapter 4.

(b) Studies on singular perturbation modelling

We consider the discrete-time system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{2.38}$$

$$\left(\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \right) \tag{2.38'}$$

The parentheses (2.38)' represent the continuous-time case. When we transform the above system into the canonical form formulated in the preceding section, we are confronted with the problems:

(i) identification of a small parameter ε,

(ii) criterion to test the validity of modelling,

(iii) efficient modelling algorithm.

The following lemma (Kando et al. [Ka.83.2], Phillips [Ph.80]) will give us useful suggestions for these problems.

Lemma 2-1 Let the following conditions be satisfied:

(A-1) $\max[2\tilde{a}_s(\tilde{a}_f+\tilde{a}_{12}\tilde{l}_0), \tilde{a}_s(\tilde{a}_f+\tilde{a}_{12}\tilde{l}_0+8\tilde{a}_s\tilde{a}_f\tilde{a}_{12}\tilde{l}_0/(\tilde{a}_f+\tilde{a}_{12}\tilde{l}_0)] < 1$

($\max[2\tilde{a}_f(\tilde{a}_s+\tilde{a}_{12}\tilde{l}_0), \tilde{a}_f(\tilde{a}_s+\tilde{a}_{12}\tilde{l}_0+8\tilde{a}_s\tilde{a}_f\tilde{a}_{12}\tilde{l}_0/(\tilde{a}_s+\tilde{a}_{12}\tilde{l}_0)] < 1$)

(A-2) $\varepsilon = \tilde{a}_s\tilde{a}_f$ (time-scale separation ratio)$<<1$

(A-3) $\|\tilde{M}_0\| \leq \tilde{a}_s\tilde{a}_{12} << 1$

($\|\tilde{M}_0\| \leq \tilde{a}_f\tilde{a}_{12} << 1$)

where $\tilde{a}_s = \|\tilde{A}_{11}^{-1}\|$, $\tilde{a}_f = \|\tilde{A}_{22}-\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}\|$

$\tilde{a}_{12} = \|\tilde{A}_{12}\|$, $\tilde{l}_0 = \|\tilde{A}_{21}\tilde{A}_{11}^{-1}\|$.

$$\left(\begin{array}{l}\tilde{a}_s = \|\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}\| = \|\tilde{A}_0\|, \quad \tilde{a}_f = \|\tilde{A}_{22}^{-1}\| \\ \tilde{a}_{12} = \|\tilde{A}_{12}\|, \quad \tilde{l}_0 = \|\tilde{A}_{22}^{-1}\tilde{A}_{21}\|.\end{array}\right)$$

Then,

(a) there exists a unique solution $\tilde{L}$ of the NARE

$$R(\tilde{L}) = -\tilde{A}_{22}\tilde{L} + \tilde{L}\tilde{A}_{11} - \tilde{L}\tilde{A}_{12}\tilde{L} + \tilde{A}_{21} = 0 \tag{2.39}$$

that possesses the magnitude

$0 \leq \|\tilde{L}\| \leq [\tilde{l}_0 + 2\tilde{a}_f\tilde{l}_0/(\tilde{a}_f+\tilde{a}_{12}\tilde{l}_0)]$

($0 \leq \|\tilde{L}\| \leq [\tilde{l}_0 + 2\tilde{a}_s\tilde{l}_0/(\tilde{a}_s+\tilde{a}_{12}\tilde{l}_0)]$)

and separates the eigenvalues of the system into slow eigenvalues:

$$\lambda(\tilde{A}_s) = \lambda(\tilde{A}_{11} - \tilde{A}_{12}\tilde{L}) \tag{2.40.a}$$

and fast eigenvalues: $\lambda(\tilde{A}_f) = \lambda(\tilde{A}_{22} + \tilde{L}\tilde{A}_{12})$ (2.40.b)

and $\tilde{L} = \tilde{L}_0 + O(\varepsilon)$; $\tilde{L}_0 = -\tilde{A}_{21}\tilde{A}_{11}^{-1}$. (2.41)

($\tilde{L} = \tilde{L}_0 + O(\varepsilon)$; $\tilde{L}_0 = \tilde{A}_{22}^{-1}\tilde{A}_{21}$.) (2.41)'

Furthermore, there exists a unique solution of a Sylvester equation

$$R(\tilde{M}) = \tilde{M}(\tilde{A}_{22}+\tilde{L}\tilde{A}_{12}) - (\tilde{A}_{11}-\tilde{A}_{12}\tilde{L})\tilde{M} + \tilde{A}_{12} = 0 \tag{2.42}$$

and $\tilde{M} = \tilde{M}_0 + O(\varepsilon)$; $\tilde{M}_0 = \tilde{A}_{11}^{-1}\tilde{A}_{12}$. (2.43)

($\tilde{M} = \tilde{M}_0 + O(\varepsilon)$; $\tilde{M}_0 = -\tilde{A}_{12}\tilde{A}_{22}^{-1}$.) (2.43)'

(b) The eigenvalues are approximated by

$$\begin{cases} \lambda(\tilde{A}_s) = \lambda(\tilde{A}_{11})(1 + O(\varepsilon)) \\ \lambda(\tilde{A}_f) = \lambda(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})(1 + O(\varepsilon)). \end{cases} \tag{2.44}$$

$$\left(\begin{cases} \lambda(\tilde{A}_s) = \lambda(\tilde{A}_0)(1 + O(\varepsilon)) \\ \lambda(\tilde{A}_f) = \lambda(\tilde{A}_{22})(1 + O(\varepsilon)). \end{cases} \right) \tag{2.44'}$$

(c) The two-time-scale property

$$\max|\lambda(\tilde{A}_f)| << \min|\lambda(\tilde{A}_s)| \tag{2.45}$$

$$(\max|\lambda(\tilde{A}_s)| << \min|\lambda(\tilde{A}_f)|) \tag{2.45'}$$

is satisfied.

(d) The coupling ratio with respect to the fast mode is small, i.e., $\rho_f = \|\tilde{M}_{12}\|/\|\tilde{M}_{22}\| << 1$, where $[\tilde{M}_{ij}]$ represents a modal matrix.

<u>Proof</u> By following the procedure shown in [Ko.75] and by applying the contraction mapping theorem, the results of (a) can be proved under the conditions (A-1) and (A-2). By using (2.40.a), (2.40.b) and (2.41), we have

$$\begin{cases} \tilde{A}_s = \tilde{A}_{11} - \tilde{A}_{12}\tilde{L} = \tilde{A}_{11}(I - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{L}_0)(I + O(\varepsilon)) \\ \tilde{A}_f = \tilde{A}_{22} + \tilde{L}\tilde{A}_{12} = (\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})(I + O(\varepsilon)). \end{cases} \quad (2.46)$$

$$\left(\begin{cases} \tilde{A}_s = \tilde{A}_{11} - \tilde{A}_{12}\tilde{L} = (\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})(I + O(\varepsilon)) \\ \tilde{A}_f = \tilde{A}_{22} + \tilde{L}\tilde{A}_{12} = (I + \tilde{L}_0\tilde{A}_{12}\tilde{A}_{22}^{-1})\tilde{A}_{22}(I + O(\varepsilon)). \end{cases} \right) \quad (2.46)'$$

Here, when the relation $\|\tilde{A}_{11}^{-1}\tilde{A}_{12}\|\,\|\tilde{L}_0\| = \|\hat{M}_0\|\,\|\tilde{L}_0\| = O(\varepsilon) << 1$ (2.47)

$$(\|\tilde{L}_0\|\,\|\tilde{A}_{12}\tilde{A}_{22}^{-1}\| = \|\tilde{L}_0\|\,\|\tilde{M}_0\| = O(\varepsilon) << 1\ ) \quad (2.47)'$$

is satisfied under the condition (A-3), the result of (b) is obtained. Furthermore, by using the expressions:

$$\begin{cases} \max|\lambda(\tilde{A}_f)| \le \|\tilde{A}_f\| = \|\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}\|(1 + O(\varepsilon)) \\ \qquad = \tilde{a}_f(1 + O(\varepsilon)) \quad (2.48.a) \\ \min|\lambda(\tilde{A}_s)| \le \|\tilde{A}_s^{-1}\|^{-1} = \|\tilde{A}_{11}^{-1}\|^{-1}(1 + O(\varepsilon)) \\ \qquad = \tilde{a}_s^{-1}(1 + O(\varepsilon)) \quad (2.48.b) \end{cases}$$

$$\left(\begin{cases} \max|\lambda(\tilde{A}_s)| \le \|\tilde{A}_s\| = \|\tilde{A}_0\|(1 + O(\varepsilon)) \\ \qquad = \tilde{a}_s(1 + O(\varepsilon)) \quad (2.48.a)' \\ \min|\lambda(\tilde{A}_f')| \le \|\tilde{A}_f^{-1}\|^{-1} = \|\tilde{A}_{22}^{-1}\|^{-1}(1 + O(\varepsilon)) \\ \qquad = \tilde{a}_f^{-1}(1 + O(\varepsilon)) \quad (2.48.b)' \end{cases} \right)$$

the result of (c) can be proved. Finally, under the condition (A-3), the result of (d) follows from the relation:

$$\begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{M}_{11} & -\tilde{M}\tilde{Q}_{22}^{-1} \\ -\tilde{L}\tilde{M}_{11} & (I+\tilde{L}\tilde{M})\tilde{Q}_{22}^{-1} \end{bmatrix} ; \quad [\tilde{Q}_{ij}] = [\tilde{M}_{ij}]^{-1}.$$

Q.E.D.

Here the parentheses represent the case of the continuous-time system.

[Remarks]

1. In the discrete-time case, we can consider the following two cases

with respect to the condition $\|\tilde{A}_{11}^{-1}\tilde{A}_{12}\|\|\tilde{L}_0\| = \|\tilde{M}_0\|\|\tilde{L}_0\| = O(\varepsilon)<<1$ ((2.47)).

(case 1) $\|\tilde{M}_0\| = O(\varepsilon)$, $\|\tilde{L}_0\| = O(1)$

We have

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \; ; \; \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{2.49}$$

Applying the boundary layer method to the singularly perturbed discrete-time system which will be discussed in Chapter 3 yields

$$\begin{cases} x_1(k,\varepsilon) = X_{10} - \varepsilon^{k+1}A_{11}^{-1}A_{12}m_{20}(k) + O(\varepsilon) \\ x_2(k,\varepsilon) = A_{21}A_{11}^{-1}X_{10}(k) + \varepsilon^k m_{20}(k) + O(\varepsilon) \end{cases} \tag{2.50}$$

where $X_{10}(k+1) = A_{11}X_{10}(k) \;\; ; \;\; X_{10}(0) = x_{10}$ (2.51)

$m_{20}(k+1) = (A_{22} - A_{21}A_{11}^{-1}A_{12})m_{20}(k) \; ; \; m_{20}(0) = x_{20}-A_{21}A_{11}^{-1}x_{10}.$ (2.52)

Here, (2.51) is a reduced system and (2.52) is a boundary layer system. In the state $x_2(k,\varepsilon)$, the boundary-layer jump phenomenon appears near $k = 0$, and the solutions of (2.49) exhibit a similar behavior to those of the singularly perturbed continuous system (2.29). Thus, $x_1(k,\varepsilon)$ corresponds to a slow variable and $x_2(k,\varepsilon)$ a fast variable.

(case 2) $\|\tilde{M}_0\| = O(1)$, $\|\tilde{L}_0\| = O(\varepsilon)$

We have

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon A_{21} & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \; ; \; \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{2.53}$$

Similarly as above, we have

$$\begin{cases} x_1(k,\varepsilon) = x_{10}(k) - \varepsilon^k A_{11}^{-1}A_{12}m_{20}(k) + O(\varepsilon) \\ x_2(k,\varepsilon) = 0 + \varepsilon^k m_{20}(k) + O(\varepsilon) \end{cases} \tag{2.54}$$

where $X_{10}(k+1) = A_{11}X_{10}(k) \;\; ; \;\; X_{10}(0) = x_{10} + A_{11}^{-1}A_{12}x_{20}$ (2.55)

$m_{20}(k+1) = (A_{22} - A_{21}A_{11}^{-1}A_{12})m_{20}(k) \;\; ; \;\; m_{20}(0) = x_{20}.$ (2.56)

The above results imply that the boundary-layer jump phenomena appear near $k = 0$ in both the states $x_1(k,\varepsilon)$ and $x_2(k,\varepsilon)$. The system (2.53) corresponds to a special case. However, if necessary, we can transform the system (2.53) into the canonical form (2.49) by using the scaling $\hat{X}(k) = [I_1, I_2/\varepsilon]X(k)$.

The condition (A-3) excludes the case such as (case 2).

In the continuous-time version, the condition (A-3) requires that the interconnection between subsystems, i.e., the magnitude of the matrices A_{ij} in the righthand side of (2.29) must be comparative.

2. The singularly perturbed systems have the following structural properties commonly ([Ka.83.1]):

(P-1) The eigenvalue relation:

$$\begin{cases} \lambda(\tilde{A}_s) = \lambda(\tilde{A}_{11})(1 + O(\varepsilon)) \\ \lambda(\tilde{A}_f) = \lambda(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})(1 + O(\varepsilon)). \end{cases}$$

$$\left(\begin{cases} \lambda(\tilde{A}_s) = \lambda(\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})(1 + O(\varepsilon)) \\ \lambda(\tilde{A}_f) = \lambda(\tilde{A}_{22})(1 + O(\varepsilon)). \end{cases} \right)$$

(P-2) The two-time-scale property:

$$\max|\lambda(\tilde{A}_f)| << \min|\lambda(\tilde{A}_s)| .$$

$$(\max|\lambda(\tilde{A}_s)| << \min|\lambda(\tilde{A}_f)| .)$$

(P-3) The coupling ratio with respect to the fast mode:

$$\rho_f = \|\tilde{M}_{12}\| / \|\tilde{M}_{22}\| << 1 \quad \text{where } [\tilde{M}_{ij}] \text{ represents a modal matrix.}$$

It is noted that the coupling for the fast mode is weak ($\rho_f = O(\varepsilon)$) but the coupling for the slow mode need not be weak ($\rho_s = \|\tilde{M}_{21}\| / \|\tilde{M}_{11}\| = O(1)$). In general, even though the system possesses the two-time-scale property, it is not necessarily weakly coupled. However, the singularly perturbed system possesses all of the above properties (P-1)-(P-3), and the norm conditions (A-1)-(A-3) shown in Lemma 2-1 guarantee these properties.

2.3.3 Modelling algorithm

When these norm conditions are satisfied, where a small parameter ε is chosen as $\varepsilon = \tilde{a}_s\tilde{a}_f$, the singularly perturbed canonical form can be obtained by a modelling algorithm shown in Fig. 2.1.

It consists of two-stages as follows:

[Stage I] Grouping and scaling

To the original system

$$d\hat{X}/dt = \hat{A}\hat{X}(t) \quad (\hat{X}(k+1) = \hat{A}\hat{X}(k)),$$

applying a permutation matrix P and a scaling diagonal matrix D which are techniques frequently used for balancing a matrix (Partlett et al. [Pa.69]), yields

$$\begin{aligned} dX/dt &= (DP)\hat{A}(P^TD^{-1})X(t) \quad (X(k+1) = (DP)\hat{A}(P^TD^{-1})X(k)) \\ &= \tilde{A}X(t). \end{aligned}$$

Here, a permutation P is chosen so that the diagonal elements of $\tilde{A}$ are re-ordered in descending (ascending) order, or the fast coupling ratio ρ_f becomes small. Since the singular perturbation theory does not demand to the extent of exact grouping within slow and fast groups, there is no necessity for requiring the complex and exquisite algorithm. By applying direct search methods, e.g., a simplex method (Konno et al. [Kon.78]), a scaling D can be determined so that the norm conditions in Lemma 2-1 be all satisfied.

[Stage II] Verification of modelling

The norm conditions are calculated and the validity of modelling is verified. After the application to several numerical examples, two-time-scale systems can be classified into the following three types:

[Type A] A nuclear reactor model (Kando et al. [Ka.71]) and a single machine model (Hassan et al. [Has.81]) belong to this category, where without scaling procedure, all norm conditions are satisfied. Thus, these models possess a strong singularly perturbed structure.

[Type B] To this type belong a boiler model (Nicholson et al. [Nc.64])

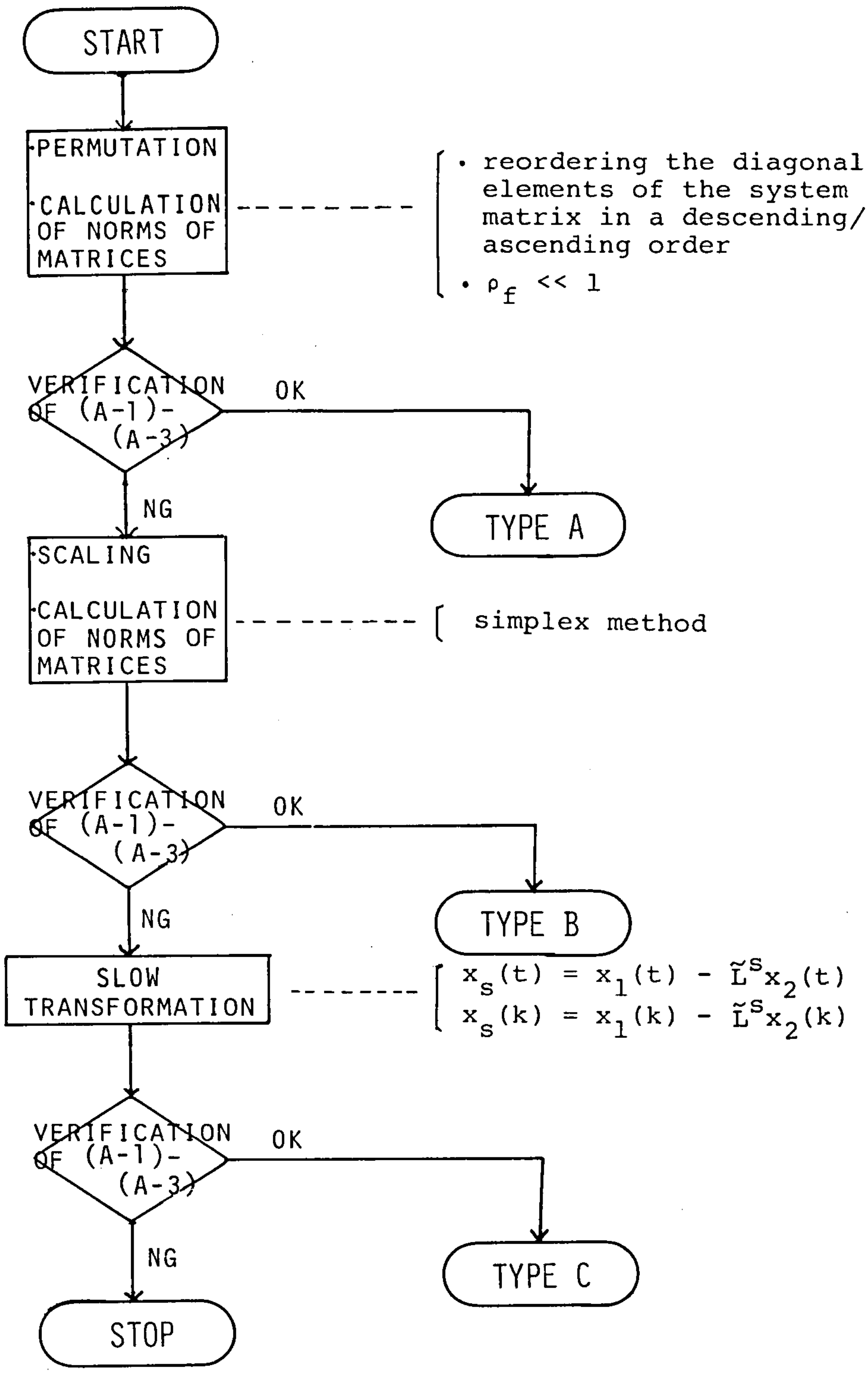

Fig. 2.1. Modelling algorithm.

and an aircraft model (Teneketzis et al. [Te.77]), which require the scaling procedure in order to satisfy the norm conditions.

[Type C] A chemical reactor model (Georgakis [Geo.77]) belongs to this type. In this model, the norm conditions are not satisfied via any permutation and scaling procedure. This model seems to demand the further co-ordinates transformation, e.g., the slow transformation

$$\begin{bmatrix} x_s(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_1 & -\tilde{L}^s \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \qquad \left(\begin{bmatrix} x_s(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} I_1 & -\tilde{L}^s \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \right).$$

Most of models we tried satisfied the norm conditions via only permutation/scaling procedure. It is noted that within these procedure, the physical identity of state variables is not destroyed by the modelling.

In this way, firstly, the continuous-time modelling is performed. Secondly, if necessary, by discretizing this continuous-time model with adequate sampling rates, the corresponding discrete-time models, i.e., the slow-time-scale or the fast-time-scale models are identified.

2.4 Concluding Remarks

In this chapter, firstly, the time-scale decomposition methods, i.e., the Riccati iteration method, the subspace iteration method and the quasi-steady-state method have been briefly reviewed and the relationships between them have been discussed.

Secondly, the problem of singular perturbation modelling which plays an important role in case of an application of the singular perturbation methods has been investigated. After discussions of the canonical forms for the discrete-time versions have been given, the modelling concepts based on the structural properties of the singularly perturbed systems have been studied and then the computer-oriented modelling algorithms have been developed.

It is noted that the time-scale decomposition methods are the concepts based on coordinates transformations, but the singular perturbation method is not necessarily coordinate-free concept. Accordingly, all of two-time-scale systems can not be always transformed to the singularly perturbed forms as indicated by the numercal examples.

The modelling concepts and algorithms shown in this chapter provide us the useful criterion to verify the validity of the singular perturbation modelling, and give us the beneficial data to predict an accuracy of solutions derived in the following system design and analysis phase.

CHAPTER 3. INITIAL VALUE PROBLEMS BASED ON TIME-SCALE DECOMPOSITION CONCEPTS

3.1 Introduction

Since the initial value problem was investigated by Tihonov [Wa.78], many studies have been extensively developed (Kokotović et al. [Ko.76], O'Malley [OM.68]) : among them, we shall cite the regular degeneration case (Levin et al. [Lev.54], Vasil'eva [Va.63]), the conditionally stable case (Levin [Lev.56], Chang et al. [Cha.69]), and the singular case (O'Malley et al. [OM.76],[OM.80]). Their main objects are studies on sufficient conditions under which the reduced solutions approximate the original solutions, and the methods to construct high-order approximate solutions, which are uniformly valid over the whole interval.

Among them, O'Malley [OM.68],[OM.71.1],[OM.74] has proposed the boundary layer methods as so-called two-time-scale asymptotic expansion methods, which provide us asymptotic series solutions composed of the outer region and the boundary-layer correction terms via recursive algorithms. Furthermore, they have been extended to three-time-scale systems (O'Malley [OM.77]). These methods provide us a powerful tool for control system design and analysis (O'Malley [OM.77]) : especially, regulator problems (O'Malley [OM.72],[OM.75.1], Yackel et al. [Ya.73]), and trajectory optimization problems (Sannuti [Sat.74],[Sat.75], Freedman et al. [Fr.76.1]). However, these methods have been mainly developed for the continuous-time version.

In view of this situation, a boundary layer method for the discrete-time version is studied in this section. So far, in the fast-time-scale version, Hoppensteadt et al. [Ho.77] and Blankenship [Bl.81] have applied its method to control system design and analysis, and they have shown that solutions of original problems can be expressed as a hybrid combination of the continuous-time slow and the discrete-time fast terms. The further discussions on these subjects will be developed in Chapter 4. In this section, the boundary layer method for the slow-time-scale version is formulated, and by utilizing it, the initial value problem is studied. It will be also shown that the asymptotic series solutions via the boundary layer method are equivalent to those via the Riccati iteration method. These results will be applied to the controller design for the discrete-time version, which will be investigated in Chapter 4.

3.2 Initial Value Problems of Discrete-Time Systems

3.2.1 Formulation of the boundary layer method

We consider the slow-time-scale version:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad ; \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{3.1}$$

where $\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix}$.

Applying the fast transformation

$$\begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} = \begin{bmatrix} I+\varepsilon HL & \varepsilon H \\ L & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} ; \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} I & -\varepsilon H \\ -L & I+\varepsilon LH \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} \tag{3.2}$$

where $A_{21} + LA_{11} - \varepsilon A_{22}L - \varepsilon LA_{12}L = 0$ (3.3)

$$-(A_{11}-\varepsilon A_{12}L)H + \varepsilon H(A_{22}+LA_{12}) + A_{12} = 0 \tag{3.4}$$

yields

$$\begin{bmatrix} x_s(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & \varepsilon A_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} \quad ; \quad \begin{bmatrix} x_s(0) \\ x_f(0) \end{bmatrix} = \begin{bmatrix} (I+\varepsilon HL)x_{10} + \varepsilon Hx_{20} \\ Lx_{10} + x_{20} \end{bmatrix} \tag{3.5}$$

where $A_s = A_{11} - \varepsilon A_{12}L$ (3.6.a)

$A_f = A_{22} + LA_{12}$. (3.6.b)

Then, we have the following theorem:

Theorem 3-1 Let the conditions (A-1)-(A-3) in Lemma 2-1 be satisfied in (3.1). Then, the initial value problem (3.1) has the following unique asymptotic series solutions:

$$\begin{cases} x_1(k,\varepsilon) = [\sum_{j=0}^{l} x_1^{(j)}(k)\varepsilon^j + O(\varepsilon^{l+1})] + \varepsilon^{k+1}[\sum_{j=0}^{l} m_1^{(j)}(k)\varepsilon^j + O(\varepsilon^{l+1})] \\ x_2(k,\varepsilon) = [\sum_{j=0}^{l} x_2^{(j)}(k)\varepsilon^j + O(\varepsilon^{l+1})] + \varepsilon^{k}[\sum_{j=0}^{l} m_2^{(j)}(k)\varepsilon^j + O(\varepsilon^{l+1})] \end{cases}$$

(3.7.a)

where i) for j = 0 (3.7.b)

$$\begin{cases} x_1^{(0)}(k+1) = A_{11}x_1^{(0)}(k) \quad ; \quad x_1^{(0)}(0) = x_{10} & (3.8.a) \\ x_2^{(0)}(k) = A_{21}A_{11}^{-1}x_1^{(0)}(k) & (3.8.b) \end{cases}$$

$$\begin{cases} m_2^{(0)}(k+1) = (A_{22} - A_{21}A_{11}^{-1}A_{12})m_2^{(0)}(k) \quad ; \quad m_2^{(0)}(0) = x_{20} - x_2^{(0)}(0) & (3.9.a) \\ m_1^{(0)}(k) = -A_{11}^{-1}A_{12}m_2^{(0)}(k) & (3.9.b) \end{cases}$$

ii) for $j \geq 1$

$$\begin{cases} x_1^{(j)}(k+1) = A_{11}x_1^{(j)}(k) + A_{12}x_2^{(j-1)}(k) \quad ; \quad x_1^{(j)}(0) = -m_1^{(j-1)}(0) & (3.10.a) \\ x_2^{(j)}(k) = A_{21}A_{11}^{-1}x_1^{(j)}(k) + (A_{22}-A_{21}A_{11}^{-1}A_{12})x_2^{(j-1)}(k-1) & (3.10.b) \end{cases}$$

$$\begin{cases} m_2^{(j)}(k+1) = (A_{22} - A_{21}A_{11}^{-1}A_{12})m_2^{(j)}(k) + A_{21}A_{11}^{-1}m_1^{(j-1)}(k+1) \\ \qquad\qquad ; \quad m_2^{(j)}(0) = -x_2^{(j)}(0) & (3.11.a) \\ m_1^{(j)}(k) = -A_{11}^{-1}A_{12}m_2^{(j)}(k) + A_{11}^{-1}m_1^{(j-1)}(k+1). & (3.11.b) \end{cases}$$

By use of the Riccati iteration algorithm, the initial value problem has an alternative expression as follows:

$$\begin{cases} x_1(k,\varepsilon) = [\sum_{j=0}^{l} x_s^{(j)}(k)\varepsilon^j + (\varepsilon^{l+1})] - \varepsilon^{k+1}[\sum_{j=0}^{l}\sum_{h=0}^{j} h_{j-h}m^{(h)}(k) + O(\varepsilon^{l+1})] \\ x_2(k,\varepsilon) = -[\sum_{j=0}^{l}\sum_{h=0}^{j} l_h x_s^{(j-h)}(k) + O(\varepsilon^{l+1})] \end{cases} \tag{3.12}$$

$$+ \varepsilon^k[\sum_{j=0}^{l}(m^{(j)}(k) + \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h} l_l h_{j-1-l-h} m^{(h)}(k)) + O(\varepsilon^{l+1})]$$

where $$x_s^{(j)}(k+1) = A_{11}x_s^{(j)}(k) - A_{12}\sum_{h=0}^{j-1} l_h x_s^{(j-1-h)}(k) \tag{3.13.a}$$

$$; \begin{cases} x_s^{(0)}(0) = x_{10} \\ x_s^{(j)}(0) = \sum_{h=0}^{j-1} h_h l_{j-1-h} x_{10} + h_{j-1}x_{20}, (j\geq 1) \end{cases} \tag{3.13.b}$$

$$m^{(j)}(k+1) = A_{22}m^{(j)}(k) + \sum_{h=0}^{j} l_h A_{12} m^{(j-h)}(k) \tag{3.14.a}$$

$$; \begin{cases} m^{(0)}(0) = l_0 x_{10} + x_{20} \\ m^{(j)}(0) = l_j x_{10} \quad (j\geq 1) \end{cases} \tag{3.14.b}$$

$$\begin{cases} l_j = [A_{22}l_{j-1} + \sum_{h=0}^{j-1} l_h A_{12} l_{j-1-h}]A_{11}^{-1} \quad ; \quad l_0 = -A_{21}A_{11}^{-1} & (3.15) \\ h_j = A_{11}^{-1}[h_{j-1}A_{22} + \sum_{h=0}^{j-1} h_{j-1-h} l_h A_{12}] + h_0 \sum_{h=0}^{j-1} l_h h_{j-1-h} & \\ \quad ; \quad h_0 = A_{11}^{-1}A_{12}. & (3.16) \end{cases}$$

<u>Proof</u> In (3.5), setting the transformation

$$x_f(k,\varepsilon) = \varepsilon^k m(k,\varepsilon) \tag{3.17}$$

yields

$$m(k+1,\varepsilon) = (A_{22} + LA_{12})m(k,\varepsilon) \quad ; \quad m(0,\varepsilon) = x_f(0,\varepsilon). \tag{3.18}$$

Accordingly, we have

$$\begin{cases} x_s(k+1,\varepsilon) = (A_{11} - A_{12}L)x_s(k,\varepsilon) \quad ; \quad x_s(0,\varepsilon) = (I+\varepsilon HL)x_{10} + \varepsilon Hx_{20} \\ m(k+1,\varepsilon) = (A_{22} + LA_{12})m(k,\varepsilon) \quad ; \quad m(0,\varepsilon) = Lx_{10} + x_{20}. \end{cases} \tag{3.19}$$

Then, by using the fast transformation (2.9), the solutions of initial value problem (3.1) can be expressed as

$$\begin{cases} x_1(k,\varepsilon) = x_s(k,\varepsilon) - \varepsilon^{k+1}H\, m(k,\varepsilon) \\ x_2(k,\varepsilon) = -Lx_s(k,\varepsilon) + \varepsilon^k(I + \varepsilon LH)m(k,\varepsilon). \end{cases} \tag{3.20}$$

This fact shows that the solutions of the singularly perturbed initial-value problem can be composed of the outer region term $(X_1(k,\varepsilon), X_2(k,\varepsilon))$ and the boundary-layer correction term $(m_1(k,\varepsilon), m_2(k,\varepsilon))$, i.e.,

$$\begin{cases} x_1(k,\varepsilon) = X_1(k,\varepsilon) + \tilde{m}_1(k,\varepsilon) \quad ; \quad \tilde{m}_1(k,\varepsilon) = \varepsilon^{k+1} m_1(k,\varepsilon) \\ x_2(k,\varepsilon) = X_2(k,\varepsilon) + \tilde{m}_2(k,\varepsilon) \quad ; \quad \tilde{m}_2(k,\varepsilon) = \varepsilon^{k} m_2(k,\varepsilon). \end{cases} \tag{3.21}$$

Here, we consider the NARE (3.3) and the Sylvester equation (3.4). By using the Kronecker product and rewriting them in the vector form

$$F_1(\mathbf{l},\varepsilon) = 0, \quad F_2(\mathbf{h},\varepsilon) = 0 \tag{3.22}$$

we have the Jacobian matrices

$$(\partial F_1/\partial \mathbf{l})|_{\varepsilon=0} = I \otimes A_{11}^T, \quad (\partial F_2/\partial \mathbf{h})|_{\varepsilon=0} = -A_{11} \otimes I \tag{3.23}$$

which are nonsingular under the conditions (A-1)-(A-3). Due to the implicit function theorem, the solutions $L(\varepsilon)$ and $H(\varepsilon)$ are analytic in ε at $\varepsilon = 0$, i.e.,

$$\begin{cases} L(\varepsilon) = \sum_{j=0}^{k} \hat{l}_j \varepsilon^j + O(\varepsilon^{k+1}) \quad ; \quad \hat{l}_0 = -A_{21}A_{11}^{-1} \\ H(\varepsilon) = \sum_{j=0}^{k} \hat{h}_j \varepsilon^j + O(\varepsilon^{k+1}) \quad ; \quad \hat{h}_0 = A_{11}^{-1}A_{12}. \end{cases} \tag{3.24}$$

On the other hand, by the following iterations, we have

$$\begin{cases} L_{k+1} = [\varepsilon A_{22}L_k + \varepsilon L_k A_{12} L_k - A_{21}]A_{11}^{-1} \quad ; \quad L_0 = -A_{21}A_{11}^{-1} \\ H_{k+1} = A_{11}^{-1}[\varepsilon H_k(A_{22} + LA_{12}) + \varepsilon A_{12}LH_k + A_{12}] \quad ; \quad H_0 = A_{11}^{-1}A_{12}. \end{cases} \tag{3.25}$$

The above iteration algorithms converge to unique dichotomic solutions L and H which satisfy (3.3) and (3.4) together with the two-time-scale property (2.45) under the conditions (A-1)-(A-3).

Then, by using mathematical induction, it can be easily shown that (Appendix. A)

$$\hat{l}_j = l_j \quad \text{and} \quad \hat{h}_j = h_j \quad ; \quad j = 0,1,\ldots,k. \tag{3.26}$$

In this way, there exists a positive ε^* such that for $^\forall\varepsilon \in [0,\varepsilon^*)$ the expansion (3.24) converges, where

$$\varepsilon^* = \min\left[\frac{1}{2a_s(a_f+a_{12}l_0)}, \frac{(a_f+a_{12}l_0)}{a_s((a_f+a_{12}l_0)^2+8a_f a_{12}l_0)}\right] \tag{3.27}$$

$$; \quad a_s = \|A_{11}^{-1}\|, \quad a_f = \|A_{22} - A_{21}A_{11}^{-1}A_{12}\|$$

$$a_{12} = \|A_{12}\|, \; l_0 = \|A_{21}A_{11}^{-1}\|.$$

Here, by using the asymptotic expansion

$$\begin{cases} x_s(k,\varepsilon) = \sum_{j=0}^{k} x_s^{(j)}(k)\varepsilon^j + O(\varepsilon^{k+1}) \\ m(k,\varepsilon) = \sum_{j=0}^{k} m^{(j)}(k)\varepsilon^j + O(\varepsilon^{k+1}) \end{cases} \tag{3.28}$$

and (3.24), we can obtain the asymptotic series solutions (3.12). The equivalence between (3.7) and (3.12) must be proved. That is, in case of setting in (3.12) as

$$\begin{cases} x_1^{(j)}(k) = x_s^{(j)}(k) \\ x_2^{(j)}(k) = -\sum_{h=0}^{j} l_h x_s^{(j-h)}(k) \end{cases} \tag{3.29.a}$$

$$\begin{cases} m_1^{(j)}(k) = -\sum_{h=0}^{j} h_{j-h} m^{(h)}(k) \\ m_2^{(j)}(k) = m^{(j)}(k) + \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h} l_l h_{j-1-h-l} m^{(h)}(k) \end{cases} \tag{3.29.b}$$

the left terms must satisfy the recursive equations (3.10.a)-(3.11.b) respectively. By using the relations (3.15) and (3.16) in the process of algebraic manipulations, these proofs can be given as follows:

[I] Eqn. (3.10.a) follows from eqn. (3.13) straightforwards.

$$\begin{aligned} \text{[II]} \quad x_2^{(j)}(k+1) &= -\sum_{h=0}^{j} l_h x_s^{(j-h)}(k+1) \\ &= -l_0 A_{11} x_s^{(j)}(k) - \sum_{h=1}^{j} l_h A_{11} x_s^{(j-h)}(k) \\ &\quad + \sum_{h=0}^{j}\sum_{l=0}^{j-h-1} l_h A_{12} l_l x_s^{(j-1-h-l)}(k) \\ &= -l_0 A_{11} x_s^{(j)}(k) - \sum_{h=0}^{j-1} [A_{22} l_h + \sum_{l=0}^{h} l_l A_{12} l_{h-l}] x_s^{(j-h-1)}(k) \\ &\quad + \sum_{h=0}^{j}\sum_{l=0}^{j-h-1} l_h A_{12} l_l x_s^{(j-1-h-l)}(k) \end{aligned}$$

$$= -l_0 A_{11} x_s^{(j)}(k) - A_{22}\sum_{h=0}^{j-1} l_h x_s^{(j-h-1)}(k)$$

$$= -l_0 [A_{11} X_1^{(j)}(k) + A_{12} X_2^{(j-1)}(k)] + [A_{22} + l_0 A_{12}] X_2^{(j-1)}(k)$$

$$= -l_0 X_1^{(j)}(k+1) + [A_{22} + l_0 A_{12}] X_2^{(j-1)}(k). \qquad (3.10.b)$$

[III] $$m_2^{(j)}(k+1) = m^{(j)}(k+1) + \sum_{h=0}^{j-1}\left(\sum_{l=0}^{j-1-h} l_l h_{j-1-h-l}\right) m^{(h)}(k+1)$$

$$= [A_{22} m^{(j)}(k) + \sum_{h=0}^{j} l_h A_{12} m^{(j-h)}(k)]$$

$$+ \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h} l_l h_{j-1-h-l} [A_{22} m^{(h)}(k) + \sum_{p=0}^{h} l_p A_{12} m^{(h-p)}(k)]$$

$$= [A_{22} + l_0 A_{12}] m^{(j)}(k) + \sum_{h=1}^{j} l_h A_{11} h_0 m^{(j-h)}(k)$$

$$+ \sum_{h=0}^{j-1}\sum_{l=1}^{j-1-h} l_l h_{j-1-h-l} A_{22} m^{(h)}(k) + \sum_{h=0}^{j-1}\sum_{l=1}^{j-1-h} l_l h_{j-1-h-l} \sum_{p=0}^{h} l_p A_{12} m^{(h-p)}(k)$$

$$+ l_0 \sum_{h=0}^{j-1} h_{j-1-h} [A_{22} m^{(h)}(k) + \sum_{p=0}^{h} l_p A_{12} m^{(h-p)}(k)]$$

$$= [A_{22} + l_0 A_{12}] m^{(j)}(k) + \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h} l_{l+1} A_{11} h_{j-1-h-l} m^{(h)}(k)$$

$$- \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h}\sum_{p=1}^{l} l_p A_{12} l_{l-p} h_{j-1-h-l} m^{(h)}(k) + l_0 \sum_{h=0}^{j-1} h_{j-1-h} [A_{22} m^{(h)}(k)$$

$$+ \sum_{p=0}^{h} l_p A_{12} m^{(h-p)}(k)] \sum_{h=0}^{j-1}\sum_{p=1}^{j-1-h} l_{j-h-p} A_{11} h_p m^{(h)}(k)$$

$$= [A_{22} + l_0 A_{12}] m^{(j)}(k) + \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h} A_{22} l_l h_{j-1-h-l} m^{(h)}(k)$$

$$+ \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h} l_0 A_{12} l_l h_{j-1-h-l} m^{(h)}(k) + l_0 \sum_{h=0}^{j-1} h_{j-1-h} [A_{22} m^{(h)}(k)$$

$$+ \sum_{p=0}^{h} l_p A_{12} m^{(h-p)}(k)]$$

$$= [A_{22} + l_0 A_{12}] [m^{(j)}(k) + \sum_{h=0}^{j-1}\sum_{l=0}^{j-1-h} l_l h_{j-1-h-l} m^{(h)}(k)]$$

$$+ l_0 \sum_{h=0}^{j-1} h_{j-1-h} [A_{22} m^{(h)}(k) + \sum_{l=0}^{h} l_l A_{12} m^{(h-l)}(k)]$$

$$= [A_{22} + l_0 A_{12}] m_2^{(j)}(k) - l_0 m_1^{(j-1)}(k+1) \qquad (3.11.a)$$

[IV] $$m_1^{(j)}(k) = -\sum_{h=0}^{j} h_{j-h} m^{(h)}(k)$$

$$= -h_0 m^{(j)}(k) - \sum_{h=0}^{j-1} h_{h+1} m^{(j-1-h)}(k)$$

$$= -h_0 m^{(j)}(k) - h_0 \sum_{h=0}^{j-1} \sum_{l=0}^{j-1-h} l_l h_{j-1-l-h} m^{(h)}(k)$$

$$- \sum_{h=0}^{j-1} h_{h+1} m^{(j-1-h)}(k) + A_{11}^{-1} \sum_{h=0}^{j-1} \sum_{l=0}^{h} h_{h-l} l_l A_{12} m^{(j-1-h)}(k)$$

$$+ h_0 \sum_{h=0}^{j-1} \sum_{l=0}^{h} l_l h_{h-l} m^{(j-1-h)}(k) - A_{11}^{-1} \sum_{h=0}^{j-1} h_{j-1-h} \sum_{l=0}^{h} l_l A_{12} m^{(h-l)}(k)$$

$$= -h_0 m^{(j)}(k) - h_0 \sum_{h=0}^{j-1} \sum_{l=0}^{j-1-h} l_l h_{j-1-l-h} m^{(h)}(k)$$

$$- A_{11}^{-1} \sum_{h=0}^{j-1} h_{j-1-h} [A_{22} m^{(h)}(k) + \sum_{l=0}^{h} l_l A_{12} m^{(h-l)}(k)]$$

$$= -h_0 m_2^{(j)}(k) + A_{11}^{-1} m_1^{(j-1)}(k+1). \qquad (3.11.b)$$

[V] Finally, the initial conditions can be obtained by using (3.13.b) and (3.14.b) as follows:

(i) for $j = 0$

$$X^{(0)}(0) = x^{(0)}(0) = x_{10}$$

$$X^{(0)}(0) + m^{(0)}(0) = -l_0 x^{(0)}(0) + m^{(0)}(0)$$

$$= x_{20}$$

(ii) for $j \geq 1$

$$x_1^{(j)}(0) + m_1^{(j-1)}(0) = x_s^{(j)}(0) - \sum_{h=0}^{j-1} h_{j-1-h} m^{(h)}(0)$$

$$= x_s^{(j)}(0) - [h_{j-1} m^{(0)}(0) + \sum_{h=1}^{j-1} h_{j-1-h} m^{(h)}(0)]$$

$$= [\sum_{h=0}^{j-2} h_h l_{j-1-h} x_{10} + h_{j-1} l_0 x_{10} + h_{j-1} x_{20}]$$

$$- h_{j-1}(l_0 x_{10} + x_{20}) - \sum_{h=1}^{j-1} h_{j-1-h} l_h x_{10}$$

$$= 0$$

$$x_2^{(j)}(0) + m_2^{(j)}(0) = -\sum_{h=0}^{j} l_h x^{(j-h)}(0) + [m^{(j)}(0) + \sum_{h=0}^{j-1} \sum_{l=0}^{j-1-h} l_l h_{j-1-l-h} m^{(h)}(0)]$$

$$= -\sum_{h=0}^{j-1} l_h [\sum_{l=0}^{j-h-1} h_l l_{j-h-1-l} x_{10} + h_{j-h-1} x_{20}] - l_j x_s^{(0)}(0)$$

$$+ l_j x_{10} + \sum_{h=1}^{j-1} \sum_{l=0}^{j-1-h} l_l h_{j-1-l-h} m^{(h)}(0) + \sum_{l=0}^{j-1} l_l h_{j-1-l} m^{(0)}(0)$$

$$= -\sum_{h=0}^{j-1} l_h \sum_{l=0}^{j-h-1} h_l l_{j-h-1-l} x_{10} - \sum_{h=0}^{j-1} l_h h_{j-h-1} x_{20}$$

$$+ \sum_{h=1}^{j-1} \sum_{l=0}^{j-1-h} l_l h_{j-1-l-h} l_h x_{10} + \sum_{l=0}^{j-1} l_l h_{j-1-l} (l_0 x_{10} + x_{20})$$

$$= -\sum_{h=0}^{j-1} l_h \sum_{l=0}^{j-h-1} h_l l_{j-h-1-l} x_{10} + \sum_{h=0}^{j-1} \sum_{l=0}^{j-1-h} l_l h_{j-1-l-h} l_h x_{10}$$

$$= 0.$$ Q.E.D.

[Remarks]

In the initial value problems, the asymptotic series solutions (3.7) can be constructed easily by solving the expressions (3.8.a)-(3.11.b) successively, which are obtained by using the asymptotic series expansions and by equating coefficients of ε^j for $j \geq 0$ on the both sides of full systems which the outer region and the boundary-layer correction terms must satisfy.

3.2.2 Reduced order models via the singular perturbation method and the dominant mode method

Amongst the many model reduction methods (Sandell et al. [San.78], Jamshidi [Jam.83], Lamba et al. [Lam.82]), that to obtain the reduced-order model which retains the dominant eigenvalues of the original system is called the dominant mode method or the modal aggregation (Jamshidi [Jm.83]). As representative reduced-order models, the first model of Davison [Da.66], the Marshall-Chidambara model [Mar.66], [Chi.67.1], the Chidambara-Davison model [Chi.67.2],[Da.68.1] and the second model of Davison [Da.68.1] are well known. A comparison study among them was given by Fossard [Fo.70]. Recent studies on this method have been reviewed by Bonzin et al. [Bo.82].

Here, we will consider the Marshal-Chidambara model of the discrete-time system (3.1). Using the modal matrix $\tilde{M}$

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \tag{3.30}$$

and transforming (3.1) into the Jordan form yields

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & \tilde{\Lambda}_2 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} u(k). \tag{3.31}$$

where $\lambda(\tilde{\Lambda}_1)$ and $\lambda(\tilde{\Lambda}_2)$ are slow and fast eigenvalues, respectively.
We assume that u(k) is a constant input, i.e., $u(k) = \bar{u}$.
From (3.31), we have

$$z_2(k+1)-z_2(k) = (\tilde{\Lambda}_2 - I)z_2(k) + G_2\bar{u}. \tag{3.32}$$

Since the fast variable $z_2(k)$ immediately attains the steady state, we have

$$\bar{z}_2(k) = \bar{z}_2(k+1) = (I - \tilde{\Lambda}_2)^{-1}\tilde{G}_2\bar{u}. \tag{3.33}$$

Substituting (3.33) into (3.30) yields

$$\bar{x}_2(k) = \tilde{M}_{21}\tilde{M}_{11}^{-1}\bar{x}_1(k) + (\tilde{M}_{22} - \tilde{M}_{21}\tilde{M}_{11}^{-1}\tilde{M}_{12})(I - \tilde{\Lambda}_2)^{-1}\tilde{G}_2\bar{u}. \tag{3.34}$$

Then, the Marshall-Chidambara model is represented as

$$\bar{x}_1(k+1) = (\tilde{A}_{11}+\tilde{A}_{12}\tilde{M}_{21}\tilde{M}_{11}^{-1})\bar{x}_1(k)+[\tilde{B}_1+\tilde{A}_{12}(\tilde{M}_{22}-\tilde{M}_{21}\tilde{M}_{11}^{-1}\tilde{M}_{12})(I-\tilde{\Lambda}_2)^{-1}\tilde{G}_2]\bar{u}. \tag{3.35}$$

It is noted that the above reduced-order model has the same steady state as the original model for a constant input, but it is a poor expression of the original model for the polynomial inputs such as a ramp (Chidambara [Chi.69]).

Here, we will consider the Marshall-Chidambara model of the following singularly perturbed discrete-time system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \bar{u}. \tag{3.36}$$

Due to the fast transformation (3.2), the Marshall-Chidambara model reduces to

$$
\begin{cases}
\bar{x}_1(k+1) = [A_{11} + O(\varepsilon)]\bar{x}_1(k) + [B_1 + O(\varepsilon)]\bar{u} \\
\bar{x}_2(k) = [\ A_{21}A_{11}^{-1} + O(\varepsilon)]\bar{x}_1(k) + [(B_2 - A_{21}A_{11}^{-1}B_1) + O(\varepsilon)]\bar{u}.
\end{cases}
\tag{3.37}
$$

On the other hand, by setting $\varepsilon = 0$ in (3.36), we have the reduced system of the singularly perturbed discrete-time system:

$$
\begin{cases}
x_1^{(0)}(k+1) = A_{11}x_1^{(0)}(k) + B_1\bar{u} \\
x_2^{(0)}(k) = A_{21}A_{11}^{-1}x_1^{(0)}(k) + (B_2 - A_{21}A_{11}^{-1}B_1)\bar{u}.
\end{cases}
\tag{3.38}
$$

Accordingly, upon neglecting the $O(\varepsilon)$-error, we obtain the equivalence between the reduced-order model via singular perturbation method and the reduced-order model via dominant mode method. This relationship implies that the reduced system corresponds to the quasi-stedy-state model after the fast mode $x_f(k)$ decayed.

3.2.3 Numerical example

As an illustrative example, we will consider a single-machine power system (Hassan [Has.81]). The system equation is given by

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \\ x_7(k+1) \end{bmatrix}
=
\begin{bmatrix}
0.9949 & 0.2444 & -0.6165\times10^{-2} & -0.9865\times10^{-4} & -0.1049\times10^{-2} & 0.9422\times10^{-4} & -0.5795\times10^{-3} \\
-0.3837\times10^{-1} & 0.9542 & -0.4948\times10^{-1} & -0.3733\times10^{-3} & -0.5590\times10^{-2} & 0.3139\times10^{-2} & -0.2686\times10^{-2} \\
-0.1020 & -0.1261\times10^{-1} & 0.7994 & 0.7580\times10^{-4} & 0.3482\times10^{-1} & 0.1357\times10^{-2} & 0.1182\times10^{-2} \\
-0.4938 & -0.1147 & 0.3781 & 0.7607\times10^{-4} & 0.1684\times10^{-1} & 0.5959\times10^{-3} & 0.8880\times10^{-3} \\
-0.2114 & -0.3861\times10^{-1} & 0.6330 & 0.6876\times10^{-4} & 0.2777\times10^{-1} & 0.1064\times10^{-2} & 0.1064\times10^{-2} \\
-0.3087 & -0.7338\times10^{-1} & -0.5320\times10^{-1} & 0.3419\times10^{-5} & -0.2129\times10^{-2} & -0.1214\times10^{-3} & 0.5565\times10^{-3} \\
-0.2321 & -0.4879\times10^{-1} & -0.4093\times10^{-1} & -0.4523\times10^{-4} & -0.1753\times10^{-2} & -0.8854\times10^{-4} & 0.1382\times10^{-2}
\end{bmatrix}
\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \end{bmatrix}
\tag{3.39}
$$

where x_1 is the rotor angle, x_2 the rotor speed, x_3 the field flux linkage, x_4 the flux linkage of armature direct axis, x_5 the flux linkage of armature quadratic axis, x_6 the flux linkage of damper direct axis, and x_7 the flux linkage of damper quadratic axis.

In this problem, the conditions (A-1)-(A-3) are all satisfied as follows:

(A-1) $\max[0.0923,\ 0.0600] < 1$

(A-2) $\varepsilon = \tilde{a}_s\tilde{a}_f = 0.0018 << 1$

(A-3) $\tilde{a}_s\tilde{a}_{12} = 0.0448 << 1$

where $\tilde{a}_s = \|\tilde{A}_{11}^{-1}\| = 1.2684$, $\tilde{a}_f = \|\tilde{A}_{22}-\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}\| = 0.1420\times 10^{-2}$

$\tilde{a}_{12} = \|\tilde{A}_{12}\| = 0.3533\times 10^{-1}$, $l_0 = \|\tilde{A}_{21}\tilde{A}_{11}^{-1}\| = 0.9914.$

For reference, the eigenvalues are shown:

(a) exact eigenvalues

$$\begin{cases} \lambda(\tilde{A}_s) = 0.8946+j0.5425\times 10^{-1},\ 0.9869 \\ \lambda(\tilde{A}_f) = 0.1313\times 10^{-2},\ 0.1106\times 10^{-3},\ -0.8522\times 10^{-6},\ -0.4594\times 10^{-7} \end{cases}$$

(b) approximate eigenvalue

$$\begin{cases} \lambda(\tilde{A}_{11}) = 0.9554+j0.4597\times 10^{-1},\ 0.8376 \\ \lambda(\tilde{A}_{11}+\tilde{A}_{12}\tilde{A}_{21}\tilde{A}_{11}^{-1}) = 0.8947+j0.5425\times 10^{-1},\ 0.9868 \\ \lambda(\tilde{A}_{22}-\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}) = 0.1313\times 10^{-2},\ 0.1144\times 10^{-3},\ -0.8523\times 10^{-6}, \\ \qquad -0.4593\times 10^{-7} \end{cases}$$

and the coupling ratio is $\rho_s = \|\tilde{M}_{12}\|/\|\tilde{M}_{22}\| = 0.0212<<1.$ Accordingly, by following the procedure of the preceding section, we can obtain the approximate solutions (3.7.a) and (3.7.b) of this problem. Figure 3.1 and 3.2 show the responses x_1 and x_5 caused by an initial perturbation of

$X(0) = [0.05,\ 0,\ 0.05,\ 0,\ 0,\ 0,\ 0]^T.$

In the fast variable x_5 (Fig.3.2), the boundary-layer jump phenemenon appears near $k = 0$, and the boundary-layer correction decays to zero fast in the interval $k\geq 1$. In this problem, the first-order approximation almost coincides with the exact solution.

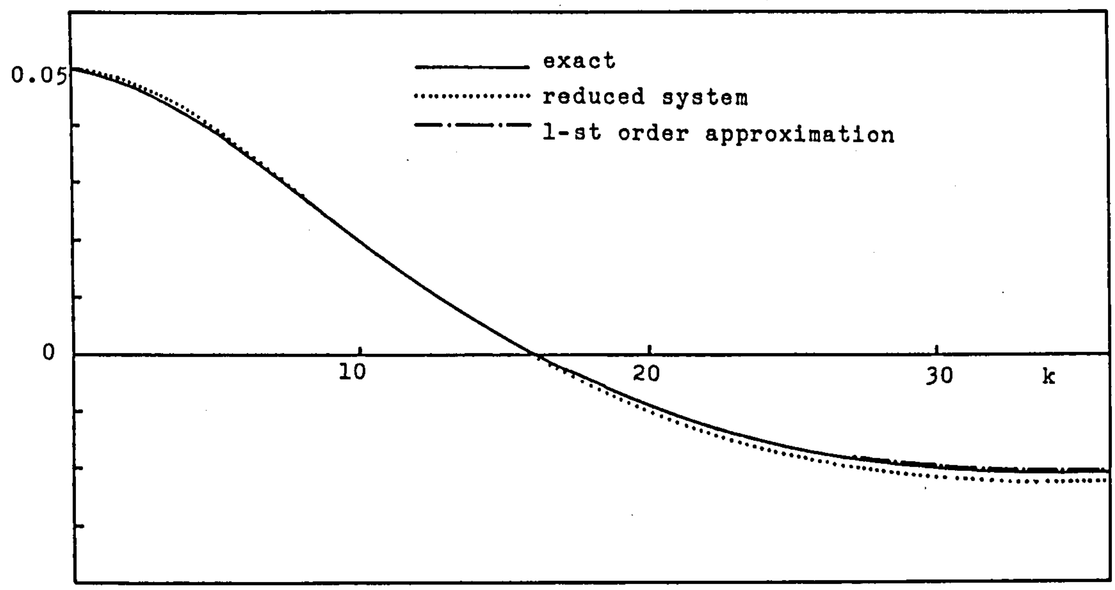

Fig. 3.1. Response of the state $x_1(k)$.

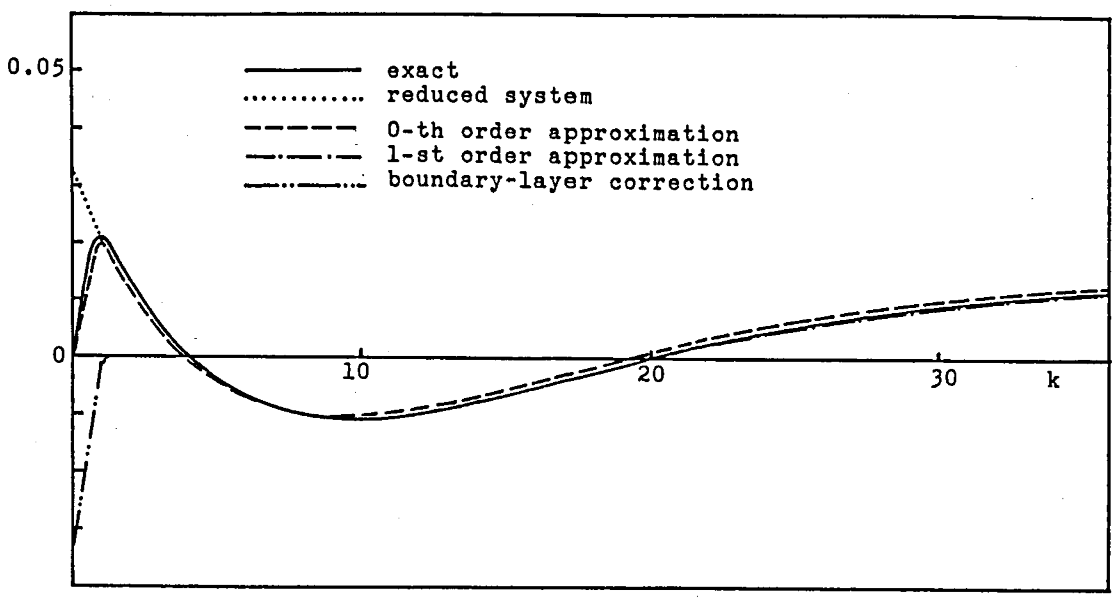

Fig. 3.2. Response of the state $x_5(k)$.

3.3 Initial Value Problems of Continuous-Time Systems

We consider the initial value problem of the continuous-time system:

$$\begin{bmatrix} dx_1/dt \\ \varepsilon dx_2/dt \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad ; \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{3.40}$$

Then, we have an alternative asymptotic series expression in addition to a conventional expression via the boundary layer method, which is shown in the following theorem.

<u>Theorem 3-2</u> Let the conditions (A-1)-(A-3) in Lemma 2-1 be satisfied in the system (3.40).

Then, the initial value problem (3.40) has the following unique asymptotic series solutions:

$$\begin{cases} x_1(t,\varepsilon) = \sum_{j=0}^{N} \xi_j(t)\varepsilon^j - \sum_{j=0}^{N}\sum_{k=0}^{j} h_k \eta_{j-k}(\tau)\varepsilon^{j+1} + O(\varepsilon^{N+1}) & (3.41.a) \\ x_2(t,\varepsilon) = -\sum_{j=0}^{N}\sum_{k=0}^{j} l_k \xi_{j-k}(t)\varepsilon^j + \sum_{j=0}^{N} [\eta_j(\tau) + \sum_{h=0}^{j-1}\sum_{k=0}^{j-1-h} l_k h_{j-1-h-k} \eta_h(\tau)]\varepsilon^j & \\ \qquad + O(\varepsilon^{N+1}) \;;\; \tau = t/\varepsilon & (3.41.b) \end{cases}$$

where

$$d\xi_j/dt = A_{11}\xi_j(t) - A_{12}\sum_{k=0}^{j} l_k \xi_{j-k}(t) \tag{3.42.a}$$

$$; \begin{cases} \xi_0(0) = x_{10} \\ \xi_j(0) = \sum_{k=0}^{j-1} h_k l_{j-1-k} x_{10} + h_{j-1}x_{20} \quad (j \geq 1) \end{cases}$$

$$d\eta_j/d\tau = A_{22}\eta_j(\tau) + \sum_{k=0}^{j-1} l_k A_{12}\eta_{j-1-k}(\tau) \tag{3.42.b}$$

$$; \begin{cases} \eta_0(0) = l_0 x_{10} + x_{20} \\ \eta_j(0) = l_j x_{10} \quad (j \geq 1) \end{cases}$$

$$\begin{cases} l_{k+1} = A_{22}^{-1}[l_k A_{11} - \sum_{j=0}^{k} l_j A_{12} l_{k-j}] \quad ; \quad l_0 = A_{22}^{-1}A_{21} & (3.43.a) \\ h_{k+1} = [A_{11}h_k - A_{12}\sum_{j=0}^{k} l_j h_{k-j} - \sum_{j=0}^{k} h_j l_{k-j} A_{12}]A_{22}^{-1} & (3.43.b) \\ \qquad ; \quad h_0 = -A_{12}A_{22}^{-1}. & \end{cases}$$

Proof By applying the boundary layer method (O'Malley [OM.71.1], [OM.74]) to (3.40), we obtain asymptotic series expansions:

$$\begin{cases} x_1(t,\varepsilon) = \sum_{j=0}^{N} X_{1j}(t)\varepsilon^j + \sum_{j=0}^{N} m_{1j}(\tau)\varepsilon^{j+1} + O(\varepsilon^{N+1})\ ;\ \tau = t/\varepsilon & (3.44.a) \\ x_2(t,\varepsilon) = \sum_{j=0}^{N} X_{2j}(t)\varepsilon^j + \sum_{j=0}^{N} m_{2j}(\tau)\varepsilon^{j} + O(\varepsilon^{N+1}) & (3.44.b) \end{cases}$$

where $(X_{1j}(t), X_{2j}(t))$ and $(m_{1j}(\tau), m_{2j}(\tau))$ satisfy

$$\begin{cases} dX_{1j}/dt = A_{11}X_{1j}(t) + A_{12}X_{2j}(t) & (3.45.a) \\ dX_{2j-1}/dt = A_{21}X_{1j}(t) + A_{22}X_{2j}(t) & (3.45.b) \end{cases}$$

$$\begin{cases} dm_{1j}/d\tau = A_{11}m_{1j-1}(\tau) + A_{12}m_{2j}(\tau) & (3.46.a) \\ dm_{2j}/d\tau = A_{21}m_{1j-1}(\tau) + A_{22}m_{2j}(\tau) & (3.46.b) \end{cases}$$

under the initial conditions :

$$\begin{cases} X_{10}(0) = x_{10} & (3.47.a) \\ X_{20}(0) + m_{20}(0) = x_{20} & (3.47.b) \end{cases}$$

$$\begin{cases} X_{1j}(0) + m_{1j-1}(0) = 0 \quad (\ j\geq 1\ ) & (3.48.a) \\ X_{2j}(0) + m_{2j}(0) = 0 \quad (\ j\geq 1\ ) & (3.48.b) \end{cases}$$

and $\lim_{\tau\to\infty} m_{1j}(\tau) = 0$, $\lim_{\tau\to\infty} m_{2j}(\tau) = 0$.

Then, similarly as the discrete-time case, the equivalence between (3.41) and (3.44) can be proved after considerable algebraic manipulations. Furthermore, it can be shown that (3.41) is equivalent to the approximate solutions via the q.s.s. method (Kokotović et al. [Ko.80],[Win.80]) shown in Section 2.2.3. Q.E.D.

3.4 Concluding Remarks

In this chapter, the initial value problem for the singularly perturbed discrete-time systems has been studied. The boundary layer method for the discrete-time version has been formulated, and then the relationship between the asymptotic series solutions via the boundary layer method and those via the Riccati iteration method has been studied. These asymptotic solutions can be obtained systematically via the recursive computational procedure. Furthermore, the effectiveness of the boundary layer method has been verified in the numerical example.

In this way, this method provides us a useful tool for large-scale system analysis and design. In the following chapter, this method will be applied to the controller designs of discrete-time systems with two-time-scale property.

CHAPTER 4. FEEDBACK CONTROLLER DESIGN OF DISCRETE-TIME SYSTEMS

4.1 Introduction

Keeping in step with the spread of reliable and inexpensive microcomputers, the problems associated with digital controls have recently aroused the interest of numerous engineers (Mahalanabis [Mah.82], Ito [It.83]). When we consider the design of such digital controllers, we are often confronted with the problems of high dimensionality and stiffness difficulties.

In order to avoid these difficulties, the studies on model reduction methods for analysis and design of discrete systems with slow and fast modes have been extensively developed in recent years (Saksena et al. [Sak.84], Javid [Jav.79]). These studies can be classified into the fast-time-scale version and the slow-time-scale version.

The former version is based on the numerical approximation such as an euler approximation (Rajagopalan et al. [Raj.81], Blankenship [Bl.81], Kimura [Ki.83], Tran et al. [Tr.83.1], Litkouhi et al. [Lit.84], Kando et al. [Ka.86.1]) or based on the quasi-steady-state concepts (Mahmoud [Mam.82.1],[Mam.82.2],[Mam.82.3],[Mam.85], Tran et al. [Tr.83.2], [Tr.83.3],[Tr.84]). The system design and analysis is performed from the viewpoint of fast-time-scale. This design policy provides us a hybrid solution of a continuous-time slow subsystem and a discrete-time fast subsystem. The continuous-time slow solution mainly dominates the behavior of the original system over whole interval and the discrete-time fast solution functions only near the initial time to correct the boundary-layer jump. Thus, although the fast-time-scale version is discrete-type apparently, the main part of design and analysis is essentially performed in the continuous-time domain.

On the other hand, the latter version coincides with the state space form (Rajagopalan et al. [Raj.80.1], Rao et al. [Rao.81], Naidu et al. [Na.81],[Na.82],[Na.85]) corresponding to the singularly perturbed difference equations (Comstock et al. [Co.76], Hsio et al. [Hs.79], Reinhardt [Re.79]). Thus, these studies based on this version are useful for design and analysis of discrete-time systems essentially represented by difference formulas such as economic, ecology and production-inventory control systems etc. (Bishop [Bi.75], Naidu et al. [Na.85]). In this version, the two-time-scale property of the discrete-time system itself and the lower sampling rate are assumed. The system behavior is analyzed from the viewpoint of slow-time-scale, and as the result, the solutions of original systems are expressed as a combination of discrete-time slow and fast subsystems (Phillips [Ph.80], Kando et al. [Ka.83.2],[Ka.83.3],[Ka.84],[Ka.85], Rao et al. [Rao.82], Naidu et al. [Na.84]).

A combination of these two versions leads to the multirate controller design naturally (Litkouhi et al. [Lit.85], Kando et al. [Ka.86.2]).

In this chapter, firstly, it will be shown that the singularly perturbed continuous-time system leads to two different discrete-time models via slow and fast sampling rates, and then the properties of these models will be discussed briefly. In Section 4.2, the stabilizing problem will be dealt with and by applying the results, the observer and the state feedback design based on the observer will be studied in Section 4.3. In Section 4.4, by using the boundary layer method discussed in Chapter 3, a near optimum controller design of the regulator problem will be investigated. An extention to multi-time-scale systems will be discussed in Section 4.5. These studies are investigated from the viewpoint of slow-time-scale. In Section 4.6, the stabilizing problem will be treated from the viewpoint of fast-time-scale, which is a counterpart of results shown in Section 4.2. On the basis of these results of two time-scale versions, the multirate digital controller design will be developed in Section 4.7.

4.1.1 Formulation of discrete-time systems with two-time-scale property

It has been shown in Section 2.3.2 that the singularly perturbed continuous-time system via slow and fast sampling rates leads to two different discrete-time formulas, i.e., the slow sampling model and the fast sampling model. In this section, the properties of these models are discussed briefly.

[I] Slow sampling model

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} A_{11}^{s}, \varepsilon A_{12}^{s} \\ A_{21}^{s}, \varepsilon A_{22}^{s} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} B_{1}^{s} \\ B_{2}^{s} \end{bmatrix} u_k \tag{4.1}$$

where $A_{11}^{s} = \exp(A_0)$, $\quad A_{12}^{s} = \exp(A_0)M_0$

$A_{21}^{s} = -L_0\exp(A_0)$, $\quad A_{22}^{s} = -L_0\exp(A_0)M_0 + \bar{A}_f$

$B_1^{s} = (\exp(A_0)-I)A_0^{-1}B_0$, $\quad B_2^{s} = -L_0(\exp(A_0)-I)A_0^{-1}B_0 - A_{22}^{-1}B_2$.

In the above, the notations A_{ij}, B_i represent the continuous-time version and A_{ij}^{s}, B_i^{s} the discrete-time version.

[Remarks]

1. The slow sampling model (4.1) coincides with the state space form (Rajagopalan et al. [Raj.80.1], Rao et al. [Rao.81], Naidu et al. [Na.81], [Na.82],[Na.85]) of the singularly perturbed difference equations (Comstock et al. [Co.76], Hsio et al. [Hs.79], Reinhardt [Re.79]). In this model, the similar properties to the singularly perturbed continuous-time system appear, e.g., the two-time-scale property of discrete-time system itself, i.e., $\max_i |\lambda_i(A_f^{s})| \ll \min_j |\lambda_j(A_s^{s})|$ where $A_s^{s} = A_{11}^{s}-\varepsilon A_{12}^{s}L^{s}$, $A_f^{s} = \varepsilon(A_{22}^{s}+L^{s}A_{12}^{s})$, and the boundary-layer jump phenomenon etc.

2. By using the boundary layer method (Kando et al. [Ka.83.2]) and by setting as

$$\begin{cases} x_1(k,\varepsilon) = X_1(k,\varepsilon) + \varepsilon^{k+1} m_1(k,\varepsilon) \\ x_2(k,\varepsilon) = X_2(k,\varepsilon) + \varepsilon^{k} m_2(k,\varepsilon) \\ u(k,\varepsilon) = U(k,\varepsilon) + \varepsilon^{k+1} v(k,\varepsilon), \end{cases} \tag{4.2}$$

we have the approximate solutions of (4.1):

$$\begin{cases} x_1(k,\varepsilon) = X_{10}(k) + O(\varepsilon) \\ x_2(k,\varepsilon) = X_{20}(k) + \varepsilon^{k} m_{20}(k) + O(\varepsilon) \end{cases} \tag{4.3}$$

where
$$\begin{cases} X_{10}(k+1) = A_{11}^{s} X_{10}(k) + B_1^{s} U_0(k) \\ X_{20}(k) = A_{21}^{s} A_{11}^{s-1} X_{10}(k) + (B_2^{s} - A_{21}^{s} A_{11}^{s-1} B_1^{s}) U_0(k-1) \end{cases} \tag{4.4}$$

$$m_{20}(k+1) = (A_{22}^{s} - A_{21}^{s} A_{11}^{s-1} A_{12}^{s}) m_{20}(k) + (B_2^{s} - A_{21}^{s} A_{11}^{s-1} B_1^{s}) v_0(k). \tag{4.5}$$

Thus, in this model, the solutions of (4.1) can be expressed as a combination of two discrete-time subsystems. However, it is noted that the fast part is treated as dead beat, and as the result, the performance degradation between the original continuous-time version and the discrete-time version is inevitable.

[II] Fast sampling model

$$\begin{bmatrix} x_{1,n+1} \\ x_{2,n+1} \end{bmatrix} = \begin{bmatrix} I+\varepsilon A_{11}^{f}, & \varepsilon A_{12}^{f} \\ A_{21}^{f}, & A_{22}^{f} \end{bmatrix} \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix} + \begin{bmatrix} \varepsilon B_1^{f} \\ B_2^{f} \end{bmatrix} u_n \tag{4.6}$$

where $x_{i,n} \triangleq x_i(nT_f,\varepsilon)$, $u_n \triangleq u(nT_f,\varepsilon)$

$$A_{11}^{f} = A_0 + M_0(I-\exp(A_{22}))L_0, \quad A_{12}^{f} = M_0(I-\exp(A_{22}))$$

$$A_{21}^{f} = (\exp(A_{22})-I)L_0, \qquad A_{22}^{f} = \exp(A_{22})$$

$$B_1^{f} = B_0 - M_0(\exp(A_{22})-I)A_{22}^{-1}B_2, \quad B_2^{f} = (\exp(A_{22})-I)A_{22}^{-1}B_2$$

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_0 = B_1 - A_{12}A_{22}^{-1}B_2.$$

In the above, the notations A_{ij}, B_i represent the continuous-time version and A^f_{ij}, B^f_i the discrete-time version.

[Remarks]

1. The eqn.(4.6) is more exact model than an euler approximation model (Blankenship [Bl.81], Rajagopalan et al. [Raj.81]). In eqn. (4.6), 「A_{22} : Hurwitz matrix」 implies $|\lambda(A^f_{22})| < 1$. However, in the euler approximation model, this is not necessarily valid because $A^f_{22} = I+A_{22}$, and thus the additional assumption $|\lambda(I+A_{22})| <1$ is required to guarantee the asymptotic stability in the initial value problem or in the open-loop controlled system.

2. The eigenvalues of the slow and fast parts of the fast sampling model (4.6) are given by

$$\begin{cases} \lambda(A^f_s) = \lambda[I + \varepsilon(A^f_{11}-A^f_{12}L^f(\varepsilon))] = \lambda(\exp(\varepsilon A_0) + O(\varepsilon)) \\ \lambda(A^f_f) = \lambda(A^f_{22} + \varepsilon L^f(\varepsilon)A^f_{12}) = \lambda(\exp(A_{22}) + O(\varepsilon)), \end{cases} \tag{4.7}$$

where $L^f(\varepsilon)$ is a dichotomic solution of the NARE

$$R(L^f) = -A^f_{22}L^f + L^f(I+\varepsilon A^f_{11}) - \varepsilon L^f A^f_{12}L^f + A^f_{21} = 0$$

and is expressed approximately as

$$L^f(\varepsilon) = L^f_0 + O(\varepsilon) \quad ; \quad L^f_0 = (A^f_{22}-I)^{-1}A^f_{21}.$$

In this version, we have the relation:

$$L^f_0 = L_0 \quad ; \quad L_0 = A^{-1}_{22}A_{21}$$

where L_0 is a dichotomic solution of the NARE (2.31.a) of the continuous-time version.

It is noted that even if the continuous-time system (2.29) possesses the two-time-scale property, i.e., $\max_i|\lambda_i(A_s)| \ll \min_j|\lambda_j(A_f/\varepsilon)|$, the fast sampling model (4.6) does not necessarily satisfy its property, i.e., $\min_i|\lambda_i(A^f_s)| \gg \max_j|\lambda_j(A^f_f)|$. (However, in the slow sampling model (4.1), the two-time-scale property is preserved.)

3. By the use of the boundary layer method (Blankenship [Bl.81], Hoppensteadt et al. [Ho.77]), the solutions of (4.6) can be expressed as

$$\begin{cases} x_1(n,\varepsilon) = X_1(t,\varepsilon) + \varepsilon m_1(n,\varepsilon) \quad ; \quad t = \varepsilon n \\ x_2(n,\varepsilon) = X_2(t,\varepsilon) + m_2(n,\varepsilon) \\ u(n,\varepsilon) = U(t,\varepsilon) + v(n,\varepsilon). \end{cases} \tag{4.8}$$

Then, we have

$$\begin{cases} x_1(n,\varepsilon) = X_{10}(t) + O(\varepsilon) \\ x_2(n,\varepsilon) = X_{20}(t) + m_{20}(n) + O(\varepsilon) \\ u(n,\varepsilon) = U_0(t) + v_0(n) + O(\varepsilon) \end{cases} \tag{4.9}$$

where $$\begin{cases} dX_{10}/dt = A_0X_{10}(t) + B_0U_0(t) \\ X_{20}(t) = -A_{22}^{-1}A_{21}X_{10}(t) - A_{22}^{-1}B_2U_0(t) \end{cases} \tag{4.10}$$

$$m_{20}(n+1) = A_{22}^{f}m_{20}(n) + B_2^{f}v_0(n). \tag{4.11}$$

The eqn. (4.10) corresponds to the reduced system of the continuous-time system (2.29). The solutions of (4.6) can be expressed as a hybrid combination of the continuous-time slow part (4.10) which dominates the system behavior over whole interval, and the discrete-time fast part (4.11) which functions only near the initial time. Thus, although the fast sampling model (4.6) is discrete-type apparently, the analysis and design is essentially performed in the continuous-time domain.

In the following Sections 4.2 - 4.4, the various control problems for the slow sampling model will be studied. For brevity, we omit the superscript s of the notations A_{ij}^{s}, B_i^{s} etc.. In Section 4.6, the state feedback controller design for the fast sampling model will be developed, which corresponds to a counterpart of the results for the slow sampling model. These results for two sampling models will lead to the multirate controller design naturally. In the last section, Section 4.7, such a multirate controller design will be developed within a framework of the decomposition-coordination principle from the temporal viewpoint.

4.2 State Feedback Controller Design for Slow Sampling Model

4.2.1 Design approaches of state feedback controllers

We consider the slow sampling model:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k); \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \quad (4.12)$$

$$y(k) = C_1 x_1(k) + C_2 x_2(k)$$

where $\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix}$; $x_1(k)\in R^{n_1}$, $x_2(k)\in R^{n_2}$, $y(k)\in R^m$, $u\in R^r$.

The above model can be obtained via the modelling procedures shown in Chapter 2.

By applying the two-stage design approach proposed by Phillips [Ph.80], the state feedback controller which allows all eigenvalues to be placed at desired locations is given by

$$u(k) = G_1 x_1(k) + \varepsilon G_2 x_2(k) \quad (4.13)$$

where
$$G_1 = G_s(I+\varepsilon ML)+\varepsilon(I+G_s A_s^{-1} B_s)G_f[\hat{K}(I+\varepsilon ML)+L] \quad (4.14.a)$$

$$G_2 = G_s M+(I+G_s A_s^{-1} B_s)G_f(I+\varepsilon\hat{K}M) \quad (4.14.b)$$

and
$$A_s = A_{11} - \varepsilon A_{12} L \quad (4.15.a)$$

$$\varepsilon A_f = \varepsilon(A_{22}+ LA_{12}) \quad (4.15.b)$$

$$B_s = B_1 + \varepsilon M(B_2 + LB_1) \quad (4.15.c)$$

$$B_f = B_2 + LB_1. \quad (4.15.d)$$

The fast transformation gain L, M and the gain $\hat{K}$ satisfy

$$R(L) = -\varepsilon A_{22}L + LA_{11} - \varepsilon LA_{12}L + A_{21} = 0 \tag{4.16}$$

$$R(M) = \varepsilon MA_f - A_sM + A_{12} = 0 \tag{4.17}$$

$$R(\hat{K}) = -\varepsilon A_f\hat{K} + \hat{K}(A_s + B_sG_s) + B_fG_s = 0. \tag{4.18}$$

By using these gains, we set the transformation matrices T and $\hat{T}$ as

$$T = \begin{bmatrix} I & -\varepsilon M \\ -L & I + \varepsilon LM \end{bmatrix} \tag{4.19}$$

$$\hat{T} = \begin{bmatrix} I & 0 \\ \hat{K} & I \end{bmatrix}. \tag{4.20}$$

Figure 4.1 shows the design approaches of state feedback controllers for the slow sampling model. In this figure, the right procedure represents the two-stage-design approach and the left procedure represents the direct approach, respectively. The latter approach will be applied in the proof of lemma and theorem shown later.
The transformation matrix P which links two approaches each other can be obtained as follows:

$$P = \begin{bmatrix} I+\varepsilon M\hat{K} & -\varepsilon M \\ 0 & I-\varepsilon\hat{K}(I+\varepsilon M\hat{K})^{-1}M \end{bmatrix} \approx \begin{bmatrix} I & -\varepsilon A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}. \tag{4.21}$$

The transformation gain $\hat{L}$ which is used in the direct approach has the following relation with the gains L, M, and $\hat{K}$ which are used in the two-stage design approach:

$$\begin{aligned} \hat{L} &= [L+(I+\varepsilon LM)\hat{K}](I+\varepsilon M\hat{K})^{-1} \\ &= (I+\varepsilon\hat{K}M)^{-1}[L+\hat{K}(I+\varepsilon ML)] = (L_0+\hat{K}_0) + O(\varepsilon) \end{aligned} \tag{4.22.a}$$

where $\hat{L}$ satisfies the following NARE:

$$R(\hat{L}) = -\varepsilon(A_{22}+B_2G_2)\hat{L}+\hat{L}(A_{11}+B_1G_1)-\varepsilon\hat{L}(A_{12}+B_1G_2)\hat{L}+(A_{21}+B_2G_1) = 0. \tag{4.22.b}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \xrightarrow{\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = T \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix}} \begin{bmatrix} x_s(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & \varepsilon A_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u(k)$$

$$\downarrow \; u(k) = G_s x_s(k) + u_f(k)$$

$$\begin{bmatrix} x_s(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_s + B_s G_s & 0 \\ B_f G_s & \varepsilon A_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u_f(k)$$

$$\downarrow \; \begin{bmatrix} x_s(k) \\ \eta(k) \end{bmatrix} = \hat{T} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix}$$

$$\begin{bmatrix} x_s(k) \\ \eta(k) \end{bmatrix} = \hat{T} T^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad \begin{bmatrix} x_s(k+1) \\ \eta(k+1) \end{bmatrix} = \begin{bmatrix} A_s + B_s G_s & 0 \\ 0 & \varepsilon A_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ \eta(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f + \hat{K} B_s \end{bmatrix} u_f(k)$$

$$\downarrow \; u_f(k) = \varepsilon (I + G_s A_s^{-1} B_s) G_f \eta(k)$$

$$\downarrow \; u(k) = G_1 x_1(k) + \varepsilon G_2 x_2(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 G_1, & \varepsilon (A_{12} + B_1 G_2) \\ A_{21} + B_2 G_1, & \varepsilon (A_{22} + B_2 G_2) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \longrightarrow \begin{bmatrix} x_s(k+1) \\ \eta(k+1) \end{bmatrix} = \begin{bmatrix} A_s + B_s G_s & \varepsilon B_s (I + G_s A_s^{-1} B_s) G_f \\ 0, & \varepsilon [A_f + (B_f + \hat{K} B_s)(I + G_s A_s^{-1} B_s) G_f] \end{bmatrix} \begin{bmatrix} x_s(k) \\ \eta(k) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{A}_{11}, & \varepsilon \hat{A}_{12} \\ \hat{A}_{21}, & \varepsilon \hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$\downarrow \; \begin{bmatrix} x_1(k) \\ \gamma(k) \end{bmatrix} = P \begin{bmatrix} x_s(k) \\ \eta(k) \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} x_1(k+1) \\ \gamma(k+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} - \varepsilon \hat{A}_{12} \hat{L}, & \varepsilon \hat{A}_{12} \\ 0, & \varepsilon (\hat{A}_{22} + \hat{L} \hat{A}_{12}) \end{bmatrix} \begin{bmatrix} x_1(k) \\ \gamma(k) \end{bmatrix}$$

$$\begin{bmatrix} x_1(k) \\ \gamma(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{L} & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = P \hat{T} T^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Fig. 4.1. State feedback controller design for the slow sampling model.

After considerable algebraic manipulations, we can prove these results (Kando et al. [Ka.84.1]). The above relation implies that the existence of $\hat{L}$ of (4.22.b) that satisfies the two-scale property:

$$\begin{cases} \lambda(\hat{A}_{slow}) = \lambda(\hat{A}_{11} - \hat{A}_{12}\hat{L}) \\ \lambda(\hat{A}_{fast}) = \lambda[\varepsilon(\hat{A}_{22} + \hat{L}\hat{A}_{12})] \end{cases}$$

is guaranteed under the following conditions:

(i) (A-1) of Lemma 2-1 (the existence condition of L)

(ii) $\lambda(\varepsilon A_f) \cap \lambda(A_s) = \phi$ (the existence condition of M)

(iii) $\lambda(\varepsilon A_f) \cap \lambda(A_s+B_sG_s) = \phi$ (the existence condition of $\hat{K}$).

[Remarks]

1. In Fig.4.1, instead of the fast transformation T ((4.19)), the following transformation can be applied:

(i) Slow transformation

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} I+\varepsilon L^s M^s & \varepsilon L^s \\ M^s & I \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} = T^s \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} \tag{4.23}$$

where $R(L^s) = A_{12} + A_{11}L^s - \varepsilon L^s A_{21} L^s - \varepsilon L^s A_{22} = 0$ (4.24.a)

$$R(M^s) = \varepsilon(A_{22}+A_{21}L^s)M^s - M^s(A_{11}-\varepsilon L^s A_{21}) + A_{21} = 0 \tag{4.24.b}$$

$$A_s = A_{11} - \varepsilon L^s A_{21} \tag{4.25.a}$$

$$\varepsilon A_f = \varepsilon(A_{22} + A_{21}L^s). \tag{4.25.b}$$

(ii) Combined transformation

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} I & \varepsilon L^s \\ -L & I \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} = T^c \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} \tag{4.26}$$

where $A_s = A_{11} - \varepsilon A_{12}L$ (4.27.a)

$$\varepsilon A_f = \varepsilon(A_{22} + A_{21}L^s) \tag{4.27.b}$$

L : fast transformation, L^s : slow transformation.

2. When we set firstly in the two stage design approach

$$u(k) = u_s(k) + \varepsilon G_f x_f(k),$$

we must modify the procedure in Fig. 4.1 slightly.

3. Note that in the two stage design approach (Fig. 4.1), we must use the state feedback control:

$$u_f(k) = \varepsilon(I + G_s A^{-1} B_s) G_f \eta(k)$$

in place of setting:

$$u_f(k) = \varepsilon G_f \eta(k).$$

The reason will be clear in the following lemma.

4.2.2 Eigenvalue assignment problem

In the singularly perturbed continuous-time systems, eigenvalue assignment problems have been investigated so far by Porter [Po.75], [Po.76] and Chow et al. [Cho.76.1]. In case of discrete-time versions, we can obtain the following results:

Lemma 4-1 Let the following conditions be satisfied:

(A-1) $0 \le \varepsilon < \min\{1/[2\hat{a}_s(\hat{a}_f+\hat{a}_{12}\hat{l}_0)], (\hat{a}_f+\hat{a}_{12}\hat{l}_0)/[\hat{a}_s[(\hat{a}_f+\hat{a}_{12}\hat{l}_0)^2+8\hat{a}_f\hat{a}_{12}\hat{l}_0]]\} = \varepsilon_1^*$

where $\hat{a}_s = \|\hat{A}_{11}^{-1}\|$, $\hat{a}_f = \|\hat{A}_{22}-\hat{A}_{21}\hat{A}_{11}^{-1}\hat{A}_{12}\|$, $\hat{a}_{12} = \|\hat{A}_{12}\|$, $\hat{l}_0 = \|\hat{A}_{21}\hat{A}_{11}^{-1}\|$.

(A-2) (A_{11}, B_1), (A_{f0}, B_{f0}) : completely controllable.

Then, the state feedback controller

$$u(k) = G_s x_1(k) + \varepsilon[G_s A_{11}^{-1} A_{12} + (I+G_s A_{11}^{-1} B_1) G_f] x_2(k) \tag{4.28}$$

assigns all eigenvalues of (4.12) approximately at the following locations:

$$\begin{cases} \lambda(A_s) = \lambda(A_{11}+B_1G_s)(1+O(\varepsilon)) \\ \lambda(\varepsilon A_f) = \lambda[\varepsilon(A_{f0}+B_{f0}G_f)](1+O(\varepsilon)) \end{cases} \tag{4.29}$$

where $A_{f0} = A_{22}-A_{21}A_{11}^{-1}A_{12}$, $B_{f0} = B_2-A_{21}A_{11}^{-1}B_1$.

Proof Substituting the controller (4.28) into the system (4.12) yields

$$\begin{bmatrix} x_1^c(k+1) \\ x_2^c(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}+B_1G_s & \varepsilon[A_{12}+B_1(G_sA_{11}^{-1}A_{12}+(I+G_sA_{11}^{-1}B_1)G_f)] \\ A_{21}+B_2G_s & \varepsilon[A_{22}+B_2(G_sA_{11}^{-1}A_{12}+(I+G_sA_{11}^{-1}B_1)G_f)] \end{bmatrix} \begin{bmatrix} x_1^c(k) \\ x_2^c(k) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{A}_{11} & \varepsilon\hat{A}_{12} \\ \hat{A}_{21} & \varepsilon\hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1^c(k) \\ x_2^c(k) \end{bmatrix} = \begin{bmatrix} \hat{\hat{A}}_{11} & \hat{\hat{A}}_{12} \\ \hat{\hat{A}}_{21} & \hat{\hat{A}}_{22} \end{bmatrix} \begin{bmatrix} x_1^c(k) \\ x_2^c(k) \end{bmatrix}. \tag{4.30}$$

Applying the fast transformation:

$$\begin{bmatrix} x_1^c(k) \\ \gamma(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{L} & I \end{bmatrix} \begin{bmatrix} x_1^c(k) \\ x_2^c(k) \end{bmatrix} \tag{4.31}$$

where $R(\hat{L}) = -\varepsilon\hat{A}_{22}\hat{L} + \hat{L}\hat{A}_{11} - \varepsilon\hat{L}\hat{A}_{12}\hat{L} + \hat{A}_{21} = 0$ (4.32)

gives

$$\begin{bmatrix} x_1^c(k+1) \\ \gamma(k+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_s & \varepsilon\hat{A}_{12} \\ R(\hat{L}) & \varepsilon\hat{A}_f \end{bmatrix} \begin{bmatrix} x_1^c(k) \\ \gamma(k) \end{bmatrix} \tag{4.33}$$

where $\hat{A}_s = \hat{A}_{11}-\varepsilon\hat{A}_{12}\hat{L}$

$\varepsilon\hat{A}_f = \varepsilon(\hat{A}_{22}+\hat{L}\hat{A}_{12})$.

Using the Kronecker product and rewriting the NARE (4.32) in vector form $f(\hat{l},\varepsilon) = 0$ yields the Jacobian matrices:

$$(\partial f/\partial\hat{l})_{\varepsilon=0} = I\otimes\hat{A}_{11}' = I\otimes(A_{11}+B_1G_s)' \ : \text{nonsingular}. \tag{4.34}$$

Due to the implicit function theorem, the solution $\hat{L}(\varepsilon)$ of the NARE

(4.32) is analytic in ε at $\varepsilon = 0$, i.e.,

$$\hat{L}(\varepsilon) = \sum_{i=0}^{k} \hat{l}_i \varepsilon^i + O(\varepsilon^{k+1}) \quad ; \quad \hat{l}_0 = -\hat{A}_{21}\hat{A}_{11}^{-1}. \tag{4.35}$$

The above Taylor expansion is equivalent to k-th iteration $\hat{L}_k$ of the following iteration algorithm, which is equivalent to the iteration (2.19)':

$$\hat{L}_{k+1} = (\varepsilon\hat{A}_{22}\hat{L}_k + \varepsilon\hat{L}_k\hat{A}_{12}\hat{L}_k - \hat{A}_{21})\hat{A}_{11}^{-1} \quad ; \quad \hat{L}_0 = -\hat{A}_{21}\hat{A}_{11}^{-1}. \tag{4.36}$$

Under the condition (A-1), the above iteration converges to a unique dichotomic solution of the NARE (4.32), which possesses the magnitude

$$0 \le \|\hat{L}\| \le [\hat{l}_0 + 2\hat{a}_f\hat{l}_0/(\hat{a}_f + \hat{a}_{12}\hat{l}_0)]$$

and satisfies the two-time-scale property

$$\begin{cases} \lambda(\hat{A}_{slow}) = \lambda(\hat{A}_{11} - \varepsilon\hat{A}_{12}\hat{L}) \\ \lambda(\hat{A}_{fast}) = \lambda[\varepsilon(\hat{A}_{22} + \hat{L}\hat{A}_{12})]. \end{cases}$$

Thus, the expansion (4.35) converges for every small positive parameter $\varepsilon \in [0, \varepsilon_1^*)$. Using (4.35) leads to

$$\begin{cases} \hat{A}_s = \hat{A}_{11} - \varepsilon\hat{A}_{12}\hat{L} = (A_{11} + B_1G_s) + O(\varepsilon) \\ \varepsilon\hat{A}_f = \varepsilon(\hat{A}_{22} + \hat{L}\hat{A}_{12}) \end{cases} \tag{4.37.a}$$

$$\begin{aligned} &= \varepsilon[A_{22} + B_2(G_sA_{11}^{-1}A_{12} + (I + G_sA_{11}^{-1}B_1)G_f] - \varepsilon(A_{21} + B_2G_s)(A_{11} + B_1G_s)^{-1}[A_{12} + \\ &\qquad B_1(G_sA_{11}^{-1}A_{12} + (I + G_sA_{11}^{-1}B_1)G_s] + O(\varepsilon^2) \\ &= \varepsilon A_{f0} + \varepsilon\{B_2[I - G_s(A_{11} + B_1G_s)^{-1}B_1](I + G_sA_{11}^{-1}B_1)G_f - \\ &\qquad A_{21}(A_{11} + B_1G_s)^{-1}B_1(I + G_sA_{11}^{-1}B_1)G_f\} + O(\varepsilon^2). \end{aligned} \tag{4.37.b}$$

Applying the following relation:

$$[I - G_s(A_{11} + B_1G_s)^{-1}B_1](I + G_sA_{11}^{-1}B_1) = I$$

yields

$$\varepsilon\hat{A}_f = \varepsilon A_{f0} + \varepsilon[B_2 - A_{21}(A_{11} + B_1G_s)^{-1}B_1(I + G_sA_{11}^{-1}B_1)]G_f + O(\varepsilon^2)$$

$$= \varepsilon(A_{f0}+B_{f0}G_f) + O(\varepsilon^2). \qquad (4.38)$$

Q.E.D.

[Remarks]

1. When we choose in the controller (4.28) as

$$\begin{cases} G_s = -(I-\hat{R}^{-1}\hat{C}'B_r)(\hat{R}+B_r'K_rB_r)^{-1}B_r'K_rA_r-\hat{R}^{-1}\hat{C}'A_r \\ G_f = -R^{-1}(B_1'Q_{12}+B_2'Q_{22})A_c \end{cases}$$

the above state feedback controller leads to the mode-decoupled near optimum controller shown in Section 4.3. Thus, the results given in this section include the results of the regulator problem.

2. In eqn. (4.28), we assume:

$$G_s = -\bar{B}_{1r}^{-1}\bar{A}_{11r}Q_{11}, \quad G_f = 0$$

where Q_{11} is a transformation matrix into the controllable canonical form, and $\bar{A}_{11r}$, $\bar{B}_{1r}$ etc. are the matices given by the nontrivial rows of A_{11} and B_1 etc. (O'Reilly [OR.81], Ichikawa [Ic.78]). Then, it is interesting to investigate whether we can obtain the mode-decoupled dead-beat controller or not. However, in the process shown in Lemma 4-1, it is required that the slow subsystem matrix $(A_{11}+B_1G_s)$ must be nonsingular. Unfortunately, this result prevents us from constructing such a reduced-order dead-beat controller via the singular pertubation approach.

4.2.3 Asymptotic stability

In the stabilization problem for the singularly perturbed system, the gain $G_f = 0$ in the controller (4.28) is an admissible choice because of $|\lambda(\varepsilon A_{f0})| \ll 1$ for sufficiently small ε. Then, we obtain as the stabilizing controller:

$$u(k) = G_sx_1(k) + \varepsilon G_sA_{11}^{-1}A_{12}x_2(k) \qquad (4.39)$$

Now we are interesting in estimating an upper bound ε^* such that for $\forall \varepsilon \in [0,\varepsilon^*)$ the resulting closed-loop system is asymptotically stable.

Here, we will investigate an asymptotic stability of the general case, and then will apply the results to the case of the closed-loop system when the controller (4.39) is implemented.

We consider the following system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \; ; \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{4.40}$$

where $\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix}.$

Then, by applying a scalar Lyapunov approach, we obtain the following stability results:

Lemma 4-2 Let the following conditions be satisfied:

(A-1) $|\lambda(\tilde{A}_{11})| < 1$.

(A-2) $|\lambda(\tilde{A}_{f0})| < 1 \quad ; \quad \tilde{A}_{f0} = \tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}$.

Then, the system (4.40) is asymptotically stable for every small parameter $\varepsilon \in [0,\varepsilon^*)$

where
$$\varepsilon^* = \frac{1}{\theta_f \|A_{f0}\| + \dfrac{\|A_{12}\| \, \|L_0\|}{(\theta_s^{-1} - \|A_{11}\|)}} \tag{4.41}$$

$$; \quad A_{f0} = A_{22} - A_{21}A_{11}^{-1}A_{12}, \quad L_0 = A_{21}A_{11}^{-1}$$

$$\theta_s = \left[\frac{\lambda_{max}(P_s)}{\lambda_{min}(Q_s)+\lambda_{max}(P_s)\|A_{11}\|^2}\right]^{1/2} , \quad \theta_f = \left[\frac{\lambda_{max}(P_f)}{\lambda_{min}(Q_f)}\right]^{1/2}.$$

Proof

Substituting the transformation:

$$\begin{bmatrix} x_1(k) \\ x_f(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{L}_0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{4.42}$$

into (4.40) yields

$$\begin{bmatrix} x_1(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11}-\tilde{A}_{12}\tilde{L}_0 & \tilde{A}_{12} \\ -\tilde{A}_{f0}\tilde{L}_0 & \tilde{A}_{f0} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_f(k) \end{bmatrix}. \tag{4.43}$$

As Lyapunov functions of decoupled systems:

$$x_1(k+1) = \tilde{A}_{11}x_1(k), \quad x_f(k+1) = \tilde{A}_{f0}x_f(k) \tag{4.44}$$

we consider the following:

$$\begin{cases} v_1(x_1(k)) = x_1'(k)P_sx_1(k) \quad ; \quad A_{11}'P_sA_{11} - P_s = -Q_s & (4.45) \\ v_f(x_f(k)) = x_f'(k)P_fx_f(k) \quad ; \quad A_{f0}'P_fA_{f0} - P_f = -Q_f. & (4.46) \end{cases}$$

The estimates for the Lyapunov functions $v_1(x_1(k))$ and $v_f(x_f(k))$ are given by

$$\begin{cases} \eta_1\|x_f\|^2 \leq v_f(x_f(k)) \leq \eta_2\|x_f\|^2 \\ \Delta v_f(x_f(k)) \leq -\eta_3\|x_f\|^2 \end{cases} \tag{4.47}$$

$$\begin{cases} \xi_1\|x_1\|^2 \leq v_1(x_1(k)) \leq \xi_2\|x_1\|^2 \\ \Delta v_1(x_1(k)) \leq -\xi_3\|x_1\|^2 \end{cases} \tag{4.48}$$

where $\xi_1 = \lambda_{min}(P_s)$, $\xi_2 = \lambda_{max}(P_s)$, $\xi_3 = \lambda_{min}(Q_s)$

$\eta_1 = \lambda_{min}(P_f)$, $\eta_2 = \lambda_{max}(P_f)$, $\eta_3 = \lambda_{min}(Q_f)$.

By using the above estimates, the differences of v_1 and v_f along the trajectory of (4.43) are given by

$$\begin{cases} \Delta v_1(x_1(k)) \leq -(\xi_3+p_sa_{11}^2)\|x_1\|^2+p_s[(a_{11}+\varepsilon a_{12}l_0)\|x_1\|+\varepsilon a_{12}\|x_f\|]^2 & (4.49.a) \\ \Delta v_f(x_f(k)) \leq -\eta_3\|x_f\|^2+p_f[\varepsilon a_f l_0\|x_1\| +\varepsilon a_f\|x_f\|]^2 & (4.49.b) \end{cases}$$

where $a_{11} = \| A_{11} \|$, $a_{12} = \| A_{12} \|$, $l_0 = \| A_{21}A_{11}^{-1} \|$

$a_f = \| A_{22}-A_{21}A_{11}^{-1}A_{12} \|$, $p_s = \lambda_{max}(P_s) = \xi_2$, $p_f = \lambda_{max}(P_f) = \eta_2$.

As a scalar Lyapunov function of the global system, we consider the following:

$$\nu(k) = d_1 v_1(k) + d_2 v_f(k) \tag{4.50}$$

where $d_1 > 0$ and $d_2 > 0$.
By utilizing (4.49.a) and (4.49.b), the difference of ν can be evaluated as follows:

$$\begin{aligned}\Delta\nu(k) &= \nu(k+1) - \nu(k) \\ &\leq -(\|x_f\|, \|x_1\|)\, S \begin{bmatrix} \|x_f\| \\ \|x_1\| \end{bmatrix}\end{aligned} \tag{4.51}$$

where

$$S = \begin{bmatrix} d_2\eta_3 - d_1p_s\varepsilon^2a_{12}^2 - d_2\varepsilon^2p_fa_f^2\,, & -d_1p_s\varepsilon(a_{11}+\varepsilon a_{12}l_0)a_{12} - d_2\varepsilon^2p_fa_f^2l_0 \\ -d_1p_s\varepsilon(a_{11}+\varepsilon a_{12}l_0)a_{12} - d_2\varepsilon^2p_fa_f^2l_0\,, & d_1(\xi_3+p_sa_{11}^2) - d_1p_s(a_{11}+\varepsilon a_{12}l_0)^2 - d_2\varepsilon^2p_fa_f^2l_0^2 \end{bmatrix}.$$

The matrix S can be represented as

$$S = C_1'DC_1 - C_2'DC_2 \tag{4.52}$$

where

$$C_1 = \begin{bmatrix} \eta_3^{1/2} & 0 \\ 0 & (\xi_3+p_sa_{11}^2)^{1/2} \end{bmatrix},\ C_2 = \begin{bmatrix} \varepsilon p_f^{1/2}a_f, & \varepsilon p_f^{1/2}a_fl_0 \\ \varepsilon p_s^{1/2}a_{12}, & p_s^{1/2}(a_{12}+\varepsilon a_{12}l_0) \end{bmatrix},\ D = \begin{bmatrix} d_2 & 0 \\ 0 & d_1 \end{bmatrix}.$$

Thus, if the matrix S is positive definite, the system (4.40) is asymptotically stable. In (4.52), a matrix (C_1-C_2) is an M-matrix (Bernussou [Be.82], Šiljak [Sil.78]) if and only if there is a positive definite diagonal matrix D such that the matrix S is positive definite (Araki [Ar.75]), where

$$C_1-C_2 = \begin{bmatrix} \eta_3^{1/2}-\varepsilon p_f^{1/2}a_f, & -\varepsilon p_f^{1/2}a_f l_0 \\ -\varepsilon p_s^{1/2}a_{12}, & (\xi_3+p_s a_{11}^2)^{1/2}-p_s^{1/2}(a_{11}+\varepsilon a_{12}l_0) \end{bmatrix}. \qquad (4.53)$$

By using the equivalent condition of M-matrix, i.e., the condition under which all principal minors are positive, an upper bound of ε can be estimated as (4.41). If necessary, the estimate ε^* can be improved by using parametrical optimization techniques (Geromel [Ger.82]).
Q.E.D.

4.2.4 Main results

By summarizing the above results, we obtain the following theorem:

Theorem 4-1 Let the following conditions be satisfied:

(A-1) $0\leq\varepsilon<\min[\varepsilon_1^*,\varepsilon_2^*] = \varepsilon^{**}$

where $\varepsilon_1^* = \min\{\dfrac{1}{2\hat{a}_s(\hat{a}_f+\hat{a}_{12}\hat{l}_0)}, \dfrac{(\hat{a}_f+\hat{a}_{12}\hat{l}_0)}{\hat{a}_s[(\hat{a}_f+\hat{a}_{12}\hat{l}_0)^2+8\hat{a}_f\hat{a}_{12}\hat{l}_0]}\}$

$$\varepsilon_2^* = \frac{1}{\hat{\theta}_f\hat{a}_f + \dfrac{\hat{a}_{12}\hat{l}_0}{(\hat{\theta}_s^{-1} - \hat{a}_{11})}}.$$

(A-2) $(\tilde{A}_{11}, B_1)$: stabilizable.

(A-3) $|\lambda(\tilde{A}_{f0})|<1$; $\tilde{A}_{f0} = \tilde{A}_{22}-\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}$.

Then, if the state feedback controller

$$u(k) = G_s x_1(k) + G_s\tilde{A}_{11}^{-1}\tilde{A}_{12}x_2(k) \qquad (4.54)$$

is implemented, there exist a controller gain G_s and a positive scalar ε^{**} such that for $\forall\varepsilon\in[0,\varepsilon^{**})$ the closed-loop system is asymptotically stable. Furthermore, the eigenvalue of the closed-loop system are approximated as follows:

$$\begin{cases} \lambda(\hat{A}_{slow}) = \lambda(\tilde{A}_{11}+B_1G_s)(1+O(\varepsilon)) \\ \lambda(\hat{A}_{fast}) = \lambda(\tilde{A}_{f0})(1+O(\varepsilon)). \end{cases}$$

Proof

Substituting the controller (4.54) into (4.12) yields a closed-loop system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}+B_1G_s, & \varepsilon(A_{12}+B_1G_sA_{11}^{-1}A_{12}) \\ A_{21}+B_2G_s, & \varepsilon(A_{22}+B_2G_sA_{11}^{-1}A_{12}) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{4.56}$$

$$= \begin{bmatrix} \hat{A}_{11} & \varepsilon\hat{A}_{12} \\ \hat{A}_{21} & \varepsilon\hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \hat{\hat{A}}_{11} & \hat{\hat{A}}_{12} \\ \hat{\hat{A}}_{21} & \hat{\hat{A}}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

In (4.56), we have

$$\begin{aligned} \hat{A}_f &= \varepsilon(\hat{A}_{22} - \hat{A}_{21}\hat{A}_{11}^{-1}\hat{A}_{12}) \\ &= \varepsilon(A_{22}+B_2G_sA_{11}^{-1}A_{12}) - \varepsilon(A_{21}+B_2G_s)(A_{11}+B_1G_s)^{-1}(A_{12}+B_1G_sA_{11}^{-1}A_{12}) \\ &= \varepsilon(A_{22}+B_2G_sA_{11}^{-1}A_{12}) - \varepsilon(A_{21}+B_2G_s)A_{11}^{-1}A_{12} \\ &= \varepsilon A_{f0} = \tilde{A}_{f0}. \end{aligned} \tag{4.57}$$

By utilizing Lemma 4-2, under the condition (A-2) and (A-3), the system (4.56) is asymptotically stable for every $\varepsilon\in[0,\varepsilon_2^*)$. Furthermore, by the use of Lemma 4-1, for every $\varepsilon\in[0,\varepsilon_1^*)$ there exists a unique dichotomic solution $\hat{L}(\varepsilon)$ which satisfies the NARE (4.32), and

$$\hat{L}(\varepsilon) = \sum_{i=0}^{k} \hat{l}_i\varepsilon^i + O(\varepsilon^{k+1}) \quad ; \quad \hat{l}_0 = -\hat{A}_{21}\hat{A}_{11}^{-1}. \tag{4.58}$$

Accordingly, the eigenvalues of the closed-loop system (4.56) can be represented by

$$\begin{cases} \lambda(\hat{A}_{slow}) = \lambda(\hat{A}_{11}-\varepsilon\hat{A}_{12}\hat{L}) = \lambda(A_{11}+B_1G_s)(1+O(\varepsilon)) & (4.59.a) \\ \lambda(\hat{A}_{fast}) = \lambda[\varepsilon(\hat{A}_{22}+\hat{L}\hat{A}_{12}) = \lambda(\varepsilon A_{f0})(1+O(\varepsilon)) = \lambda(\tilde{A}_{f0})(1+O(\varepsilon)). & (4.59.b) \end{cases}$$

Q.E.D.

[Remarks]

The state feedback controller (4.54) can be expressed in terms of the original system matrices $\tilde{A}_{ij}$ without requiring the calculation of the gains L, M and $\hat{K}$. Furthermore, if the condition (A-1) is rewritten in terms of norms of the closed-loop system matrices $\hat{A}_{ij}$ etc., a small parameter ε need not be identified explicitly. These remarks can be also applied to Lemma 4-1.

4.2.5 Numerical example

We consider a ninth order boiler model (Nicholson [Nc.64]). By discretizing with a sampling period T = 1.5 sec, and by using a permutation: $P^T = [e_9, e_3, e_8, e_7, e_6, e_4, e_1, e_2, e_5]$
where e_i is a unit vector
and a scaling: D = diag [1,1,1,1,0.002,45,200,1,0.005],
the discrete-time model is obtained as follows:

$$\begin{cases} X(k+1) = \tilde{A}\,X(k) + \tilde{B}\,U(k) \\ Y(k) = \tilde{C}\,X(k) \end{cases} \tag{4.60}$$

where X(k) is a 9 dimensional state vector, u(k) is a 5 dimensional control vector, and Y(k) is a 3 dimensional output vector, and the physical meanings of them are given in Ref. [Nc.64] .

The system matrices of the above model are given by

$$\tilde{A} = \begin{bmatrix}
1.0 & -0.39089\times10^{-3} & 0.61645\times10^{-2} & 0.16388\times10^{-2} & -0.52739\times10^{-3} & -0.25469 & -0.19459\times10^{-2} & -0.15434\times10^{-3} & -0.50214\times10^{-2} \\
0.0 & 0.97658 & -0.57734\times10^{-2} & -0.82093\times10^{-2} & -0.32036\times10^{-1} & 0.45219\times10^{-3} & 0.39993\times10^{-1} & 0.16631\times10^{-1} & -0.32052\times10^{-3} \\
0.0 & 0.10330\times10^{-2} & 0.90345 & 0.43017\times10^{-2} & 0.25860\times10^{-1} & -0.14738 & 0.11663\times10^{-2} & 0.20085\times10^{-2} & 0.64715\times10^{-1} \\
0.0 & 0.42524\times10^{-2} & 0.14370\times10^{-1} & 0.72193 & 0.13381 & -0.88713\times10^{-3} & 0.70201\times10^{-2} & 0.29023\times10^{-3} & 0.70937\times10^{-3} \\
0.0 & 0.68198\times10^{-2} & 0.15561 & 0.21738 & 0.71666 & -0.14094\times10^{-1} & 0.42542\times10^{-1} & 0.11568\times10^{-3} & 0.90480\times10^{-2} \\
0.0 & -0.51834\times10^{-3} & 0.91461\times10^{-2} & 0.32486\times10^{-2} & -0.19377\times10^{-2} & 0.65222 & -0.26019\times10^{-2} & -0.20841\times10^{-3} & -0.67591\times10^{-2} \\
0.0 & -0.13436 & 0.14212 & 0.20163 & 0.76593 & -0.11326\times10^{-1} & 0.38299\times10^{-1} & -0.35863\times10^{-2} & 0.79476\times10^{-2} \\
0.0 & 0.88639 & -0.15849\times10^{-1} & -0.21289\times10^{-1} & -0.16849\times10^{-1} & 0.18446\times10^{-2} & 0.40449\times10^{-1} & 0.15932\times10^{-1} & -0.10479\times10^{-2} \\
0.0 & 0.83357 & 0.48242\times10^{-1} & -0.23445\times10^{-1} & -0.91587\times10^{-1} & -0.34662 & -0.34593\times10^{-2} & 0.11285\times10^{-2} & 0.13434\times10^{-1}
\end{bmatrix} \tag{4.61.a}$$

$$
\tilde{B} = \begin{bmatrix}
0.45812\times10^{-2} & 0.22603\times10^{-4} & 0.18184\times10^{-4} & 0.31241\times10^{-2} & 0.16046\times10^{-4} \\
-0.11822 & -0.74327\times10^{-4} & 0.44697\times10^{-6} & 0.26711\times10 & -0.13225\times10^{-4} \\
0.23725\times10^{-1} & -0.41293\times10^{-1} & -0.19716\times10^{-3} & 0.10985\times10^{-1} & 0.47359\times10^{-2} \\
-0.11849\times10^{-1} & 0.24466\times10^{-3} & -0.79643\times10^{-6} & 0.34689\times10 & 0.25539\times10^{-4} \\
-0.14398 & 0.15212\times10^{-2} & -0.14823\times10^{-4} & 0.51085 & 0.41591\times10^{-3} \\
0.60384\times10^{-2} & -0.46526\times10^{-3} & 0.24596\times10^{-4} & 0.66058\times10^{-2} & 0.23863\times10^{-4} \\
-0.58110 & 0.17756\times10^{-2} & -0.11304\times10^{-4} & 0.13938 & 0.33220\times10^{-3} \\
-0.35241 & -0.53001\times10^{-4} & 0.20670\times10^{-5} & 0.19848\times10 & -0.54949\times10^{-4} \\
0.11031 & -0.51667\times10^{-2} & -0.73324\times10^{-3} & -0.23706\times10^{-1} & 0.30978\times10^{-3}
\end{bmatrix} \tag{4.61.b}
$$

$$
\tilde{C} = \begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.005 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}. \tag{4.61.c}
$$

After the state variables were grouped into $n_1 = 6$ and $n_2 = 3$, the norms of the subsystem matrices were calculated as follows:

$$
\tilde{a}_s = \| \tilde{A}_{11}^{-1} \| = 1.8635, \quad \tilde{a}_f = \| \tilde{A}_{f0} \| = 0.005953
$$
$$
\tilde{a}_{12} = \| \tilde{A}_{12} \| = 0.06776, \quad \tilde{l}_0 = \| \tilde{L}_0 \| = 1.1062.
$$

By following the modelling procedure shown in chapter 2, we obtain

(A-1) $\max[2\tilde{a}_s(\tilde{a}_f+\tilde{a}_{12}\tilde{l}_0)=0.23299, \tilde{a}_s(\tilde{a}_f+\tilde{a}_{12}\tilde{l}_0)+8\tilde{a}_s\tilde{a}_f\tilde{a}_{12}\tilde{l}_0/(\tilde{a}_f+\tilde{a}_{12}\tilde{l}_0)=0.3015]<1.$

(A-2) $\varepsilon = \tilde{a}_s\tilde{a}_f = 0.01109 \ll 1.$

(A-3) $\tilde{a}_s\tilde{a}_{12} = 0.1263 \ll 1.$

Furthermore, the exact eigenvalues are shown by

$$
\lambda(\tilde{A}) = \{1.0,\ 0.9859,\ 0.9581,\ 0.8639,\ 0.6597,\ 0.5651,
$$
$$
0.5814\times10^{-2},\ -0.9034\times10^{-4} \pm j0.1720\times10^{-2}\}
$$

and the approximate eigenvalues are shown by

$$
\begin{cases}
\lambda(\tilde{A}_{11}) = \{1.0,\ 0.9678,\ 0.9550,\ 0.8467,\ 0.6560,\ 0.5453\} \\
\lambda(\tilde{A}_{f0}) = \{0.5881\times10^{-2},\ -0.1429\times10^{-3} \pm j0.1780\times10^{-2}\}.
\end{cases}
$$

These results imply that the discrete model (4.60) possesses the singularly perturbed structure.

After the controller gain G_s which places the slow eigenvalues at the desired locations $\{0.98, 0.96, 0.94, 0.92, 0.90, 0.88\}$ were calculated via the usual eigenvalue assignment algorithm, e.g., the unity rank approach (Munro [Mun.79]), the state feedback controller of (4.54) was given as follows:

$$u(k) = \tilde{G}_1 x_1(k) + \tilde{G}_2 x_2(k) \tag{4.62}$$

where

$$\tilde{G}_1 = \begin{bmatrix} -0.36442\times10 & -0.71696\times10^{-2} & 0.15639 & 0.19033 & -0.17601 & -0.38906 \\ -0.36442\times10 & -0.71696\times10^{-2} & 0.15639 & 0.19033 & -0.17601 & -0.38906 \\ -0.36442\times10 & -0.71696\times10^{-2} & 0.15639 & 0.19033 & -0.17601 & -0.38906 \\ -0.36442\times10 & -0.71696\times10^{-2} & 0.15639 & 0.19033 & -0.17601 & -0.38906 \\ -0.36442\times10 & -0.71696\times10^{-2} & 0.15639 & 0.19033 & -0.17601 & -0.38906 \end{bmatrix}$$

$$\tilde{G}_2 = \begin{bmatrix} 0.57035\times10^{-3} & 0.14155\times10^{-2} & 0.46258\times10^{-1} \\ 0.57035\times10^{-3} & 0.14155\times10^{-2} & 0.46258\times10^{-1} \\ 0.57035\times10^{-3} & 0.14155\times10^{-2} & 0.46258\times10^{-1} \\ 0.57035\times10^{-3} & 0.14155\times10^{-2} & 0.46258\times10^{-1} \\ 0.57035\times10^{-3} & 0.14155\times10^{-2} & 0.46258\times10^{-1} \end{bmatrix}.$$

The eigenvalues of the closed-loop system when the above controller was implemented were

$$\lambda(\tilde{A}_c) = \{0.9784,\ 0.9624,\ 0.9486,\ 0.9075 \pm j0.01748,\ 0.8758,\ 0.5820\times10^{-2},\ -0.8988\times10^{-4} \pm j0.1718\times10^{-2}\}. \tag{4.63}$$

If necessary, these results can be improved by taking into account the higher order terms in the state feedback controller (4.54). After we calculate the matrix norms, we have the following results corresponding to the conditions given in Theorem 4-1

(A-1) $\max[0.32217,\ 0.075184] < 1$.

(A-2) $(\tilde{A}_1,\ B_1)$: stabilizable.

(A-3) $\lambda(\tilde{A}_{f0}) = \{0.58809\times10^{-2},\ -0.1429\times10^{-3} \pm j0.1780\times10^{-2}\} < 1$.

Thus, the conditions (A-1)-(A-3) are all satisfied.

4.3 Design of Observers and Stabilizing Feedback Controllers Based on Observers for Slow Sampling Model

4.3.1 Observer design

We consider the slow sampling model (4.12).

In the discrete-time observer, it is well known (Willems [Wil.80]) that an essential distinction should be made between the following two problems:

(i) full-order observers of prediction type constructed from the measurements $y(k-1), y(k-2), \ldots, y(0)$,

(ii) full-order observers of current type constructed from the measurements $y(k), y(k-1), \ldots, y(0)$.

These problems will be treated via the singular perturbation theory in this section.

(a) Full-order observers of prediction type

Firstly, we will investigate a full-order observer for the system (4.12):

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} [y(k) - C_1\hat{x}_1(k) - C_2\hat{x}_2(k)] \tag{4.64}$$

which is similar to that of continuous-time systems.

The observation errors

$$\begin{cases} e_1(k) = x_1(k) - \hat{x}_1(k) \\ e_2(k) = x_2(k) - \hat{x}_2(k) \end{cases}$$

must satisfy

$$\begin{bmatrix} e_1(k+1) \\ e_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}-K_1C_1, & \varepsilon A_{12}-K_1C_2 \\ A_{21}-K_2C_1, & \varepsilon A_{22}-K_2C_2 \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} \tag{4.65}$$

By noting that the observer design problem is dual to the eigenvalue assignment problem, the observer design approaches for singularly perturbed discrete-time systems are summarized in Figure 4.2. In this figure, the transformation matrix P which links two approaches is given as follows:

$$\begin{bmatrix} I+M'\hat{K}', & -M' \\ 0, & I-\hat{K}'(I+M'\hat{K}')^{-1}M' \end{bmatrix} \approx \begin{bmatrix} (A'_{11}+C'_1G'_s)(A'_{11}+C'_{s0}G'_s)^{-1}, & -M'_0 \\ 0, & I+C'_2G'_s(A'_{11}+C'_1G'_s)^{-1}M'_0 \end{bmatrix}.$$

A fast transformation $\hat{L}$ used in the direct approach satisfies

$$\hat{L} = (I+\hat{K}M)^{-1}[\varepsilon L+\hat{K}(I+\varepsilon ML)] = [\varepsilon L+(I+\varepsilon LM)\hat{K}](I+M\hat{K})^{-1} \qquad (4.67)$$

which implies an explicit relation between two approaches.
These results can be obtained in the similar way to the preceding section.

On the basis of the design procedure shown in Fig.4.2, the gains K_1 and K_2 of (4.64) can be obtained as follows:

$$\begin{cases} K_1 = -(I+\varepsilon LM)G_s - \varepsilon[(I+\varepsilon LM)\hat{K} + \varepsilon L]G_f(I+C_sA_s^{-1}G_s)] \\ K_2 = -MG_s - \varepsilon(I+M\hat{K})G_f(I+C_sA_s^{-1}G_s) \end{cases} \qquad (4.68)$$

where $R(L) = -\varepsilon LA_{22} + A_{11}L - \varepsilon LA_{21}L + A_{12} = 0 \qquad (4.69)$

$$R(M) = \varepsilon A_fM - MA_s + A_{21} = 0 \qquad (4.70)$$

$$R(\hat{K}) = -\varepsilon\hat{K}A_f + (A_s+G_sC_s)\hat{K} + G_sC_f = 0 \qquad (4.71)$$

$$A_s = A_{11}-\varepsilon LA_{21}, \quad A_f = A_{22}+A_{21}L \qquad (4.72)$$

$$C_s = C_1+C_fM, \quad C_f = C_2+\varepsilon C_1L \qquad (4.73)$$

and the gains G_s and G_f which satisfy $(A_s+G_sC_s)$ and $(A_f+G_fC_f)$ are chosen so that they place the slow and fast eigenvalues at the desired locations, respectively.
Then, we obtain the following lemma:

$$\begin{bmatrix} e_1(k+1) \\ e_2(k+1) \end{bmatrix} = \begin{bmatrix} A'_{11} & A'_{21} \\ \varepsilon A'_{12} & \varepsilon A'_{22} \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} + \begin{bmatrix} C'_1 \\ C'_2 \end{bmatrix} v(k)$$

$$\xrightarrow{\begin{bmatrix} e_s(k) \\ e_f(k) \end{bmatrix} = \begin{bmatrix} I + \varepsilon M'L' & M' \\ \varepsilon L' & I \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} = T' \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}}$$

$$\begin{bmatrix} e_s(k+1) \\ e_f(k+1) \end{bmatrix} = \begin{bmatrix} A'_s & 0 \\ 0 & \varepsilon A'_f \end{bmatrix} \begin{bmatrix} e_s(k) \\ e_f(k) \end{bmatrix} + \begin{bmatrix} C'_s \\ C'_f \end{bmatrix} v(k)$$

$$\Big\downarrow \; v(k) = G'_s e_s(k) + v_f(k)$$

$$\begin{bmatrix} e_s(k+1) \\ e_f(k+1) \end{bmatrix} = \begin{bmatrix} A'_s + C'_s G'_s & 0 \\ C'_f G'_s & \varepsilon A'_f \end{bmatrix} \begin{bmatrix} e_s(k) \\ e_f(k) \end{bmatrix} + \begin{bmatrix} C'_s \\ C'_f \end{bmatrix} v_f(k)$$

$$\Big\downarrow \; \begin{bmatrix} e_s(k) \\ \eta(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{K}' & I \end{bmatrix} \begin{bmatrix} e_s(k) \\ e_f(k) \end{bmatrix} = \hat{T}' \begin{bmatrix} e_s(k) \\ e_f(k) \end{bmatrix}$$

$$\begin{bmatrix} e_s(k+1) \\ \eta(k+1) \end{bmatrix} = \begin{bmatrix} A'_s + C'_s G'_s & 0 \\ 0 & \varepsilon A'_f \end{bmatrix} \begin{bmatrix} e_s(k) \\ \eta(k) \end{bmatrix} + \begin{bmatrix} C'_s \\ C'_f + \hat{K}'C'_s \end{bmatrix} v_f(k)$$

$$\Big\downarrow \; v_f(k) = \varepsilon(I + G'_s A'^{-1}_s C'_s) G'_f \eta(k)$$

$$\Big\downarrow \; v(k) = -K'_1 e_1(k) - K'_2 e_2(k)$$

$$\begin{bmatrix} e_1(k+1) \\ e_2(k+1) \end{bmatrix} = \begin{bmatrix} A'_{11} - C'_1 K'_1 & A'_{21} - C'_1 K'_2 \\ \varepsilon A'_{12} - C'_2 K'_1 & \varepsilon A'_{22} - C'_2 K'_2 \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{A}'_{11} & \hat{A}'_{21} \\ \hat{A}'_{12} & \hat{A}'_{22} \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} e_s(k) \\ \eta(k) \end{bmatrix} = \hat{T}'T' \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}}$$

$$\begin{bmatrix} e_s(k+1) \\ \eta(k+1) \end{bmatrix} = \begin{bmatrix} A'_s + C'_s G'_s & \varepsilon C'_s(I + G'_s A'^{-1}_s C'_s) G'_f \\ 0 & \varepsilon A'_f + \varepsilon(C'_f + \hat{K}'C'_s)(I + G'_s A'^{-1}_s C'_s) G'_f \end{bmatrix} \begin{bmatrix} e_s(k) \\ \eta(k) \end{bmatrix}$$

$$\Big\downarrow \; \begin{bmatrix} e_1(k) \\ \gamma(k) \end{bmatrix} = P \begin{bmatrix} e_s(k) \\ \eta(k) \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} e_1(k) \\ \gamma(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{L} & I \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} = P\hat{T}'T' \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}}$$

$$\begin{bmatrix} e_1(k+1) \\ \gamma(k+1) \end{bmatrix} = \begin{bmatrix} \hat{A}'_{11} - \hat{A}'_{21}\hat{L} & \hat{A}'_{21} \\ 0 & \hat{A}'_{22} + \hat{L}\hat{A}'_{21} \end{bmatrix} \begin{bmatrix} e_1(k) \\ \gamma(k) \end{bmatrix}$$

Fig. 4.2. Observer design for singularly perturbed discrete-time systems.

Lemma 4-3 Let the following conditions be satisfied:

(A-1) (C_{s0}, A_{11}) : completely observable

where $C_{s0} = C_1+C_2A_{21}A_{11}^{-1}$.

(A-2) (C_2, A_{f0}) : completely observable

where $A_{f0} = A_{22}-A_{21}A_{11}^{-1}A_{12}$.

Let us select

$$\begin{cases} K_1 = -(I-\varepsilon L_0M_0)G_s + \varepsilon(A_{11}+G_sC_{s0})^{-1}G_sC_2G_f(I+C_{s0}A_{11}^{-1}G_s) \\ K_2 = -(M_0+\varepsilon M_1)G_s - \varepsilon[I-M_0(A_{11}+G_sC_{s0})^{-1}G_sC_2]G_f(I+C_{s0}A_{11}^{-1}G_s) \end{cases} \tag{4.74}$$

where $L_0 = A_{11}^{-1}A_{12}$, $M_0 = A_{21}A_{11}^{-1}$, $M_1 = (A_{f0}M_0-M_0L_0A_{21})A_{11}^{-1}$.

Then, there exist gain matrices G_s and G_f, and exists a positive scalar ε^* such that for every $\varepsilon\in[0,\varepsilon^*)$, the error system (4.65) is asymptotically stable.

Proof

Applying the slow transformation:

$$\gamma(k) = e_1(k) - \hat{L}e_2(k) \tag{4.75}$$

to (4.65) yields

$$\begin{bmatrix} \gamma(k+1) \\ e_2(k+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11}-\hat{L}\hat{A}_{21} & R(\hat{L}) \\ \hat{A}_{21} & \hat{A}_{22}+\hat{A}_{21}\hat{L} \end{bmatrix} \begin{bmatrix} \gamma(k) \\ e_2(k) \end{bmatrix} \tag{4.76}$$

where $R(\hat{L}) = -\hat{L}\hat{A}_{22} + \hat{A}_{11}\hat{L} - \hat{L}\hat{A}_{21}\hat{L} + \hat{A}_{12} = 0.$ (4.77)

By setting in (4.77) as

$$\varepsilon\hat{D} = \hat{L} - \hat{L}_0 \quad ; \quad \hat{L}_0 = -(A_{11}+G_sC_1)^{-1}G_sC_2 \tag{4.78}$$

we obtain a nonsymmetric algebraic Riccati-type equation on $\hat{D}$:

$$-\hat{D}F_1 + F_2\hat{D} - \varepsilon\hat{D}(A_{21}-K_2C_1)\hat{D} + F_3 = 0 \tag{4.79}$$

where $F_1 = (\epsilon A_{22}-K_2C_2) + (A_{21}-K_2C_1)\hat{L}_0$

$$F_2 = (A_{11}-K_1C_1) - \hat{L}_0(A_{21}-K_2C_1)$$

$$F_3 = A_{12} - \hat{L}_0A_{22} - (L_0M_0+\hat{L}_0M_1)G_s(C_2+C_1\hat{L}_0).$$

Similarly as Section 4.2.2, by utilizing the implicit function theorem, we see that the solution $\hat{D}(\epsilon)$ of the NARE (4.79) is analytic in ϵ at $\epsilon = 0$, i.e.,

$$\hat{D} = \hat{D}_0 + \epsilon\hat{D}_1 +\ldots+ \epsilon^k\hat{D}_k + O(\epsilon^{k+1}). \tag{4.80}$$

Then, there exists a positive scalar ϵ_1^* such that the above expansion converges for every $\epsilon\epsilon[0,\epsilon_1^*)$. In this way, we have

$$\hat{L} = \hat{L}_0 + \epsilon\hat{L}_1 + O(\epsilon^2) \tag{4.81}$$

where $\hat{L}_0 = -(A_{11}+G_sC_1)^{-1}G_sC_2$

$$\hat{L}_1 = \hat{D}_0 = -\alpha(A_{11}+G_sC_1)^{-1}G_sC_2A_{f0}-\alpha(A_{11}+G_sC_1)^{-1}(A_{11}+G_sC_{s0})A_{12}$$
$$+\alpha(L_0M_0+\hat{L}_0M_1)G_s(C_2+C_1\hat{L}_0)$$

$$\alpha = (A_{11}+G_sC_1)^{-1}(A_{11}+G_sC_{s0})^{-1}(A_{11}+G_sC_1).$$

Using (4.81) gives

$$\hat{A}_{slow} = \hat{A}_{11} - \hat{L}\hat{A}_{12}$$
$$= (A_{11}+G_sC_1) + (A_{11}+G_sC_1)^{-1}G_sC_2A_{21}A_{11}^{-1}(A_{11}+G_sC_1) + O(\epsilon)$$
$$= (A_{11}+G_sC_1)^{-1}(A_{11}+G_sC_{s0})(A_{11}+G_sC_1) + O(\epsilon). \tag{4.82}$$

By using the relation:

$$\begin{cases} [I-C_1(A_{11}+G_sC_1)^{-1}G_s][I+C_1A_{11}^{-1}G_s] = I \\ [I-M_0(A_{11}+G_sC_{s0})^{-1}G_sC_2][I+M_0A_{11}^{-1}G_s(I+C_1A_{11}^{-1}G_s)^{-1}C_2] = I \end{cases} \tag{4.83}$$

we obtain

$$\hat{A}_{fast} = \hat{A}_{22} + \hat{A}_{21}\hat{L}$$

$$= M_0G_sC_2 + (A_{21}+M_0G_sC_1)\hat{L}_0 + \varepsilon[A_{22}+M_1G_sC_2+M_1G_sC_1\hat{L}_0+(A_{21}+M_0G_sC_1)\hat{L}_1] +$$
$$\varepsilon[I-M_0(A_{11}+G_sC_{s0})^{-1}G_sC_2]G_f(I+C_{s0}A_{11}^{-1}G_s)(C_2+C_1\hat{L}_0) + O(\varepsilon^2)$$
$$= \varepsilon A_{f0} + \varepsilon[M_1G_s(C_2+C_1\hat{L}_0)-M_0(A_{11}+G_sC_{s0})^{-1}G_sC_2A_{f0}+$$
$$M_0(A_{11}+G_sC_{s0})^{-1}(A_{11}+G_sC_1)L_0M_0G_s(C_2+C_1\hat{L}_0)-$$
$$M_0(A_{11}+G_sC_{s0})^{-1}G_sC_2M_1G_s(C_2+C_1\hat{L}_0)] +$$
$$\varepsilon[I+M_0A_{11}^{-1}G_s(I+C_1A_{11}^{-1}G_s)^{-1}C_2]^{-1}G_fC_2[I+M_0A_{11}^{-1}G_s(I+C_1A_{11}^{-1}G_s)^{-1}C_2]+O(\varepsilon^2)$$
$$= \varepsilon[I+M_0A_{11}^{-1}G_s(I+C_1A_{11}^{-1}G_s)^{-1}C_2]^{-1}A_{f0}[I+M_0A_{11}^{-1}G_s(I+C_1A_{11}^{-1}G_s)^{-1}C_2]+$$
$$\varepsilon[I+M_0A_{11}^{-1}G_s(I+C_1A_{11}^{-1}G_s)^{-1}C_2]^{-1}G_fC_2[I+M_0A_{11}^{-1}G_s(I+C_1A_{11}^{-1}G_s)^{-1}C_2] +O(\varepsilon^2)$$
$$= \varepsilon[I+M_0(A_{11}+G_sC_1)^{-1}G_sC_2]^{-1}(A_{f0}+G_fC_2)[I+M_0(A_{11}+G_sC_1)^{-1}G_sC_2] + O(\varepsilon^2). \tag{4.84}$$

Thus, (4.76) can be rewrited as follows:

$$\begin{bmatrix} \gamma(k+1) \\ e_2(k+1) \end{bmatrix} = \begin{bmatrix} (A_{11}+G_sC_1)^{-1}(A_{11}+G_sC_{s0})(A_{11}+G_sC_1)+O(\varepsilon), & O(\varepsilon^2) \\ (A_{21}-K_2C_1), & \varepsilon[I+M_0(A_{11}+G_sC_1)^{-1}G_sC_2]^{-1}(A_{f0}+G_fC_2)[I+M_0(A_{11}+G_sC_1)^{-1}G_sC_2]+O(\varepsilon^2) \end{bmatrix} \begin{bmatrix} \gamma(k) \\ e_2(k) \end{bmatrix}. \tag{4.85}$$

From the above results, the eigenvalues of the error system (4.65) are approximated by

$$\begin{cases} \lambda(\hat{A}_{slow}) = \lambda(A_{11}+G_sC_{s0})[1 + O(\varepsilon)] \\ \lambda(\hat{A}_{fast}) = \lambda[\varepsilon(A_{f0}+G_fC_2)][1 + O(\varepsilon)]. \end{cases} \tag{4.86}$$

On the other hand, by utilizing Lemma 4-2, there exists a positive scalar ε_2^* such that the above system (4.85) is asymptotically stable for every $\varepsilon \in [0,\varepsilon_2^*)$. Q.E.D.

(b) Full-order observers of current type

Lemma 4-3 implies that the states $x_1(k)$ and $x_2(k)$ can be reconstructed from the measurements $y(k-1),y(k-2),\ldots,y(0)$. If the present output measurement $y(k)$ is available for the estimate of the present states, the following full-order observer can be constructed:

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}, & \varepsilon A_{12} \\ A_{21}, & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \{y(k+1) - [C_1, C_2] \begin{bmatrix} A_{11}, & \varepsilon A_{12} \\ A_{21}, & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} - [C_1, C_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k)\} \tag{4.87}$$

which is different from the classical type of observer considered in the previous section.

Then, we have the following results:

Lemma 4-4 Let the following conditions be satisfied:

(A-1) (C_{s0}, A_{11}) and (C_2, A_{f0}) is completely observable when $\tilde{A}$ is nonsingular, or

(A-1)' (C_{s0}, A_{11}) is completely observable and (D_{f0}, A_{f0}) is detectable when $\tilde{A}$ is singular, where

$$\tilde{A} = \begin{bmatrix} A_{11}, & \varepsilon A_{12} \\ A_{21}, & \varepsilon A_{22} \end{bmatrix}, \quad D_{f0} = C_2 A_{f0}.$$

If we select

$$\begin{cases} H_1 = -(I-\varepsilon L_0 M_0)G_s + \varepsilon(A_{11}+G_s D_{s0})^{-1} G_s D_{f0} G_f (I+D_{s0}A_{11}^{-1}G_s) \\ H_2 = -(M_0+\varepsilon M_1)G_s - \varepsilon[I-M_0(A_{11}+G_s D_{s0})^{-1} G_s D_{f0}] G_f (I+D_{s0}A_{11}^{-1}G_s) \end{cases} \tag{4.88}$$

where $D_{s0} = C_{s0}A_{11}$, $D_{f0} = C_2 A_{f0}$

then there exist gain matrices G_s and G_f, and a positive scalar ε^* such that for every $\varepsilon \in [0, \varepsilon^*)$ the error system is asymptotically stable.

Proof

The error system satisfies

$$\begin{bmatrix} e_1(k+1) \\ e_2(k+1) \end{bmatrix} = \{ \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} [C_1, C_2] \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix} \} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}$$

$$= [\tilde{A} - \tilde{H}\tilde{C}\tilde{A}] \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}. \tag{4.89}$$

If $\widetilde{A}$ is nonsingular, the observability of the pair $(\widetilde{C}\widetilde{A}, \widetilde{A})$ is equivalent to the observability of the pair $(\widetilde{C}, \widetilde{A})$. If $\widetilde{A}$ is singular and the pair $(\widetilde{C}, \widetilde{A})$ is observable, the pair $(\widetilde{C}\widetilde{A}, \widetilde{A})$ is detectable and all unobservable modes of the pair $(\widetilde{C}\widetilde{A}, \widetilde{A})$ are dead-beat (Willems [Wil.80]). Here, for brevity, we set

$$\widetilde{D} = [D_1, D_2] = \widetilde{C}\widetilde{A}. \tag{4.90}$$

Applying the design procedure shown in Fig. 4.2 leads to

$$\begin{aligned}
[D_s, \varepsilon D_f] &= [D_1, D_2]\begin{bmatrix} I+\varepsilon LM & \varepsilon L \\ M & I \end{bmatrix} \\
&= [C_1, C_2]\begin{bmatrix} I+\varepsilon LM & \varepsilon L \\ M & I \end{bmatrix}\begin{bmatrix} I & -\varepsilon L \\ -M & I+\varepsilon ML \end{bmatrix}\begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix}\begin{bmatrix} I+\varepsilon LM & \varepsilon L \\ M & I \end{bmatrix} \\
&= [C_s, C_f]\begin{bmatrix} A_s & 0 \\ 0 & \varepsilon A_f \end{bmatrix} \\
&= [C_s A_s, \varepsilon C_f A_f].
\end{aligned} \tag{4.91}$$

Thus, the observability of the pair $(\widetilde{C}\widetilde{A}, \widetilde{A})$ is equivalent to the observability of the pair (D_s, A_s) and (D_f, A_f). The results of Lemma 4-4 follow by setting D_s and D_f in place of C_s and C_f straightfowards.

Q.E.D.

[Remarks]

1. Since in Lemma 4-3 and Lemma 4-4, we have $|\lambda(\varepsilon A_{f0})| < 1$ for sufficiently small ε, the observer gain $G_f = 0$ is an admissible choice.
2. In Lemma 4-3 and Lemma 4-4, the nonsingularity of the matrix $(A_{11}+G_s C_{s0})$ is required. This requirement prevents us from constructing the dead-beat observers via the singular perturbation approach.

(c) Reduced-order observer

Choosing $G_f = 0$, and setting $\varepsilon = 0$ in the full-order observer (4.64) yields a reduced-order observer of possessing dimension n_1

as follows:

$$\begin{cases} \hat{x}_{1r}(k+1) = A_{11}\hat{x}_{1r}(k)+B_1u(k)+K_r[y(k)-C_{s0}\hat{x}_{1r}(k)-E_{s0}u(k-1)] \\ \hat{x}_{2r}(k) = A_{21}A_{11}^{-1}\hat{x}_{1r}(k)+B_{f0}u(k-1) \end{cases} \tag{4.92}$$

where $K_r = -G_s$, $B_{f0} = B_2-A_{21}A_{11}^{-1}B_1$, $C_{s0} = C_1+C_2A_{21}A_{11}^{-1}$, $E_{s0} = C_2B_{f0}$.

Here, we consider the reduced system:

$$\begin{cases} x_{1r}(k+1) = A_{11}x_{1r}(k) + B_1u(k) \\ x_{2r}(k) = A_{21}A_{11}^{-1}x_{1r}(k) + B_{f0}u(k-1) \\ y_r(k) = C_{s0}x_{1r}(k) + E_{s0}u(k-1). \end{cases} \tag{4.93}$$

The above system is given by setting $\varepsilon = 0$ in the full system (4.12), and represents the slow behavior. If we set $y_r(k)$ in place of $y(k)$ in (4.92), the above observer corresponds to a full-order observer for the reduced system (4.93). When we use the above observer to estimate the states of the full system (4.12), some observation errors are caused by the difference between the reduced and full system. These errors can be estimated in the following lemma.

<u>Lemma 4-5</u> Let the following conditions be satisfied:

(A-1) (C_{s0}, A_{11}) : completely observable.

Then, the observation errors $e_1(k)$ and $e_2(k)$ satisfy:

$$e_1(k) = (A_{11}+G_sC_{s0})^k e_1(0)+\sum_{\tau=0}^{k-1}(A_{11}+G_sC_{s0})^{k-1-\tau}G_sC_2(\varepsilon A_{f0})^{\tau}x_2(0)+O(\varepsilon) \tag{4.94.a}$$

$$e_2(k) = A_{21}A_{11}^{-1}e_1(k)+(\varepsilon A_{f0})^k x_2(0)+O(\varepsilon) \tag{4.94.b}$$

where $e_1(k) = x_1(k) - \hat{x}_{1r}(k)$

$e_2(k) = x_2(k) - \hat{x}_{2r}(k)$, $A_{f0} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Proof

A modification of (4.12) yields

$$x_2(k+1) = \varepsilon A_{f0}x_2(k)+A_{21}A_{11}^{-1}x_1(k+1)+B_{f0}u(k).$$

From the above expression, we have

$$\begin{aligned} x_2(k) &= (\varepsilon A_{f0})^k x_2(0) + \sum_{\tau=0}^{k-1}(\varepsilon A_{f0})^{k-1-\tau}[A_{21}A_{11}^{-1}x_1(\tau+1) + B_{f0}u(\tau)] \\ &= (\varepsilon A_{f0})^k x_2(0) + [A_{21}A_{11}^{-1}x_1(k) + B_{f0}u(k-1)] + \\ &\quad \sum_{\tau=1}^{k-1}(\varepsilon A_{f0})^{k-\tau}[A_{21}A_{11}^{-1}x_1(\tau) + B_{f0}u(\tau-1)] \\ &= (\varepsilon A_{f0})^k x_2(0) + [A_{21}A_{11}^{-1}x_1(k) + B_{f0}u(k-1)] + \varepsilon R_1 \end{aligned} \quad (4.95)$$

where $R_1 = \sum_{\tau=1}^{k-1}\varepsilon^{k-1-\tau}A_{f0}^{k-\tau}[A_{21}A_{11}^{-1}x_1(\tau) + B_{f0}u(\tau-1)]$.

The estimation error $e_1(k)$ must satisfy

$$e_1(k+1) = A_{11}e_1(k)+\varepsilon A_{12}x_2(k)+G_s[C_1x_1(k)+C_2x_2(k)-C_{s0}\hat{x}_{1r}(k)-E_{s0}u(k-1)]. \quad (4.96)$$

Substitution (4.95) into the above equation gives

$$e_1(k+1) = (A_{11}+G_sC_{s0})e_1(k)+G_sC_2(\varepsilon A_{f0})^k x_2(0)+\varepsilon R_2 \ ; \ R_2 = A_{12}x_2(k)+R_1. \quad (4.97)$$

Eqn. (4.94.a) follows from the above system.

Furthermore, by using (4.92) and (4.95), we have

$$e_2(k) = x_2(k)-\hat{x}_{2r}(k) = A_{21}A_{11}^{-1}e_1(k)+(\varepsilon A_{f0})^k x_2(0)+\varepsilon R_1. \quad (4.98)$$ Q.E.D.

In this way, by selecting a gain matrix G_s such that $|\lambda(A_{11}+G_sC_{s0})| < 1$, we can construct an ε-asymptotic observer (Locatelli [Lo.76]) of possessing dimension n_1 for the full system (4.12), where $O(\varepsilon)$ observation errors occur asymptotically. If the time-scale separation ratio ε is sufficiently small, these errors can be neglected relative to the state estimate and so the reduced-order observer (4.92) can be used practically.

4.3.2 Stabilizing controllers based on observers

In case of implementing linear state feedback, all state variables are not always available for feedback signals. In order to stabilize such systems with inaccessible states, an observer-based controller must be designed.

In this section, we consider such an observer-based state feedback controller:

$$u(k) = G_c\hat{x}_1(k) + \varepsilon G_c A_{11}^{-1}A_{12}\hat{x}_2(k) \qquad (4.99)$$

which is based on the full-order observer presented in the preceding section:

$$\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} [y(k) - C_1\hat{x}_1(k) - C_2\hat{x}_2(k)]$$

$$(4.100.a)$$

where $K_1 = -(I-\varepsilon L_0 M_0)G_s$

$$(4.100.b)$$

$K_2 = -(M_0+\varepsilon M_1)G_s.$

These observer gains are given by choosing $G_f = 0$ in (4.74) because of $|\lambda(\varepsilon A_{f0})|<1$.

Substituting the controller (4.99) into the system (4.12) and using the observer (4.100) yields a closed-loop system:

$$\begin{bmatrix} x_1(k+1) \\ \hat{x}_1(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}, & B_1G_c \\ K_1C_1, & (A_{11}+B_1G_c-K_1C_1) \end{bmatrix} \begin{bmatrix} x_1(k) \\ \hat{x}_1(k) \end{bmatrix} + \begin{bmatrix} \varepsilon A_{12}, & \varepsilon B_1G_cA_{11}^{-1}A_{12} \\ K_1C_2, & (\varepsilon A_{12}+\varepsilon B_1G_cA_{11}^{-1}A_{12}-K_1C_2) \end{bmatrix} \begin{bmatrix} x_2(k) \\ \hat{x}_2(k) \end{bmatrix}$$

$$\begin{bmatrix} x_2(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{21}, & B_2G_c \\ K_2C_1, & (A_{21}+B_2G_c-K_2C_1) \end{bmatrix} \begin{bmatrix} x_1(k) \\ \hat{x}_1(k) \end{bmatrix} + \begin{bmatrix} \varepsilon A_{22}, & \varepsilon B_2G_cA_{11}^{-1}A_{12} \\ K_2C_2, & (\varepsilon A_{22}+\varepsilon B_2G_cA_{11}^{-1}A_{12}-K_2C_2) \end{bmatrix} \begin{bmatrix} x_2(k) \\ \hat{x}_2(k) \end{bmatrix}.$$

$$(4.101)$$

Then, we obtain the following results.

<u>Theorem 4-2</u> Let the following conditions be satisfied:

(A-1) (A_{11}, B_1) : completely controllable.

(A-2) (C_{s0}, A_{11}) : completely observable.

Then, there exist gain matrices G_c and G_s and a positive scalar ε^* such that for every $\varepsilon\in[0,\varepsilon^*)$ the closed-loop system (4.101) is asymptotically stable.

Proof

Substituting the observation errors:

$$\begin{cases} e_1(k) = x_1(k) - \hat{x}_1(k) \\ e_2(k) = x_2(k) - \hat{x}_2(k) \end{cases} \tag{4.102}$$

into (4.101) yields

$$\begin{bmatrix} x_1(k+1) \\ e_1(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}+B_1G_c, & -B_1G_c \\ 0, & A_{11}-K_1C_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ e_1(k) \end{bmatrix} + \begin{bmatrix} \varepsilon(A_{11}+B_1G_c)A_{11}^{-1}A_{12}, & -\varepsilon B_1G_cA_{11}^{-1}A_{12} \\ 0, & \varepsilon A_{12}-K_1C_2 \end{bmatrix} \begin{bmatrix} x_2(k) \\ e_2(k) \end{bmatrix}$$

$$\begin{bmatrix} x_2(k+1) \\ e_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{21}+B_2G_c, & -B_2G_c \\ 0, & A_{21}-K_2C_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ e_1(k) \end{bmatrix} + \begin{bmatrix} \varepsilon(A_{22}+B_2G_cA_{11}^{-1}A_{12}), & -\varepsilon B_2G_cA_{11}^{-1}A_{12} \\ 0, & \varepsilon A_{22}-K_2C_2 \end{bmatrix} \begin{bmatrix} x_2(k) \\ e_2(k) \end{bmatrix}. \tag{4.103}$$

For simplicity, we rewrite (4.103) as follows:

$$\begin{cases} y_1(k+1) = F_{11}y_1(k) + F_{12}y_2(k) \\ y_2(k+1) = F_{21}y_1(k) + F_{22}y_2(k). \end{cases} \tag{4.104}$$

Applying the fast transformation:

$$\begin{bmatrix} y_1(k+1) \\ \eta\ (k+1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{L} & I \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \tag{4.105}$$

produces

$$\begin{bmatrix} y_1(k+1) \\ \eta\ (k+1) \end{bmatrix} = \begin{bmatrix} F_{slow} & F_{12} \\ R(\hat{L}) & F_{fast} \end{bmatrix} \begin{bmatrix} y_1(k) \\ \eta\ (k) \end{bmatrix} \tag{4.106}$$

where $$\begin{cases} F_{slow} = F_{11} - F_{12}\hat{L} \\ F_{fast} = F_{22} + \hat{L}F_{12} \end{cases}$$

$$R(\hat{L}) = -F_{22}\hat{L} + \hat{L}F_{11} - \hat{L}F_{12}\hat{L} + F_{21} = 0. \qquad (4.108)$$

By following the similar procedure to the proof of Lemma 4-3, we obtain

$$\hat{L}(\varepsilon) = \hat{L}_0 + \varepsilon\hat{L}_1 + O(\varepsilon^2) \qquad (4.109)$$

where
$$\hat{L}_0 = \begin{bmatrix} \hat{L}^0_{11} & \hat{L}^0_{12} \\ \hat{L}^0_{21} & \hat{L}^0_{22} \end{bmatrix} = \begin{bmatrix} -(A_{21}+B_2G_c)\alpha_c^{-1}, & [B_2-(A_{21}+B_2G_c)\alpha_c^{-1}B_1]G_c\alpha_o^{-1} \\ 0, & -A_{21}A_{11}^{-1} \end{bmatrix} \qquad (4.110)$$

$$\hat{L}_1 = \begin{bmatrix} \hat{L}^1_{11} & \hat{L}^1_{12} \\ \hat{L}^1_{21} & \hat{L}^1_{22} \end{bmatrix} \begin{bmatrix} -A_{f0}(A_{21}+B_2G_c)(A_{11}+B_1G_c)^{-2}, & \hat{L}^1_{12} \\ 0, & -A_{f0}M_0A_{11}^{-1} \end{bmatrix} \qquad (4.111)$$

$$;\ \hat{L}^1_{12} = [A_{f0}\hat{L}^0_{12}-B_2G_cL_0\hat{L}^0_{22}+\hat{L}^0_{12}L_0M_0G_sC_1-\hat{L}^0_{11}B_1G_cL_0\hat{L}^0_{22}+ \hat{L}^0_{12}(A_{12}-L_0M_0G_sC_2)\hat{L}^0_{22}+\hat{L}^1_{11}B_1G_c+\hat{L}^0_{12}G_sC_2\hat{L}^1_{22}](A_{11}+G_sC_{s0})^{-1}$$

$$\alpha_c = (A_{11}+B_1G_c), \quad \alpha_o = (A_{11}+G_sC_{s0}).$$

Substituting (4.109) into (4.106) yields

$$\begin{bmatrix} y_1(k+1) \\ \eta\ (k+1) \end{bmatrix} = \left\{ \begin{bmatrix} F_{slow}, & F_{12} \\ 0, & F_{fast} \end{bmatrix} + O(\varepsilon^2) \right\} \begin{bmatrix} y_1(k) \\ \eta\ (k) \end{bmatrix} \qquad (4.112)$$

where
$$F_{slow} = \begin{bmatrix} (A_{11}+B_1G_c)+O(\varepsilon), & -B_1G_c+O(\varepsilon) \\ 0, & (A_{11}+G_sC_{s0})+O(\varepsilon) \end{bmatrix}$$

$$F_{fast} = \begin{bmatrix} \varepsilon A_{f0}, & \hat{L}^0_{12}G_sC_2+O(\varepsilon) \\ 0, & \varepsilon A_{f0} \end{bmatrix}.$$

Accordingly, for sufficiently small ε the closed-loop system (4.103) or (4.101) is asymptotically stable, and the eigenvalues are approximated by

$$\begin{cases} \lambda(F_{slow}) = \lambda(A_{11}+B_1G_c)\cap\lambda(A_{11}+G_sC_{s0})[1+O(\varepsilon)] \\ \lambda(F_{fast}) = \lambda(\varepsilon A_{f0})\cap\lambda(\varepsilon A_{f0})[1+O(\varepsilon)]. \end{cases} \qquad (4.113)$$

Q.E.D.

If we use the full-order observer of current type instead of the observer of prediction type, we can obtain the similar results by modifying the above Theorem slightly.

4.3.3 Numerical example

We consider the ninth order boiler model, which was treated in the previous Section 4.2.5.

Firstly, the controller gain G_c of $\lambda(A_{11}+B_1G_c)$ which places the slow eigenvalues at the desired locations {0.99, 0.98, 0.97, 0.96, 0.95, 0.94} were calculated and then, by using the unity rank approach, the state feedback controller (4.99) based on the full-order observer was obtained as follows:

$$u(k) = [G_c,\ G_cA_{11}^{-1}A_{12}]\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\end{bmatrix}$$

$$= \begin{bmatrix} -0.56941\times10^{-1} & -0.14259\times10^{-1} & 0.29592 & 0.27772 & -0.25413 \\ -0.56941\times10^{-1} & -0.14259\times10^{-1} & 0.29592 & 0.27772 & -0.25413 \\ -0.56941\times10^{-1} & -0.14259\times10^{-1} & 0.29592 & 0.27772 & -0.25413 \\ -0.56941\times10^{-1} & -0.14259\times10^{-1} & 0.29592 & 0.27772 & -0.25413 \\ -0.56941\times10^{-1} & -0.14259\times10^{-1} & 0.29592 & 0.27772 & -0.25413 \end{bmatrix}$$

$$\begin{matrix} -0.66173\times10 & 0.85758\times10^{-2} & 0.30018\times10^{-2} & 0.97893\times10^{-1} \\ -0.66173\times10 & 0.85758\times10^{-2} & 0.30018\times10^{-2} & 0.97893\times10^{-1} \\ -0.66173\times10 & 0.85758\times10^{-2} & 0.30018\times10^{-2} & 0.97893\times10^{-1} \\ -0.66173\times10 & 0.85758\times10^{-2} & 0.30018\times10^{-2} & 0.97893\times10^{-1} \\ -0.66173\times10 & 0.85758\times10^{-2} & 0.30018\times10^{-2} & 0.97893\times10^{-1} \end{matrix}\Bigg] \begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\end{bmatrix}. \quad (4.114)$$

On the other hand, by calculating the gain G_s of $\lambda(A_{11}+G_sC_{s0})$ which places the slow eigenvalues at the desired locations {0.93, 0.92, 0.91, 0.86, 0.66, 0.56}, the observer gains K_1 and K_2 of (4.100) were

$$\begin{bmatrix} K_1 \\ \hline K_2 \end{bmatrix} = \begin{bmatrix} 0.8972298 & 0.8972298 & 0.8972298 \\ -0.7708214 & -0.7708214 & -0.7708214 \\ -0.4252098 & -0.4252098 & -0.4252098 \\ -0.7263677 & -0.7263677 & -0.7263677 \\ -0.9154802 & -0.9154802 & -0.9154802 \\ 0.2084034\times10^{-2} & 0.2084034\times10^{-2} & 0.2084034\times10^{-2} \\ \hline -0.8385936 & -0.8385936 & -0.8385936 \\ -0.7001732 & -0.7001732 & -0.7001732 \\ 0.7388870\times10^{-1} & 0.7388870\times10^{-1} & 0.7388870\times10^{-1} \end{bmatrix}. \quad (4.115)$$

When the state feedback controller (4.114) based on the full-order observer (4.100) and (4.115) was implemented to the boiler model (4.60), the eigenvalues of the closed-loop system were

$$\begin{cases} \lambda(F_{slow}) = \{0.9913,\ 0.9772\pm j0.8892\times10^{-2},\ 0.9530,\ 0.9485,\ 0.9428\}\cup \\ \qquad\qquad \{0.9390,\ 0.9117\pm j0.2104\times10^{-1},\ 0.8611,\ 0.6599,\ 0.5547\} \\ \lambda(F_{fast}) = \{0.5815\times10^{-2},\ -0.9050\times10^{-4}\pm j0.1720\times10^{-2}\}\cup \\ \qquad\qquad \{0.5815\times10^{-2},\ 0.8548\times10^{-2},\ -0.6883\times10^{-2}\}. \end{cases} \tag{4.116}$$

If necessary, these results can be improved by taking into account the higher order terms in the feedback controller gain (4.99) and the observer gain (4.100).

4.4 Near Optimum Controller Design of Regulator Problem for Slow Sampling Model

4.4.1 Problem formulation

We consider the regulator problem for the slow sampling model (4.12).

The performance index to be minimized is

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\{[x_1'(k), x_2'(k)]\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22}\end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k)\end{bmatrix} + u'(k)Ru(k)\} \tag{4.117}$$

where $Q \geq 0$ and $R > 0$ and it is assumed that $(\tilde{A}, B)$ is a stabilizable pair and that $(Q^{1/2}, \tilde{A})$ is a detectable pair.

In this chapter, the systematic procedure for the mode-decoupled near optimum controller design will be developed via the boundary layer method for the slow sampling model formulated in Chapter 3, and some existence conditions under which such a design can be performed will be studied.

Introducing the Hamiltonian and applying the variational calculus gives the following canonical equation:

$$\begin{cases} x_1(k+1,\varepsilon) = A_{11}x_1(k,\varepsilon)+\varepsilon A_{12}x_2(k,\varepsilon)-S_{11}p_1(k+1,\varepsilon)-S_{12}p_2(k+1,\varepsilon) \\ x_2(k+1,\varepsilon) = A_{21}x_1(k,\varepsilon)+\varepsilon A_{22}x_2(k,\varepsilon)-S_{12}'p_1(k+1,\varepsilon)-S_{22}p_2(k+1,\varepsilon) \\ p_1(k,\varepsilon) = Q_{11}x_1(k,\varepsilon)+Q_{12}x_2(k,\varepsilon)+A_{11}'p_1(k+1,\varepsilon)+A_{21}'p_2(k+1,\varepsilon) \\ p_2(k,\varepsilon) = Q_{12}'x_1(k,\varepsilon)+Q_{22}x_2(k,\varepsilon)+\varepsilon A_{12}'p_1(k+1,\varepsilon)+\varepsilon A_{22}'p_2(k+1,\varepsilon) \end{cases} \tag{4.118}$$

where $S_{11} = B_1R^{-1}B_1'$, $S_{12} = B_1R^{-1}B_2'$, $S_{22} = B_2R^{-1}B_2'$.

The optimal control is given by

$$u^o(k,\varepsilon) = -R^{-1}B_1'p_1(k+1,\varepsilon)-R^{-1}B_2'p_2(k+1,\varepsilon). \tag{4.119}$$

Here, by using the boundary layer method, we shall seek the asymptotic series solutions which are composed of the outer-region terms $(X_i(k,\varepsilon)$, $P_i(k,\varepsilon))$ and the boundary-layer correction terms $(\tilde{m}_i(k,\varepsilon),\tilde{\rho}_i(k,\varepsilon))$, that is,

$$\begin{cases} x_1(k,\varepsilon) = X_1(k,\varepsilon)+\tilde{m}_1(k,\varepsilon) \ ; \ \tilde{m}_1(k,\varepsilon) = \varepsilon^{k+1}m_1(k,\varepsilon) & (4.120.a) \\ x_2(k,\varepsilon) = X_2(k,\varepsilon)+\tilde{m}_2(k,\varepsilon) \ ; \ \tilde{m}_2(k,\varepsilon) = \varepsilon^{k}m_2(k,\varepsilon) & (4.120.b) \\ p_1(k,\varepsilon) = P_1(k,\varepsilon)+\tilde{\rho}_1(k,\varepsilon) \ ; \ \tilde{\rho}_1(k,\varepsilon) = \varepsilon^{k}\rho_1(k,\varepsilon) & (4.120.c) \\ p_2(k,\varepsilon) = P_2(k,\varepsilon)+\tilde{\rho}_2(k,\varepsilon) \ ; \ \tilde{\rho}_2(k,\varepsilon) = \varepsilon^{k}\rho_2(k,\varepsilon) & (4.120.d) \end{cases}$$

where $(X_i(k,\varepsilon)$, $P_i(k,\varepsilon))$ and $(m_i(k,\varepsilon)$, $\rho_i(k,\varepsilon))$ have asymptotic series expansions in ε. Then, the optimal control is expressed as

$$u^o(k,\varepsilon) = U(k,\varepsilon)+\varepsilon^{k+1}v(k,\varepsilon) \tag{4.121}$$

where $U(k,\varepsilon) = -R^{-1}B_1'P_1(k+1,\varepsilon)-R^{-1}B_2'P_2(k+1,\varepsilon)$

$v(k,\varepsilon) = -R^{-1}B_1'\rho_1(k+1,\varepsilon)-R^{-1}B_2'\rho_2(k+1,\varepsilon)$.

In the following section, the near optimum controller design will be developed via the boundary layer method, and some existence conditions under which such a design can be performed will be investigated.

4.4.2 Near optimum controller design and existence conditions

(a) Slow subproblem

Firstly, we will consider the finite-time regulator problem, and then we will extend the results to the infinite-time problem.

We consider the following performance index:

$$J = \frac{1}{2}[x_1'(N), x_2'(N)]\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}\begin{bmatrix} x_1(N) \\ x_2(N) \end{bmatrix} +$$

$$\frac{1}{2}\sum_{k=0}^{N-1}[(x_1'(k), x_2'(k))\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u'(k)Ru(k)]. \tag{4.122}$$

The outer-region term $(X_i(k,\varepsilon),\ P_i(k,\varepsilon))$ must satisfy (4.118). Setting $\varepsilon = 0$ in (4.118) yields the following reduced canonical equations which the leading terms $(X_{i0}(k),\ P_{i0}(k))$ satisfy

$$\begin{cases} X_{10}(k+1) = A_{11}X_{10}(k) - S_{11}P_{10}(k+1) - S_{12}P_{20}(k+1) & (4.123.a) \\ X_{20}(k+1) = A_{21}X_{10}(k) - S_{12}'P_{10}(k+1) - S_{22}P_{20}(k+1) & (4.123.b) \\ P_{10}(k) = Q_{11}X_{10}(k) + Q_{12}X_{20}(k) + A_{11}'P_{10}(k+1) + A_{21}'P_{20}(k+1) & (4.123.c) \\ P_{20}(k) = Q_{12}'X_{10}(k) + Q_{22}X_{20}(k). & (4.123.d) \end{cases}$$

Here we will assume the following conditions:

(H-1) A_{11} is nonsingular

(H-2) $(I+B_1\hat{R}^{-1}\hat{C}_1')$ is nonsingular

(H-3) $\hat{Q}$ is a positive semi-definite matrix

where $\hat{R} = R + B_f'Q_{22}B_f, \quad B_f = B_2 - A_{21}A_{11}^{-1}B_1$ (4.124.a)

$$\hat{C} = (Q_{12} + A_{11}'^{-1}A_{21}'Q_{22})B_f \tag{4.124.b}$$

$$\hat{Q} = (I, A_{11}'^{-1}A_{21}')\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}\begin{bmatrix} I \\ A_{21}A_{11}^{-1} \end{bmatrix} - \hat{C}\hat{R}^{-1}\hat{C}'. \tag{4.124.c}$$

Under these conditions, by using tha variable transformation:

$$\lambda_{10}(k) = (I+B_1\hat{R}^{-1}\hat{C}')'(P_{10}(k) + A_{11}'^{-1}A_{21}'P_{20}(k)) \tag{4.125}$$

and after considerable algebraic manipulations, we obtain the following reduced canonical equations:

$$\begin{cases} X_{10}(k+1) = A_r X_{10}(k) - B_r \hat{R}^{-1} B_r' \lambda_{10}(k+1) \\ \lambda_{10}(k) = \hat{Q} X_{10}(k) + A_r' \lambda_{10}(k+1) \end{cases} \tag{4.126}$$

where $A_r = (I+B_1\hat{R}^{-1}\hat{C}')^{-1}A_{11}$, $B_r = (I+B_1\hat{R}^{-1}\hat{C}')^{-1}B_1$.

The slow control part is given by

$$U_0(k) = -R^{-1}B_1'P_{10}(k+1) - R^{-1}B_2'P_{20}(k+1).$$

By using (4.123) and (4.125), we have

$$U_0(k) = -\hat{R}^{-1}B_r'\lambda_{10}(k+1) - \hat{R}^{-1}\hat{C}'X_{10}(k+1) = W_0(k) - \hat{R}^{-1}\hat{C}'X_{10}(k+1). \tag{4.127}$$

In the reduced canonical equations (4.126), setting the Riccati transformation:

$$\lambda_{10}(k) = K_r(k)X_{10}(k) \tag{4.128}$$

gives the Riccati difference equation:

$$K_r(k) = A_r'K_r(k+1)A_r - A_r'K_r(k+1)B_r[\hat{R}+B_r'K_r(k+1)B_r]^{-1}B_r'K_r(k+1)A_r + \hat{Q}$$
$$;\quad K_r(k+1) = \hat{Q}. \tag{4.129}$$

In the infinite-time problem, the above equation leads to the algebraic Riccati equation:

$$K_r = A_r'K_rA_r - A_r'K_rB_r[\hat{R}+B_r'K_rB_r]^{-1}B_r'K_rA_r + \hat{Q}. \tag{4.130}$$

A unique positive-semidefinite solution of (4.130) exists under the condition:

(H-4) (C_r, A_r, B_r) is detectable and stabilizable where $\hat{Q} = C_r'C_r$.

From the above results, we obtain the following lemma.

Lemma 4-6 Let the conditions (H-1)-(H-4) be satisfied:

Then, the slow control part of the optimal control $u^o(k,\varepsilon)$ is given by

$$
\begin{aligned}
U_0(k) &= W_0(k)-\hat{R}^{-1}\hat{C}'X_{10}(k) \\
&= -[(I-\hat{R}^{-1}\hat{C}'B_r)(\hat{R}+B_r'K_rB_r)^{-1}B_r'K_rA_r+\hat{R}^{-1}\hat{C}'A_r]X_{10}(k) \\
&= -G_1X_{10}(k) \qquad (4.131)
\end{aligned}
$$

where $W_0(k) = -[\hat{R}+B_r'K_rB_r]^{-1}B_r'K_rA_rX_{10}(k)$, which

corresponds to the control of the following slow subproblem:

$$
\left\{
\begin{aligned}
&X_{10}(k+1) = A_rX_{10}(k)+B_rW_0(k) \quad ; \quad X_{10}(0) = x_{10} \\
&J = 1/2\sum_{k=0}^{\infty}[X_{10}'(k)\hat{Q}X_{10}(k)+W_0'(k)\hat{R}W_0(k)].
\end{aligned}
\right. \qquad (4.132)
$$

The slow control shown above dominates the system behavior over the whole interval.

[Remarks]

The slow subproblem given above can be also obtained in the following manner. Setting $\varepsilon = 0$ in (4.12) and (4.122) yields the following reduced problem:

$$
\left\{
\begin{aligned}
&X_{10}(k+1) = A_{11}X_{10}(k)+B_1U_0(k) \\
&X_{20}(k) = A_{21}A_{11}^{-1}X_{10}(k)+B_1U_0(k-1)
\end{aligned}
\right. \qquad (4.133.a)
$$

$$
\begin{aligned}
J = {} & 1/2[X_{10}'(N)(\hat{Q}+\hat{C}\hat{R}^{-1}\hat{C}')X_{10}(N)+U_0'(N-1)B_f'Q_{22}B_fU_0(N-1)+2X_{10}'(N)\hat{C}U_0(N-1)] \\
& +1/2\sum_{k=0}^{N-1}[X_{10}'(k)(\hat{Q}+\hat{C}\hat{R}^{-1}\hat{C}')X_{10}(k)+2X_{10}'(k)\hat{C}U_0(k-1)+ \\
& \qquad U_0'(k-1)B_f'Q_{22}B_fU_0(k-1)+U_0'(k)RU_0(k)]. \qquad (4.133.b)
\end{aligned}
$$

Transforming the control variable

$$U_0(k) = W_0(k)-\hat{R}^{-1}\hat{C}'X_{10}(k+1)$$

in (4.133.a) and (4.133.b) leads to the slow subproblem (4.132). Here, introducing the Hamiltonian and applying the variational calculus yields the canonical equations (4.126) corresponding to the slow subproblem.

(b) Fast subproblem

The boundary-layer correction terms $(\tilde{m}_i(k,\varepsilon),\tilde{\rho}_i(k,\varepsilon))$ must satisfy the following equations:

$$\begin{cases} \varepsilon m_1(k+1,\varepsilon) = A_{11}m_1(k,\varepsilon)+A_{12}m_2(k,\varepsilon)-S_{11}\rho_1(k+1,\varepsilon)-S_{12}\rho_2(k+1,\varepsilon) \\ m_2(k+1,\varepsilon) = A_{21}m_1(k,\varepsilon)+A_{22}m_2(k,\varepsilon)-S_{12}'\rho_1(k+1,\varepsilon)-S_{22}\rho_2(k+1,\varepsilon) \\ \rho_1(k,\varepsilon) = \varepsilon Q_{11}m_1(k,\varepsilon)+Q_{12}m_2(k,\varepsilon)+\varepsilon A_{11}'\rho_1(k+1,\varepsilon)+\varepsilon A_{21}'\rho_2(k+1,\varepsilon) \\ \rho_2(k,\varepsilon) = \varepsilon Q_{12}'m_1(k,\varepsilon)+Q_{22}m_2(k,\varepsilon)+\varepsilon^2 A_{12}'\rho_1(k+1,\varepsilon)+\varepsilon^2 A_{22}'\rho_2(k+1,\varepsilon). \end{cases} \tag{4.134}$$

Setting $\varepsilon = 0$ in the above equation and assuming the condition

(H-5) $[I+\tilde{B}_f R^{-1}(B_1'Q_{12}+B_2'Q_{22})]$: nonsingular

gives

$$\begin{cases} \tilde{m}_{20}(k+1) = \tilde{A}_c\tilde{m}_{20}(k) \quad ; \quad \tilde{m}_{20}(0) = x_{20}-X_{20}(0) & (4.135.a) \\ \tilde{m}_{10}(k) = -\tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{m}_{20}(k)+\tilde{A}_{11}^{-1}B_1R^{-1}(B_1'Q_{12}+B_2'Q_{22})\tilde{m}_{20}(k+1) & (4.135.b) \end{cases}$$

where $\tilde{A}_c = [I+\tilde{B}_f R^{-1}(B_1'Q_{12}+B_2'Q_{22})]^{-1}\tilde{A}_f$,

$\tilde{A}_f = \tilde{A}_{22}-\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}$, $\tilde{B}_f = B_2-\tilde{A}_{21}\tilde{A}_{11}^{-1}B_1$.

Also, the terms $(\tilde{\rho}_{10}(k),\tilde{\rho}_{20}(k))$ are given by

$$\begin{cases} \tilde{\rho}_{10}(k) = Q_{12}\tilde{m}_{20}(k) & (4.136.a) \\ \tilde{\rho}_{20}(k) = Q_{22}\tilde{m}_{20}(k). & (4.136.b) \end{cases}$$

Here, under the condition

(H-6) $|\lambda(A_c)|<<1$,

$\tilde{m}_{i0}(k)$ decays to zero as $k\to\infty$.

Thus, we obtain the following lemma.

Lemma 4-7 Let the conditions (H-5) and (H-6) be satisfied.

Then, the leading terms $(\tilde{m}_{i0}(k), \tilde{\rho}_{i0}(k)$ which decay to zero as $k \to \infty$ exist, and satisfy the boundary-layer initial value problem (4.135) and (4.136). Furthermore, the fast control part of the optimal control $u^o(k,\varepsilon)$ is given by

$$\begin{aligned}\tilde{v}^o &= -R^{-1}[B_1'\tilde{\rho}_{10}(k+1)+B_2'\tilde{\rho}_{20}(k+1)] \\ &= -R^{-1}[B_1'Q_{12}+B_2'Q_{22}]\tilde{A}_c\tilde{m}_{20}(k) \\ &= -\tilde{G}_2\tilde{m}_{20}(k) \qquad (4.137)\end{aligned}$$

where $\tilde{G}_2 = \varepsilon G_2$, $\tilde{m}_{20}(k) = \varepsilon^k m_{20}(k)$.

The fast control shown above functions only near the boundary-layer.

(c) Near optimum controller design

Coordinating the slow and fast controls yields

$$\begin{aligned}u^o(k,\varepsilon) &= -R^{-1}[B_1'P_{10}(k+1)+B_2'P_{10}(k+1)]-\varepsilon^{k+1}R^{-1}[B_1'\rho_{10}(k+1)+B_2'\rho_{20}(k+1)]+O(\varepsilon) \\ &= U_0(k)+\varepsilon^{k+1}v_0(k)+O(\varepsilon) \qquad (4.138)\end{aligned}$$

where the slow control $U_0(k)$ and the fast control $\varepsilon^{k+1}v_0(k)$ are given by (4.131) and (4.137), that is,

$$U_0(k) = -G_1X_{10}(k), \quad \varepsilon^{k+1}v_0(k) = -\varepsilon^{k+1}G_2m_{20}(k)$$

where $G_1 = (I-\hat{R}^{-1}\hat{C}'B_r)(\hat{R}+B_r'K_rB_r)^{-1}B_r'K_rA_r+\hat{R}^{-1}\hat{C}'A_r$

$G_2 = R^{-1}(B_1'Q_{12}+B_2'Q_{22})A_c$.

Furthermore, by using the following relations:

$$X_{10}(k) \approx x_1(k,\varepsilon)-\varepsilon^{k+1}m_{10}(k) \qquad (4.139.a)$$

$$X_{20}(k) = [A_{21}A_{11}^{-1}-B_fR^{-1}(B_r'K_r+C')]X_{10}(k)$$

$$= G_3X_{10}(k) \approx G_3(x_1(k,\varepsilon)-\varepsilon^{k+1}m_{10}(k)) \qquad (4.139.b)$$

$$m_{10}(k) = -(A_{11}^{-1}A_{12}-A_{11}^{-1}B_1G_2)m_{20}(k) \qquad (4.139.c)$$

$$\varepsilon^k m_{20}(k) \approx x_2(k,\varepsilon)-X_{20}(k) \qquad (4.139.d)$$

and by trasforming the composite control (4.138) to the feedback form of the state variables, we obtain finally

$$u^s(k,\varepsilon) = -G_1x_1(k,\varepsilon)-\varepsilon[G_1A_{11}^{-1}A_{12}+(I-G_1A_{11}^{-1}B_1)G_2]x_2(k,\varepsilon). \qquad (4.140)$$

In this way, it has been shown that by decomposing the global problem into the sobproblems defined at respective time-scales via the boundary layer method and by coordinating these subproblems of possessing lower demension in a proper way, the mode-decoupled feedback law can be obtained.

(d) Suboptimal responses

When the control (4.140) is implemented, the suboptimally controlled system is given by

$$\begin{cases} x_1^s(k+1,\varepsilon) = \hat{A}_{11}x_1^s(k,\varepsilon) + \varepsilon\hat{A}_{12}x_2^s(k,\varepsilon) & ; \quad x_1^s(0,\varepsilon) = x_{10} \\ x_2^s(k+1,\varepsilon) = \hat{A}_{21}x_1^s(k,\varepsilon) + \varepsilon\hat{A}_{22}x_2^s(k,\varepsilon) & ; \quad x_2^s(0,\varepsilon) = x_{20} \end{cases} \qquad (4.141)$$

where $\hat{A}_{11} = A_{11} - B_1G_1$

$$\hat{A}_{12} = \{A_{12} - B_1[G_1A_{11}^{-1}A_{12} + (I-G_1A_{11}^{-1}B_1)G_2]\}$$

$$\hat{A}_{21} = A_{21} - B_2G_1$$

$$\hat{A}_{22} = \{A_{22} - B_2[G_1A_{11}^{-1}A_{12} + (I-G_1A_{11}^{-1}B_1)G_2]\}.$$

Now, we have an interest in predicting the behavior of (4.141) by using the responses of the slow subproblem and the fast subproblem. We consider the following fast transformation:

$$\hat{x}_f(k,\varepsilon) = \hat{L}x_1^s(k,\varepsilon) + x_2^s(k,\varepsilon) \qquad (4.142.a)$$

where $R(\hat{L}) = -\varepsilon\hat{A}_{22}\hat{L} + \hat{L}\hat{A}_{11} - \varepsilon\hat{L}\hat{A}_{12}\hat{L} + \hat{A}_{21} = 0.$ (4.142.b)

Then, for every $\varepsilon\in[0,\varepsilon_1^*)$, we have

$$\hat{L} = \hat{L}_0 + O(\varepsilon) \quad ; \quad \hat{L}_0 = -\hat{A}_{21}\hat{A}_{11}^{-1} \tag{4.143}$$

where $\varepsilon_1^* = \min\{1/[2\hat{a}_s(\hat{a}_f+\hat{a}_{12}\hat{l}_0)],\ (\hat{a}_f+\hat{a}_{12}\hat{l}_0)/[\hat{a}_s[(\hat{a}_f+\hat{a}_{12}\hat{l}_0)^2+8\hat{a}_f\hat{a}_{12}\hat{l}_0)]]\}$

$$\hat{a}_s = \|\hat{A}_{11}^{-1}\|,\ \hat{a}_f = \|\hat{A}_{22}-\hat{A}_{21}\hat{A}_{11}^{-1}\hat{A}_{12}\|,\ \hat{a}_{12} = \|\hat{A}_{12}\|,\ \hat{l}_0 = \|\hat{A}_{21}\hat{A}_{11}^{-1}\|. \tag{4.144}$$

It is noted that $\hat{A}_{11}$ is nonsingular since the two-time-scale property is satisfied in (4.141). Substituting (4.142.a) into (4.141) yields

$$\begin{cases} x_1^s(k+1,\varepsilon) = (\hat{A}_{11}-\varepsilon\hat{A}_{12}\hat{L}_0)x_1^s(k,\varepsilon) + \varepsilon\hat{A}_{12}\hat{x}_f(k,\varepsilon) \ ; \ x_1^s(0,\varepsilon) = x_{10} \\ \hat{x}_f(k+1,\varepsilon) = -\varepsilon(\hat{A}_{22}+\hat{L}_0\hat{A}_{12})\hat{L}_0x_1^s(k,\varepsilon) + \varepsilon(\hat{A}_{22}+\hat{L}_0\hat{A}_{12})\hat{x}_f(k,\varepsilon) \\ \qquad ; \ \hat{x}_f(0,\varepsilon) = \hat{L}_0x_{10}+x_{20}. \end{cases} \tag{4.145}$$

Here, applying the following relation:

$$A_{11}^{-1} = \hat{A}_{11}^{-1}(I-B_1G_1A_{11}^{-1}) = (I-A_{11}^{-1}B_1G_1)\hat{A}_{11}^{-1}$$

leads to

$$\begin{aligned}\hat{A}_{22}+\hat{L}_0\hat{A}_{12} &= \{A_{22}-B_2[G_1A_{11}^{-1}A_{12}+(I-G_1A_{11}^{-1}B_1)G_2]\}-(A_{21}-B_2G_1)\hat{A}_{11}^{-1}\{A_{12}- \\ &\qquad B_1[G_1A_{11}^{-1}A_{12}+(I-G_1A_{11}^{-1}B_1)G_2]\} \\ &= A_{22}-B_2G_1A_{11}^{-1}A_{12}-B_2(I-G_1A_{11}^{-1}B_1)G_2-A_{21}[A_{11}^{-1}A_{12}- \\ &\qquad \hat{A}_{11}^{-1}B_1(I-G_1A_{11}^{-1}B_1)G_2]+B_2G_1[A_{11}^{-1}A_{12}-\hat{A}_{11}^{-1}B_1(I-G_1A_{11}^{-1}B_1)G_2] \\ &= A_f-B_f(I-G_1A_{11}^{-1}B_1)G_2-B_fG_1\hat{A}_{11}^{-1}B_1(I-G_1A_{11}^{-1}B_1)G_2 \\ &= A_f-B_fG_2 \\ &= A_c.\end{aligned} \tag{4.146}$$

Furthermore, applying the following transformation:

$$\hat{x}_s(k,\varepsilon) = x_1^s(k,\varepsilon) + \varepsilon\hat{H}_0\hat{x}_f(k,\varepsilon) \quad ; \quad \hat{H}_0 = \hat{A}_{11}^{-1}\hat{A}_{12} \tag{4.147}$$

to (4.145) yields

$$\begin{cases} \hat{x}_s(k+1,\varepsilon) = (\hat{A}_{11}-\varepsilon\hat{A}_{12}\hat{L}_0-\varepsilon^2\hat{H}_0A_c\hat{L}_0)\hat{x}_s(k,\varepsilon)+ \\ \qquad \varepsilon[\hat{A}_{12}-(\hat{A}_{11}-\varepsilon\hat{A}_{12}\hat{L}_0)\hat{H}_0+\varepsilon\hat{H}_0A_c+\varepsilon^2\hat{H}_0A_c\hat{L}_0\hat{H}_0]\hat{x}_f(k,\varepsilon) \\ \hat{x}_f(k+1,\varepsilon) = -\varepsilon A_c\hat{L}_0\hat{x}_s(k,\varepsilon)+\varepsilon(A_c+\varepsilon A_c\hat{L}_0\hat{H}_0)\hat{x}_f(k,\varepsilon) \end{cases} \tag{4.148}$$

$$;\ \hat{x}_s(0,\varepsilon) = x_{10}+\varepsilon\hat{H}_0(\hat{L}_0x_{10}+x_{20}),\ \hat{x}_f(0,\varepsilon) = \hat{L}_0x_{10}+x_{20}.$$

The above system can be regarded as the singularly perturbed initial value problem discussed in Chapter 3. Thus, in the similar way to Chapter 3, the initial value problem (4.148) has asymptotic series solutions:

$$\begin{cases} \hat{x}_s(k,\varepsilon) = \hat{X}_s(k,\varepsilon) + \varepsilon^{k+1}\hat{m}_s(k,\varepsilon) \\ \hat{x}_f(k,\varepsilon) = \hat{X}_f(k,\varepsilon) + \varepsilon^{k}\hat{m}_f(k,\varepsilon) \end{cases} \tag{4.149}$$

where the leading terms $(X_{s0}(k),X_{f0}(k))$ satisfy

$$\begin{cases} \hat{X}_{s0}(k+1) = \hat{A}_{11}\hat{X}_{s0}(k) = (A_{11}-B_1G_1)\hat{X}_{s0}(k) \quad ; \quad \hat{X}_{s0}(0) = x_{10} \\ \hat{X}_{f0}(k) = 0 \end{cases} \tag{4.150}$$

and the boundary layer correction terms $(\hat{m}_{s0}(k),\hat{m}_{f0}(k))$ satisfy

$$\begin{cases} \hat{m}_{f0}(k+1) = A_c\hat{m}_{f0}(k) \quad ; \quad \hat{m}_{f0}(0) = \hat{L}_0x_{10}+x_{20} \\ \hat{m}_{s0}(k) = 0. \end{cases} \tag{4.151}$$

Here, we consider the responses of the slow subproblem (4.132) or (4.133.a):

$$\begin{cases} X_{10}(k+1) = [A_r-B_r(\hat{R}+B_r'K_rB_r)^{-1}B_r'K_rA_r]X_{10}(k) \\ \qquad = (A_{11}-B_1G_1)X_{10}(k) \quad ; \quad X_{10}(0) = x_{10} \qquad (4.152.a) \\ X_{20}(k) = (A_{21}-B_2G_1)(A_{11}-B_1G_1)^{-1}X_{10}(k) \qquad (4.152.b) \end{cases}$$

$$= -\hat{L}_0X_{10}(k) \qquad (4.152.b)$$

where $K_r = A_r'K_rA_r - A_r'K_rB_r(\hat{R}+B_r'K_rB_r)^{-1}B_r'K_rA_r+\hat{Q}$.

The solution of the boundary layer initial value problem is given by

$$m_{20}(k+1) = A_c m_{20}(k) \quad ; \quad m_{20}(0) = x_{20}-X_{20}(0) = \hat{L}_0 x_{10}+x_{20}. \tag{4.153}$$

By comparing (4.150),(4.151) with (4.152),(4.153), we obtain the following relations:

$$\hat{X}_{s0}(k) = X_{10}(k), \quad \hat{m}_{f0}(k) = m_{20}(k). \tag{4.154}$$

In this way, the suboptimal responses of (4.141) are approximated by the responses of the slow and fast subproblems as follows:

$$\begin{cases} x_1(k,\varepsilon) = X_{10}(k) + O(\varepsilon) & (4.155.a) \\ x_2(k,\varepsilon) = X_{20}(k) + \varepsilon^k m_{20}(k) + O(\varepsilon) & \end{cases}$$

$$= -\hat{L}_0 X_{10}(k) + \varepsilon^k m_{20}(k) + O(\varepsilon). \tag{4.155.b}$$

(e) Asymptotic stability

By using the results of Lemma 4-2, the asymptotic stability of the suboptimal controlled system (4.141) is investigated.

Under the conditions (H-1)-(H-4), the slow subproblem (4.132) is asymptotically stable. Accordingly, we have

(C-1) $|\lambda(\hat{A}_{11})| < 1$.

On the other hand, from (4.146), we have

$$\hat{A}_f = \hat{A}_{22} - \hat{A}_{21}\hat{A}_{11}^{-1}\hat{A}_{12} = A_c. \tag{4.156}$$

Accordingly, under the condition (H-5) and (H-6), we have

(C-2) $|\lambda(\varepsilon\hat{A}_f)| < 1$.

Here, we use the results of Lemma 4-2. Then, it can be shown that under the conditions (C-1) and (C-2), i.e., (H-1)-(H-6), the suboptimally controlled system (4.141) is asymptotically stable for every $\varepsilon\in[0,\varepsilon_2^*)$,

$$\text{where}\quad \varepsilon_2^* = \frac{1}{\hat{\theta}_f\|\hat{A}_f\| + \dfrac{\|\hat{A}_{12}\|\,\|\hat{L}_0\|}{(\hat{\theta}_s^{-1} - \|\hat{A}_{11}\|)}} \tag{4.157}$$

$$;\quad \hat{\theta}_s = [\lambda_{max}(\hat{P}_s)/\lambda_{min}(\hat{Q}_s)+\lambda_{max}(\hat{P}_s)\|\hat{A}_{11}\|^2]^{1/2},\ \hat{\theta}_f = [\lambda_{max}(\hat{P}_f)/\lambda_{min}(\hat{Q}_f)]^{1/2}$$

$$\hat{A}_{11}'\hat{P}_s\hat{A}_{11} - \hat{P}_s = -\hat{Q}_s,\qquad \tilde{A}_c'\hat{P}_f\tilde{A}_c - \hat{P}_f = -\hat{Q}_f.$$

4.4.3 Main results

We summarize the above results as follows:

Theorem 4-3 Let the following conditions be satisfied:

(A-1) $0 \le \varepsilon < \varepsilon^*$

where $\varepsilon^* = \min[\varepsilon_1^*, \varepsilon_2^*]$; $\varepsilon_1^* = (4.144)$, $\varepsilon_2^* = (4.157)$.

(A-2) $(\tilde{C}, \tilde{A}_{11}, B_1)$: detectable and stabilizable, where $\tilde{C}=C_1+C_2\tilde{A}_{21}\tilde{A}_{11}^{-1}$.

(A-3) $|\lambda(\tilde{A}_c)|<<1$, where $\tilde{A}_c = [I+\tilde{B}_fR^{-1}(B_1'Q_{12}+B_2'Q_{22})]^{-1}\tilde{A}_f$.

(A-4) $(I+B_1\hat{R}^{-1}\hat{C})$, $(I+\tilde{B}_fR^{-1}(B_1'Q_{12}+B_2'Q_{22}))$; nonsingular.

Then, the suboptimal feedback law is given by

$$u^s(k,\varepsilon) = -\tilde{G}_1x_1(k,\varepsilon) - [\tilde{G}_1\tilde{A}_{11}^{-1}\tilde{A}_{12}+(I-\tilde{G}_1\tilde{A}_{11}^{-1}B_1)\tilde{G}_2]x_2(k,\varepsilon) \tag{4.158}$$

$$\text{where}\quad \tilde{G}_1 = (I-\hat{R}^{-1}\hat{C}'\tilde{B}_r)(\hat{R}+\tilde{B}_r'\tilde{K}_r\tilde{B}_r)^{-1}\tilde{B}_r'\tilde{K}_r\tilde{A}_r+\hat{R}^{-1}\hat{C}'\tilde{A}_r$$

$$\tilde{G}_2 = R^{-1}(B_1'Q_{12}+B_2'Q_{22})\tilde{A}_c.$$

Furthermore, the suboptimally controlled system is asymptotically stable, and the corresponding responses satisfy

$$\begin{cases} x_1^s(k,\varepsilon) = X_{10}(k) + O(\varepsilon) \\ x_2^s(k,\varepsilon) = X_{20}(k) + m_{20}(k) + O(\varepsilon) \end{cases} \tag{4.159}$$

where
$$\begin{cases} X_{10}(k+1) = (\tilde{A}_{11}-B_1\tilde{G}_1)X_{10}(k) \\ \qquad = [\tilde{A}_r-\tilde{B}_r(\hat{R}+\tilde{B}_r'\tilde{K}_r\tilde{B}_r)^{-1}\tilde{B}_r'\tilde{K}_r\tilde{A}_r]X_{10}(k) \quad ; \quad X_{10}(0) = x_{10} \\ X_{20}(k) = (\tilde{A}_{21}-B_2\tilde{G}_1)(\tilde{A}_{11}-B_1\tilde{G}_1)^{-1}X_{10}(k) \\ \tilde{m}_{20}(k+1) = \tilde{A}_c m_{20}(k) \quad ; \quad \tilde{m}_{20}(0) = x_{20}-X_{20}(0) \\ \qquad ; \ \tilde{A}_c = [I+\tilde{B}_f R^{-1}(B_1'Q_{12}+B_2'Q_{22})]^{-1}(\tilde{A}_{22}-\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}) \\ \tilde{m}_{10}(k) = -\tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{m}_{20}(k)+\tilde{A}_{11}^{-1}B_1R^{-1}(B_1'Q_{12}+B_2'Q_{22})\tilde{m}_{20}(k+1). \end{cases}$$

[Remarks]

1. The suboptimal control (4.158) and the existence conditions can be expressed in terms of the original system matrices $\tilde{A}_{ij}$, B_i etc., i.e., they does not involve ε explicitly and does not depend on the scaling matrix used in the phase of singular perturbation modelling.
2. If we choose the state feedback gain G_s and G_f in (4.28) as $G_s = -G_1$ and $G_f = G_2$, the results lead to the suboptimal control (4.158). Accordingly, the above results correspond to the special case of the state feedback controller design studied in Section 4.2.
3. After the singular pertubation modelling is employed, the condition (H-1) is automatically satisfied because of the two-time-scale property.
4. We can prove that the matrix $\hat{Q}$ of (4.124.c) is positive-semidefinite as follows. By using the maximum rank decomposition, we set

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} = \begin{bmatrix} C_1'C_1 & C_1'C_2 \\ C_2'C_1 & C_2'C_2 \end{bmatrix} \tag{4.160.a}$$

and $\hat{C} = (Q_{12}+\tilde{A}_{11}'^{-1}\tilde{A}_{21}'Q_{22})\tilde{B}_f = \tilde{C}'(C_2\tilde{B}_f) \ ; \ \tilde{C} = C_1+C_2\tilde{A}_{21}\tilde{A}_{11}^{-1}.$ (4.160.b)

Substituting the above equation into (4.124.c) gives

$$\hat{Q} = \tilde{C}'[I-(C_2\tilde{B}_f)\hat{R}^{-1}(C_2\tilde{B}_f)']\tilde{C}. \tag{4.161}$$

Applying the inverse matrix formula yields

$$\hat{R}^{-1} = [R+(C_2\tilde{B}_f)'(C_2\tilde{B}_f)]^{-1}$$
$$= R^{-1}-R^{-1}(C_2\tilde{B}_f)'[I+(C_2\tilde{B}_f)R^{-1}(C_2\tilde{B}_f)']^{-1}(C_2\tilde{B}_f)R^{-1}. \quad (4.162)$$

By using the above expression and after algebraic manipulations, we obtain

$$\hat{Q} = \tilde{C}'[I+(C_2\tilde{B}_f)R^{-1}(C_2\tilde{B}_f)']^{-1}\tilde{C} \text{ ; positive semidefinite} \quad (4.163)$$

where $\tilde{C} = C_1+C_2\tilde{A}_{21}\tilde{A}_{11}^{-1}$.

In this way, it has been shown that the condition (H-3) is self-evident.

5. A pair (A, B) is said to be stabilizable if and only if $w'B = 0$ and $w'A = \lambda w$ for $|\lambda| \geq 1$ implies $w = 0$. Using this definition, it can be shown that the conditions (H-4) and (A-2) are equivalent, where

(H-4) $(\tilde{C}_r, \tilde{A}_r, \tilde{B}_r)$; stabilizable and detectable

(A-2) $(C_1, \tilde{A}_{11}, B_1)$: stabilizable and detectable.

In the similar way, the conditions ⌜$(C_2, \tilde{A}_f, \tilde{B}_f)$: detectable/stabilizable⌟ and ⌜$(C_2, \tilde{A}_c, \tilde{B}_c)$: detectable/stabilizable⌟ are equivalent, where $\tilde{B}_c = [I+\tilde{B}_f R^{-1}(B_1'Q_{12}+B_2'Q_{22})]^{-1}\tilde{B}_f$. Thus, (A-3) $|\lambda(\tilde{A}_c)| << 1$ is a strong condition. Furthermore, the condition (A-2) and (A-3) guarantee the stabilizability and the detectability of the full system for small ε.

6. In the special case of $\tilde{B}_f = 0$, we have

(A-3)' $|\lambda(\tilde{A}_f)| << 1$

instead of the condition (A-3). In this case, the condition (A-4) is automatically satisfied.

4.4.4 Special penalty problem

In the singularly perturbed discrete-time system (4.12), the fast mode is approximately expressed by $\lambda(\varepsilon A_{f0})$, i.e., it is treated as dead beat.

Accordingly, it is reasonable for us to select the weighting matrix Q in the performance index (4.117) as follows:

$$Q = \begin{bmatrix} Q_{11} & \varepsilon Q_{12} \\ \varepsilon Q_{12}' & \varepsilon^2 Q_{22} \end{bmatrix}, \qquad (4.164)$$

The above index implies that the penalty to the fast variable x_2 is $O(\varepsilon)$ in comparison with one to the slow variable x_1.
In this case, we obtain the following results.

Corollary 4-1 Let the following conditions (A-1)-(A-3) be satisfied:

(A-1) $0 \le \varepsilon < \min[\varepsilon_1^*, \varepsilon_2^*]$

where $\varepsilon_1^* = (4.144)$, $\varepsilon_2^* = (4.157)$

$$\hat{A}_{11} = A_{11} - B_1 L_1, \quad \hat{A}_{12} = A_{12} - B_1 L_1 A_{11}^{-1} A_{12}$$

$$\hat{A}_{21} = A_{21} - B_2 L_1, \quad \hat{A}_{22} = A_{22} - B_2 L_1 A_{11}^{-1} A_{12}$$

$$\hat{A}_{22} - \hat{A}_{21} \hat{A}_{11}^{-1} \hat{A}_{12} = \hat{A}_f = A_{22} - A_{21} A_{11}^{-1} A_{12}.$$

(A-2) $(C_1, \tilde{A}_{11}, B_1)$: detectable/stabilizable, where $Q_{11} = C_1' C_1$.

(A-3) $|\lambda(\tilde{A}_f)| \ll 1$, where $\tilde{A}_f = \tilde{A}_{22} - \tilde{A}_{21} \tilde{A}_{11}^{-1} \tilde{A}_{12}$.

Then, the suboptimal control is given by

$$u^s(k,\varepsilon) = -\tilde{L}_1 x_1(k,\varepsilon) - \tilde{L}_1 \tilde{A}_{11}^{-1} \tilde{A}_{12} x_2(k,\varepsilon) \qquad (4.165)$$

where $\tilde{L}_1 = (R + B_1' \tilde{K}_{11} B_1)^{-1} B_1' \tilde{K}_{11} \tilde{A}_{11}^{-1}$.

Furthermore, the suboptimally controlled system is asymptotically stable, and the corresponding suboptimal responses are approximated by

$$\begin{cases} x_1(k,\varepsilon) = X_{10}(k) + O(\varepsilon) \\ x_2(k,\varepsilon) = X_{20}(k) + \tilde{m}_{20}(k) + O(\varepsilon) \end{cases} \qquad (4.166)$$

where $X_{i0}(k)$ and $\tilde{m}_{i0}(k)$ represent the responses of the following slow and the fast subproblems, respectively.

[Slow subproblem]

$$X_{10}(k+1) = \tilde{A}_{11} X_{10}(k) + B_1 U_0(k) \quad ; \quad X_{10}(0) = x_{10} \qquad (4.167.a)$$

$$J = 1/2\sum_{k=0}^{\infty}[X_{10}'(k)Q_{11}X_{10}(k) + U_0'(k)RU_0(k)]. \tag{4.167.b}$$

The slow control part $U_0(k)$ is given by

$$U_0(k) = -(R+B_1'\widetilde{K}_{11}B_1)^{-1}B_1'\widetilde{K}_{11}\widetilde{A}_{11}X_{10}(k) \tag{4.168}$$

where $\widetilde{K}_{11} = \widetilde{A}_{11}'\widetilde{K}_{11}\widetilde{A}_{11}-\widetilde{A}_{11}'\widetilde{K}_{11}B_1(R+B_1'\widetilde{K}_{11}B_1)^{-1}B_1'\widetilde{K}_{11}\widetilde{A}_{11}+Q_{11}$

and the corresponding responses satisfy

$$\begin{cases} X_{10}(k+1) = (\widetilde{A}_{11} - B_1\widetilde{L}_1) \quad ; \quad X_{10}(0) = x_{10} \\ X_{20}(k) = (\widetilde{A}_{21} - B_2\widetilde{L}_1)(\widetilde{A}_{11} - B_1\widetilde{L}_1)^{-1}X_{10}(k). \end{cases} \tag{4.169}$$

[Fast subproblem]

$$\begin{cases} \widetilde{m}_{20}(k+1) = \widetilde{A}_f\widetilde{m}_{20}(k) \quad ; \quad \widetilde{m}_{20}(0) = x_{20}-X_{20}(0) \\ \widetilde{m}_{10}(k) = -\widetilde{A}_{11}^{-1}\widetilde{A}_{12}\widetilde{m}_{20}(k) \end{cases} \tag{4.170}$$

where $\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ A_{21} & \varepsilon A_{22} \end{bmatrix}.$

[Remarks]

1. If the original system (4.12) is asymptotically stable, the matrices $\widetilde{A}_{11}$ and $(\widetilde{A}_{22}-\widetilde{A}_{21}\widetilde{A}_{11}^{-1}\widetilde{A}_{12})$ are asymptotically stable for a sufficiently small ε. As any asymptotically stable system is detectable and stabilizable, the conditions (A-2) and (A-3) are automatically satisfied in this case.
2. The conditions (A-2) and (A-3) guarantee that the original system (4.12) is detectable and stabilizable for a sufficiently small ε.

4.4.5 Numerical example

We consider a point reactor model with one delayed neutron group (Weaver [We.68]). Linearizing and normalizing the kinetic equations with respect to the desired values (n_d, c_d) gives

$$\begin{cases} dx_1/dt = -\tilde{a}_1 x_1 + \tilde{a}_1 x_2 \\ dx_2/dt = \tilde{a}_2 x_1 - \tilde{a}_2 x_2 + \tilde{b}\, u \end{cases} \tag{4.171}$$

where $x_1 = \delta c/c_d$, $x_2 = \delta n/n_d$, $u = \delta k$,

$\tilde{a}_1 = \lambda$, $\tilde{a}_2 = \beta/l$, $\tilde{b} = 1/l$

and the notations follow the conventional practice.

Descretizing the above system with a sampling period T = 0.2 sec yields the following discrete model:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.991255 & 0.874479\times10\text{-}2 \\ 0.717521 & 0.282478 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.447232\times10 \\ 0.590190\times10^2 \end{bmatrix} u(k). \tag{4.172.a}$$

The performance index is

$$J = 1/2\sum_{k=0}^{\infty}[X'(k)QX(k) + u'(k)Ru(k)]. \tag{4.172.b}$$

By following the modelling procedure shown in Chapter 2, the norm conditions were calculated as follows:

(H-1) $\max[0.56994,\ 0.33491] < 1$

(H-2) $\varepsilon = \|\tilde{A}_{11}^{-1}\|\,\|\tilde{A}_f\| = 0.27858 << 1$

(H-3) $\|\tilde{A}_{11}^{-1}\|\,\|\tilde{A}_{12}\| = 0.88219\times10^{-2} << 1.$

The exact eigenvalues are shown by

$$\lambda(A) = 1.0,\ 0.2737$$

and the approximate eigenvalues are shown by

$$\begin{cases} \lambda(\tilde{A}_{11}) = 0.99125 \\ \lambda(\tilde{A}_{11} - \tilde{A}_{12}\tilde{L}_0) = 0.99758\ ;\ \tilde{L}_0 = \tilde{A}_{21}\tilde{A}_{11}^{-1} \\ \lambda(\tilde{A}_f) = 0.27615. \end{cases}$$

Accordingly, the norm conditions are all satisfied in the system (4.171).

[I] General penalty case

On the basis of Theorem 4-3, the suboptimal control and cost were caluculated as follows:

$$\begin{cases} u^s(k) = -0.1195236\times10^{-1}x_1(k) - 0.1732521\times10^{-2}x_2(k) \\ J^s = 0.8852338\times10^{-1} \end{cases} \tag{4.173}$$

and the optimal ones were

$$\begin{cases} u^o(k) = -0.1216293\times10^{-1}x_1(k) - 0.1801313\times10^{-2}x_2(k) \\ J^o = 0.8850150\times10^{-1}. \end{cases} \tag{4.174}$$

Here, the weight matrices $Q = \mathrm{diag}[1,\ 1]$, $R = 6.2\times10^3$ were used.
In order to improve the approximation, the matrix $(\tilde{A}_{11}-\tilde{A}_{12}\tilde{L}_0)$ were used in place of $\tilde{A}_{11}$.
The Riccati solution of the slow subproblem (4.132) was

$$K_r = 0.167149\times10^2 \tag{4.175}$$

and the optimal one was

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} = \begin{bmatrix} 0.1628357\times10^2 & 0.181925 \\ 0.181925 & 0.1052878\times10 \end{bmatrix}. \tag{4.176}$$

It is noted that the relation $K_{11} \approx K_r$ is satisfied, which is similar to the result of the continuous-time version (Kando et al. [Ka.81.3]).
The condition (A-1) in Theorem 4-3 is

(A-1) $0\leq\varepsilon<\min[0.4888,\ 1.0954]$

and other conditions are all satisfied. The suboptimal and optimal responses are shown in Figure 4.3.

[II] Special penalty case

On the basis of Corollary 4-1, the suboptimal control and cost were calculated as follows:

$$\begin{cases} u^s(k,\varepsilon) = -0.115480\times10^{-1}x_1(k)-0.101876\times10^{-3}x_2(k) \\ J^s = 0.839345 \end{cases} \tag{4.177}$$

and the optimal ones were

$$\begin{cases} u^o(k) = -0.121663\times10^{-1}x_1(k)-0.394873\times10^{-3}x_2(k) \\ J^o = 0.837873 \end{cases} \tag{4.178}$$

where the weight matrices $Q = \text{diag}[10,\ 1]$ and $R = 6.2\times10^4$ were used. The condition (A-1) in Corollary 4-1 are

(A-1) $0\le\varepsilon<\min[0.4888,\ 1.0954]$

and other conditions are all satisfied also. The suboptimal and optimal responses are given in Figure 4.4. Thus, the suboptimal control (4.165) yields acceptable results in case of $Q_{11}\gg Q_{12}$, Q_{22}.

On the other hand, by following Theorem 4-3, the suboptimal control (4.158) were calculated as follows:

$$\begin{cases} u^s(k) = -0.118554\times10^{-1}x_1(k)-0.341418\times10^{-3}x_2(k) \\ J^s = 0.838189. \end{cases} \tag{4.179}$$

In this way, the suboptimal control (4.158) provides us satisfactory results independently of the choice of the weight matrix Q.

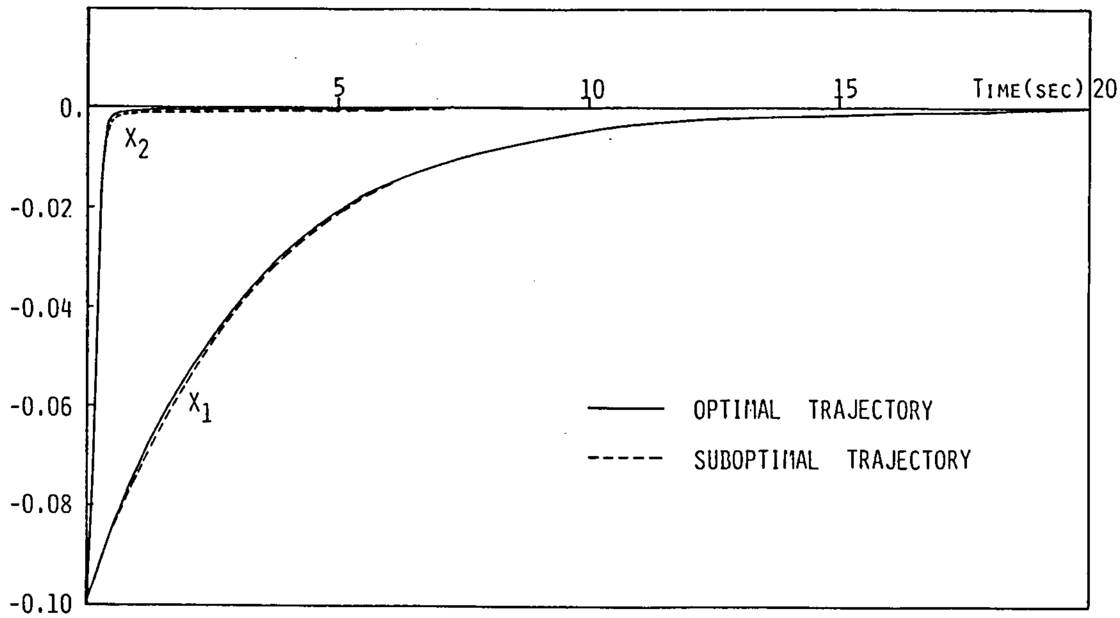

Fig. 4.3. Optimal and suboptimal responses.
(general penalty case)

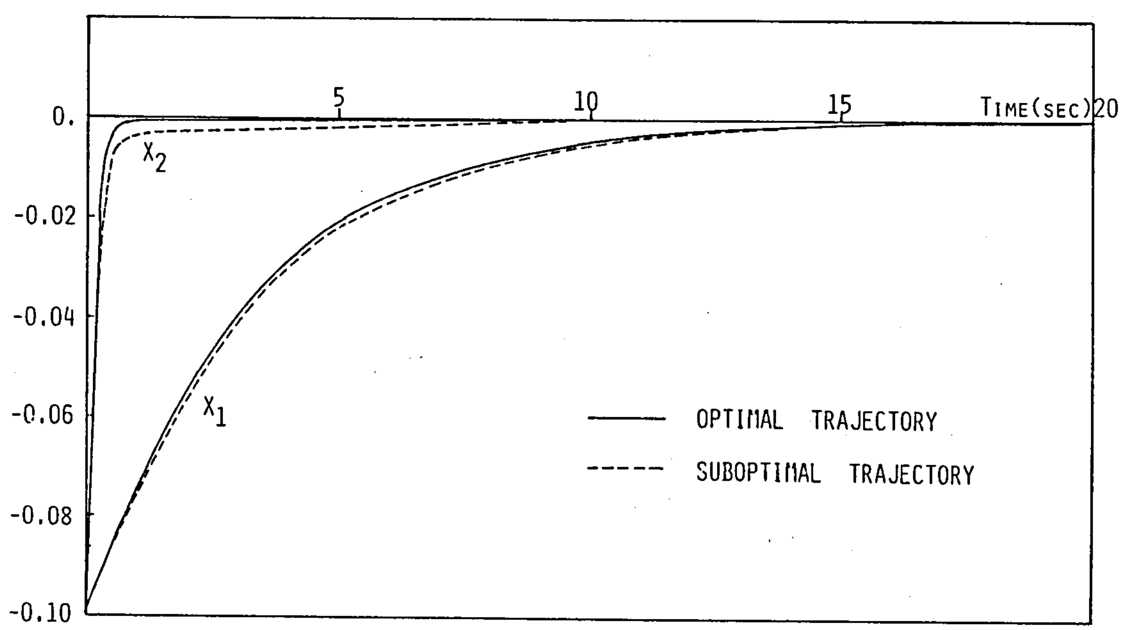

Fig. 4.4. Optimal and suboptimal responses.
(special penalty case)

4.5 Extension to Multi-Time-Scale Systems

In this section, modelling concepts for discrete-time systems with multi-time-scale property will be studied.

Now, we consider the following norm conditions:

$$\text{(H-1)}\quad \max\left\{2\tilde{a}_s^{(j)}\left(\tilde{a}_f^{(j)}+\tilde{a}_{12}^{(j)}\tilde{l}_0^{(j)}\right),\ \tilde{a}_s^{(j)}\left[\tilde{a}_f^{(j)}+\tilde{a}_{12}^{(j)}\tilde{l}_0^{(j)}+\frac{8\tilde{a}_f^{(j)}\tilde{a}_{12}^{(j)}\tilde{l}_0^{(j)}}{\left(\tilde{a}_f^{(j)}+\tilde{a}_{12}^{(j)}\tilde{l}_0^{(j)}\right)}\right]\right\}<1.$$

$$\text{(H-2)}\quad \epsilon_j = \tilde{a}_s^{(j)}\tilde{a}_f^{(j)} \ll 1.$$

$$\text{(H-3)}\quad \tilde{a}_s^{(j)}\tilde{a}_{12}^{(j)} \ll 1 \quad ; \quad j = 1,\ldots,N$$

where

$$\tilde{a}_s^{(j)} = \|(\tilde{\alpha}_{11}^{(j-1)})^{-1}\|,\quad \tilde{a}_f^{(j)} = \|\tilde{\alpha}_j\|,\quad \tilde{a}_{12}^{(j)} = \|\tilde{\alpha}_{12}^{(j-1)}\|,\quad \tilde{l}_0^{(j)} = \|\tilde{\alpha}_{21}^{(j-1)}(\tilde{\alpha}_{11}^{(j-1)})^{-1}\|$$

$$\left[\begin{array}{c|c} \tilde{\alpha}_{11}^{(0)} & \tilde{\alpha}_{12}^{(0)} \\ \hline \tilde{\alpha}_{21}^{(0)} & \tilde{\alpha}_{22}^{(0)} \end{array}\right] = \left[\begin{array}{c|ccc} \tilde{A}_{11} & \tilde{A}_{12} & \cdots & \tilde{A}_{1,N+1} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} & \cdots & \tilde{A}_{2,N+1} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{N+1,1} & \tilde{A}_{N+1,2} & \cdots & \tilde{A}_{N+1,N+1} \end{array}\right]$$

$$\tilde{\alpha}_0 = \tilde{\alpha}_{11}^{(0)} = \tilde{A}_{11}$$

$$\tilde{\alpha}_1 = \tilde{\alpha}_{22}^{(0)} - \tilde{\alpha}_{21}^{(0)}(\tilde{\alpha}_{11}^{(0)})^{-1}\tilde{\alpha}_{12}^{(0)} = \begin{bmatrix} \tilde{\alpha}_{11}^{(1)} & \tilde{\alpha}_{12}^{(1)} \\ \tilde{\alpha}_{21}^{(1)} & \tilde{\alpha}_{22}^{(1)} \end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tilde{\alpha}_j = \tilde{\alpha}_{22}^{(j-1)} - \tilde{\alpha}_{21}^{(j-1)}(\tilde{\alpha}_{11}^{(j-1)})^{-1}\tilde{\alpha}_{12}^{(j-1)} = \begin{bmatrix} \tilde{\alpha}_{11}^{(j)} & \tilde{\alpha}_{12}^{(j)} \\ \tilde{\alpha}_{21}^{(j)} & \tilde{\alpha}_{22}^{(j)} \end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tilde{\alpha}_N = \tilde{\alpha}_{22}^{(N-1)} - \tilde{\alpha}_{21}^{(N-1)}(\tilde{\alpha}_{11}^{(N-1)})^{-1}\tilde{\alpha}_{12}^{(N-1)}.$$

Then, similarly as the case of two-time-scale systems, it can be proved that the original discrete-time system possesses a multi-time-scale property and the eigenvalues are approximated by

$$\lambda(\tilde{A}) = \lambda(\tilde{\alpha}_{11}^{(0)}) \cup \lambda(\tilde{\alpha}_{11}^{(1)}) \cup \ldots \cup \lambda(\tilde{\alpha}_{11}^{(N-1)}) \cup \lambda(\tilde{\alpha}_N)[1 + O(\epsilon_M)] \quad (4.180)$$

$$;\ \epsilon_M = \max[\epsilon_1,\ldots,\epsilon_N].$$

Thus, by applying the modelling algorithm developed in Section 2.3 repeatedly, the multi-time-scale system can be identified.

[Example]

We consider the single-machine power system (3.39). After the state variables were grouped into $n_1 = 3$ and $n_2 = 4$, the above norm conditions were calculated as follows:

(H-1) $\max[0.0923,\ 0.0600]<1$

(H-2) $\varepsilon_1 = 0.0018 \ll 1$

(H-3) $\tilde{a}_s^{(1)}\tilde{a}_{12}^{(1)} = 0.0448 \ll 1$

where $\tilde{a}_s^{(1)} = 0.1268\times 10,\quad \tilde{a}_f^{(1)} = 0.1420\times 10^{-2}$
$\tilde{a}_{12}^{(1)} = 0.3533\times 10^{-1},\quad \tilde{l}_0^{(1)} = 0.9914.$

Furthermore, grouping the fast state x_2 into $n_2 = 2$ and $n_3 = 2$ yields

(H-1) $\max[0.4464,\ 0.4443]<1$

(H-2) $\varepsilon_2 = 0.0323 \ll 1$

(H-3) $\tilde{a}_s^{(2)}\tilde{a}_{12}^{(2)} = 0.1737 \ll 1$

where $\tilde{a}_s^{(2)} = 0.8832\times 10^{4},\quad \tilde{a}_f^{(2)} = 0.3659\times 10^{-5}$
$\tilde{a}_{12}^{(2)} = 0.1967\times 10^{-4},\quad \tilde{l}_0^{(2)} = 1.0989.$

The exact eigenvalues are shown by

$$\lambda(\tilde{A}) = \{0.98688,\ 0.89465+j0.54256\times 10^{-1},\ 0.13135\times 10^{-2},\ 0.11056\times 10^{-3},\ -0.85215\times 10^{-6},\ -0.45937\times 10^{-7}\} \tag{4.181}$$

and the approximate eigenvalues are shown by

$$\begin{cases} \lambda(\tilde{\alpha}_{11}^{(0)}) = \lambda(\tilde{A}_{11}) = \{0.95544+j0.45969\times 10^{-1},\ 0.83761\} \\ \lambda(\tilde{A}_{11}-\tilde{A}_{12}\tilde{L}_0) = \{0.98687,\ 0.89464+j0.54249\times 10^{-1}\} \\ \lambda(\tilde{\alpha}_{11}^{(0)}) = \{0.13243\times 10^{-2},\ 0.11436\times 10^{-3}\} \\ \lambda(\tilde{\alpha}_2) = \{-0.87625\times 10^{-6},\ -0.44316\times 10^{-7}\}. \end{cases} \tag{4.182}$$

Thus, this model can be regarded as a three-time-scale system.

On the basis of such modelling concepts, the boundary layer method for multi-time-scale systems can be formulated, and by using its method, the feedback controller design discussed in this chapter can be extended to these multi-time-scale systems.

4.6 State Feedback Controller Design for Fast Sampling Model

4.6.1 Design approaches of state feedback controllers

In this section, we consider the fast sampling model (4.6) given in Section 4.1, i.e.,

$$\begin{bmatrix} x_1(n+1,\varepsilon) \\ x_2(n+1,\varepsilon) \end{bmatrix} = \begin{bmatrix} I+\varepsilon A_{11}^f, \varepsilon A_{12}^f \\ A_{21}^f \;, \; A_{22}^f \end{bmatrix} \begin{bmatrix} x_1(n,\varepsilon) \\ x_2(n,\varepsilon) \end{bmatrix} + \begin{bmatrix} \varepsilon B_1^f \\ B_2^f \end{bmatrix} u(n,\varepsilon). \tag{4.6}$$

Then, by using the two stage design approach (Phillips [Ph.80]), we can obtain the state feedback controller which places all eigenvalues at desired locations as follows:

$$\begin{aligned} u(n,\varepsilon) &= [G_s(I+\varepsilon M^f L^f) + G_f(K^f(I+\varepsilon M^f L^f)+L^f)]x_1(n,\varepsilon) + \\ &\quad [G_f(I+\varepsilon K^f M^f) + \varepsilon G_s M^f]x_2(n,\varepsilon) \\ &= G_1 x_1(n,\varepsilon) + G_2 x_2(n,\varepsilon) \end{aligned} \tag{4.183}$$

where L^f is a dichotomic solution of the following nonsymmetric algebraic Ricaati equation (NARE):

$$R(L^f) = -A_{22}^f L^f + L^f(I+\varepsilon A_{11}^f) - \varepsilon L^f A_{12}^f L^f + A_{21}^f = 0 \tag{4.184}$$

and M^f and K^f satisfy the Sylvester equations:

$$R(M^f) = M^f A_f^f - (I+\varepsilon A_s^f)M^f + A_{12}^f = 0 \tag{4.185}$$

$$R(K^f) = -A_f^f K^f + K^f[I+\varepsilon(A_s^f+B_s^f G_s)] + B_f^f G_s = 0 \tag{4.186}$$

where $I + \varepsilon A_s^f = I + \varepsilon(A_{11}^f - A_{12}^f L^f)$, $A_f^f = A_{22}^f + \varepsilon L^f A_{12}^f$ (4.187.a)

$B_s^f = B_1^f + M^f B_f^f$, $B_f^f = B_2^f + \varepsilon L^f B_1^f$. (4.187.b)

[Remarks]

By using the system matrices A_{ij}^f of (4.6), the slow and fast eigenvalues of the fast sampling model are approximately given by

$$\begin{cases} \lambda(A_{slow}^f) = \lambda[I + \varepsilon(A_{11}^f - A_{12}^f L^f)] = \lambda[\exp(\varepsilon A_0) + O(\varepsilon)] \\ \lambda(A_{fast}^f) = \lambda(A_{22}^f + \varepsilon L^f A_{12}^f) = \lambda[\exp(A_{22}) + O(\varepsilon)] \,. \end{cases} \quad (4.188)$$

Furthermore, we have the following relations between the NARE (2.31.a) and (4.184), and between the Sylvester equations (2.31.b) and (4.185) of the continuous-time and discrete-time versions:

$$\begin{cases} L_0^f = L_0 \quad ; \quad L_0^f = (A_{22}^f - I)^{-1} A_{21}^f, \quad L_0 = A_{22}^{-1} A_{21} & (4.189.a) \\ M_0^f = M_0 \quad ; \quad M_0^f = -A_{12}^f (A_{22}^f - I)^{-1}, \quad M_0 = -A_{12} A_{22}^{-1} & (4.189.b) \end{cases}$$

and furthermore

$$K_0^f = K_0 \quad ; \quad K_0^f = (A_{22}^f - I)^{-1} B_2^f G_s, \quad K_0 = A_{22}^{-1} B_2 G_s \quad (4.189.c)$$

where $-A^f K^f + K^f[I + \varepsilon(A_s^f + B_s^f G_s)] + B_f^f G_s = 0$ and $K^f = K_0^f + O(\varepsilon)$.

It is noted that even if the continuous-time system (2.29) possesses the two-time scale property, i.e.,

$$\max_i |\lambda_i(A_s)| \ll \min_j |\lambda_j(A_f/\varepsilon)|$$

the fast sampling model (4.6) does not necessarily satisfy its property $\min_i |\lambda_i(A_s^f)| \gg \max_j |\lambda_j(A_f^f)|$. That is, the separation ratio of eigenvalues of discrete-time system itself is not necessarily small. However, in contrast with the fast sampling model, its property is preserved in the slow sampling model (4.1).

Here, we consider the following lemma.

Lemma 4-8 Let the following conditions be satisfied:

(A-1) $$0 \le \varepsilon < \min\left[\frac{1}{2a_f(a_s+a_{12}l_0)}, \frac{a_s+a_{12}l_0}{a_f[(a_s+a_{12}l_0)^2+8a_s a_{12} l_0]}\right] = \varepsilon_1^*$$

where $a_s = \|A_{11}^f - A_{12}^f L_0^f\|$, $a_{12} = \|A_{12}^f\|$, $a_f = \|(A_{22}^f - I)^{-1}\|$

$l_0 = \|(A_{22}^f - I)^{-1} A_{21}^f\|$.

(A-2) $\lambda(A_f^f) \cap \lambda[I + \varepsilon(A_s^f + B_s^f G_s)] = \emptyset$.

Then, there exists a unique dichotomic solution L^f of the NARE (4.184) that possesses the magnitude:

$$0 \le \|L^f\| \le [l_0 + \frac{2a_s l_0}{a_s + a_{12} l_0}] \tag{4.190}$$

and satisfies the two-time-scale separation property:

$$\max_i |\lambda_i(A_{fast}^f)| < \min_j |\lambda_j(A_{slow}^f)| \tag{4.191}$$

where
$$\begin{cases} \lambda(A_{slow}^f) = \lambda[I + \varepsilon(A_{11}^f - A_{12}^f L^f)] = \lambda[I + \varepsilon A_s^f] \\ \lambda(A_{fast}^f) = \lambda(A_{22}^f + \varepsilon L^f A_{12}^f) = \lambda(A_f^f) \end{cases} \tag{4.192}$$

and $$L^f = L_0^f + O(\varepsilon) \quad ; \quad L_0^f = (A_{22}^f - I)^{-1} A_{21}^f. \tag{4.193}$$

Furthermore, there exist a unique solution M^f of (4.185) and a unique solution K^f of (4.186).

Proof

We consider

$$\begin{bmatrix} x_1(n+1,\varepsilon) \\ x_2(n+1,\varepsilon) \end{bmatrix} = [A^f - \alpha I] \begin{bmatrix} x_1(n,\varepsilon) \\ x_2(n,\varepsilon) \end{bmatrix} \tag{4.194}$$

where A^f represents the system matrix of the system (4.6).

The shift of A^f matrix does not change the L^f solution. It proves the rate of convergence of the computation algorithm of the L^f, i.e., the

Riccati iteration, and the rate of convergence of the Riccati iteration depends on the separation ratio of eigenvalues of coefficient matrix (Anderson et al. [An.81]). The above shift changes its separation ratio.

By setting $\alpha = 1$ in (4.194), we have the following Riccati iteration:

$$L^f_{k+1} = (A^f_{22}-I)^{-1}[\varepsilon L^f_k(A^f_{11}-A^f_{12}L^f_k)+A^f_{21}] \quad ; \quad L^f_0 = (A^f_{22}-I)^{-1}A^f_{21}.$$

Then, by following the procedure shown in Kokotovic [Ko.75], it can be shown that under the condition (A-1), the Riccati iteration L^f_k converges to a dichotomic solution L^f which satisfies (4.190)-(4.193). Under the two-time-scale separation property (4.191), there exists a unique solution M^f of (4.185). In the similar way, there exists a unique solution K^f of (4.186) under the condition (A-2), which requires that we must assign the slow eigenvalues away from the fast eigenvalues of the open-loop system. Q.E.D.

Fig. 4.5 shows design approaches of state feedback controllers for the fast sampling model, where the superscript f is omitted for brevity. In this fiure, the right procedure represents the two stage design approach and the left procedure represents the direct approach. The transformation T and $\hat{T}$ are given as

$$T = \begin{bmatrix} I \;, & -\varepsilon M^f \\ -L^f, & I+\varepsilon L^f M^f \end{bmatrix} \qquad \hat{T} = \begin{bmatrix} I, & 0 \\ K^f, & I \end{bmatrix}. \tag{4.195}$$

The transformation P which links two approaches is obtained as follows:

$$P = \begin{bmatrix} I+\varepsilon M^f K^f, & -\varepsilon M^f \\ 0, & I-\varepsilon K^f(I+\varepsilon M^f K^f)^{-1}M^f \end{bmatrix} \approx \begin{bmatrix} I, & A^f_{12}(A^f_{22}-I)^{-1} \\ 0, & I \end{bmatrix}. \tag{4.196}$$

On the other hand, the gain $\hat{L}$ used in the direct approach has the following relation with the gain L^f, M^f, and K^f used in the two stage design approach:

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} I+\varepsilon A_{11}, & \varepsilon A_{12} \\ A_{21}, & A_{22} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} \varepsilon B_1 \\ B_2 \end{bmatrix} u(n) \xrightarrow{\begin{bmatrix} x_s(n) \\ x_f(n) \end{bmatrix} = T^{-1} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}} \begin{bmatrix} x_s(n+1) \\ x_f(n+1) \end{bmatrix} = \begin{bmatrix} I+\varepsilon A_s, & 0 \\ 0, & A_f \end{bmatrix} \begin{bmatrix} x_s(n) \\ x_f(n) \end{bmatrix} + \begin{bmatrix} \varepsilon B_s \\ B_f \end{bmatrix} u(n)$$

$$\Big\downarrow \; u(n) = G_s x_s(n) + u_f(n)$$

$$\begin{bmatrix} x_s(n+1) \\ x_f(n+1) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(A_s+B_sG_s), & 0 \\ B_fG_s, & A_f \end{bmatrix} \begin{bmatrix} x_s(n) \\ x_f(n) \end{bmatrix} + \begin{bmatrix} \varepsilon B_s \\ B_f \end{bmatrix} u_f(n)$$

$$\Big\downarrow \; \begin{bmatrix} x_s(n) \\ \eta(n) \end{bmatrix} = \hat{T} \begin{bmatrix} x_s(n) \\ x_f(n) \end{bmatrix}$$

$$\begin{bmatrix} x_s(n+1) \\ \eta(n+1) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(A_s+B_sG_s), & 0 \\ 0, & A_f \end{bmatrix} \begin{bmatrix} x_s(n) \\ \eta(n) \end{bmatrix} + \begin{bmatrix} \varepsilon B_s \\ B_f+\varepsilon KB_s \end{bmatrix} u_f(n)$$

$$\Big\downarrow \; u_f(n) = G_f \eta(n)$$

$$\begin{bmatrix} x_s(n+1) \\ \eta(n+1) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(A_s+B_sG_s), & \varepsilon B_sG_f \\ 0, & (A_f+B_fG_f)+\varepsilon KB_sG_f \end{bmatrix} \begin{bmatrix} x_s(n) \\ \eta(n) \end{bmatrix}$$

$$\Big\downarrow \; \begin{bmatrix} x_1(n) \\ \gamma(n) \end{bmatrix} = P \begin{bmatrix} x_s(n) \\ \eta(n) \end{bmatrix}$$

$$\begin{bmatrix} x_1(n+1) \\ \gamma(n+1) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(\hat{A}_{11}-\hat{A}_{12}\hat{L}), & \varepsilon\hat{A}_{12} \\ 0, & \hat{A}_{22}+\varepsilon\hat{L}\hat{A}_{12} \end{bmatrix} \begin{bmatrix} x_1(n) \\ \gamma(n) \end{bmatrix}$$

$$\Big\downarrow \; u(n) = G_1x_1(n)+G_2x_2(n), \quad \begin{cases} G_1 = G_s+G_f(L+K)+\varepsilon(G_s+G_fK)ML \\ G_2 = G_f+\varepsilon(G_s+G_fK)M \end{cases}$$

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(A_{11}+B_1G_1), & \varepsilon(A_{12}+B_1G_2) \\ A_{21}+B_2G_1, & A_{22}+B_2G_2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} \xrightarrow{\begin{bmatrix} x_s(n) \\ \eta(n) \end{bmatrix} = \hat{T}T^{-1} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}}$$

$$= \begin{bmatrix} I+\varepsilon\hat{A}_{11}, & \varepsilon\hat{A}_{12} \\ \hat{A}_{21}, & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} \xrightarrow{\begin{bmatrix} x_1(n) \\ \gamma(n) \end{bmatrix} = \begin{bmatrix} I, & 0 \\ \hat{L}, & I \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = P\hat{T}T^{-1} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}}$$

Fig. 4.5. State feedback controller design for the fast sampling model.

$$\hat{L} = [L^f + (I+\varepsilon L^f M^f)K^f](I+\varepsilon M^f K^f)^{-1}$$
$$= (I+\varepsilon K^f M^f)^{-1}[L^f + K^f(I+\varepsilon M^f L^f)] \tag{4.197}$$

where $\hat{L}$ is a dichotomic solution of the NARE:

$$R(\hat{L}) = -(A_{22}^f+B_2^fG_2)\hat{L}+\hat{L}[I+\varepsilon(A_{11}^f+B_1^fG_1)]-\varepsilon\hat{L}(A_{12}^f+B_1^fG_2)\hat{L}+(A_{21}^f+B_2^fG_1) = 0 \tag{4.198}$$

and satisfies the two-time-scale separation property of a closed-loop system:

$$\max|\lambda_i(\hat{A}_{fast})| < \min|\lambda_j(\hat{A}_{slow})| \tag{4.199.a}$$

$$\text{where}\begin{cases}\hat{A}_{slow} = I + \varepsilon[(A_{11}^f+B_1^fG_1)-(A_{12}^f+B_1^fG_2)\hat{L}] \\ \hat{A}_{fast} = (A_{22}^f+B_2^fG_2) + \varepsilon\hat{L}\ (A_{12}^f+B_1^fG_2).\end{cases} \tag{4.199.b}$$

Especially, from (4.197), we have the expression:

$$\hat{L}_0 = L_0^f + K_0^f \quad ; \quad L_0^f = (A_{22}^f-I)^{-1}A_{21}^f, \quad K_0^f = (A_{22}^f-I)^{-1}B_2^fG_s \tag{4.200.a}$$

that is,

$$[(A_{22}^f+B_2^fG_2)-I]^{-1}(A_{21}^f+B_2^fG_1) = (A_{22}^f-I)^{-1}(A_{21}^f+B_2^fG_s) \tag{4.200.b}$$

where $G_1 = G_s + G_f(A_{22}^f-I)^{-1}(A_{21}^f+B_2^fG_s)$

$G_2 = G_f.$

The above formula will be utilized later.

Thus, the conditions (A-1) and (A-2) guarantee the existence of a dichotomic solution $\hat{L}$.

4.6.2 Eigenvalue assignment problem

In the continous-time version, eigenvalue assignment and stabilizing feedback control problems have been investigated by numerous reserachers, i.e., Porter [Po.74],[Po.75],[Po.76],[Po.77.1], Chow et al. [Cho.76.1], Suzuki et al. [Su.76].

On the other hand, in Section 4.2, design problems for the discrete-time version have been studied from the viewpoint of slow-time-scale. In this section, similar problems will be treated from the viewpoint of fast-time-scale, which correspond to a counterpart of the results of Section 4.2. These studies for slow-time-scale and fast-time-scale versions are useful for us to investigate the multirate controller design, which will be shown in the next chapter.

By following the design procedure shown in Fig. 4.5, we obtain a state feedback controller:

$$\begin{aligned}
u(n,\varepsilon) &= u_{s0}(n) + u_{f0}(n) \\
&= G_s x_{s0}(n) + G_f \eta_0(n) \\
&= [G_s, G_f]\begin{bmatrix} I, & 0 \\ K_0^f, & I \end{bmatrix}\begin{bmatrix} I, & 0 \\ L_0^f, & I \end{bmatrix}\begin{bmatrix} x_1(n,\varepsilon) \\ x_2(n,\varepsilon) \end{bmatrix} \\
&= [G_s + G_f(A_{22}^f - I)^{-1}(A_{21}^f + B_2^f G_s)]x_1(n,\varepsilon) + G_f x_2(n,\varepsilon) \\
&= G_1 x_1(n,\varepsilon) + G_2 x_2(n,\varepsilon) \qquad (4.201)
\end{aligned}$$

where $u_{s0}(n)$ and $u_{f0}(n)$ are controls of the following slow and fast subsystems:

[Slow subsystem]

$$x_{s0}(n+1) = (I+\varepsilon A_0)x_{s0}(n) + \varepsilon B_0 u_{s0}(n).$$

[Fast subsystem]

$$\eta_0(n+1) = A_{22}^f \eta_0(n) + B_2^f u_{f0}(n).$$

[Remarks]

By following the composite control approach, we can obtain the same results as above. First, by combining both controllers of the slow subsystem (4.10) and the fast subsystem (4.11) for the fast sampling model (4.6), we have

$$u_c(n,\varepsilon) = U_0(t) + v_0(n)$$
$$= G_s X_{10}(t) + G_f m_{20}(n).$$

Second, by using the expressions (4.9), and by trnsforming the above expression into the state feedback form of the state variables $x_1(n,\varepsilon)$ and $x_2(n,\varepsilon)$, we obtain the same result as (4.201) as follows:

$$u_c(n,\varepsilon) = G_s x_1(n,\varepsilon) + G_f(x_2(n,\varepsilon) - X_{20}(t))$$
$$= G_s x_1(n,\varepsilon) + G_f[x_2(n,\varepsilon) + A_{22}^{-1}A_{21}X_{10}(t) + A_{22}^{-1}B_2G_sX_{10}(t)]$$
$$= [G_s + G_f(A_{22}^f - I)^{-1}(A_{21}^f + B_2^f G_s)]x_1(n,\varepsilon) + G_f x_2(n,\varepsilon).$$

The two stage design approach is useful to put such state feedback controller designs in a theoretical framework.

Implementing the above controller (4.201) to the system (4.6) yields a closed-loop system:

$$\begin{bmatrix} x_1^c(n+1,\varepsilon) \\ x_2^c(n+1,\varepsilon) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(A_{11}^f + B_1^f G_1), & \varepsilon(A_{12}^f + B_1^f G_2) \\ A_{21}^f + B_2^f G_1, & A_{22}^f + B_2^f G_2 \end{bmatrix} \begin{bmatrix} x_1^c(n,\varepsilon) \\ x_2^c(n,\varepsilon) \end{bmatrix}$$
$$= \begin{bmatrix} I+\varepsilon\hat{A}_{11}, & \varepsilon\hat{A}_{12} \\ \hat{A}_{21}, & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1^c(n,\varepsilon) \\ x_2^c(n,\varepsilon) \end{bmatrix}. \tag{4.202}$$

Then, we obtain the following lemma.

<u>Lemma 4-9</u> Let the following conditions be satisfied:

(A-1) $0 \le \varepsilon < \varepsilon_1^*$
where ε_1^* is defined in Lemma 4-8.

(A-2) $\lambda(A_{22}^f) \cap \lambda[I + \varepsilon(A_0 + B_0 G_s)] = \emptyset$
where $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 = B_1 - A_{12}A_{22}^{-1}B_2$ (continuous-time sense).

(A-3) (A_0, B_0) : completely controllable (continuous-time sense) and
(A_{22}^f, B_2^f) : completely controllable (discrete-time sense).

Then, the state feedback controller (4.201) assigns all eigenvalues of the system (4.6) approximately at the following locations:

$$\begin{cases} \lambda(\hat{A}_{slow}) = \lambda[I + \varepsilon(A_0+B_0G_s) + O(\varepsilon^2)] \\ \lambda(\hat{A}_{fast}) = \lambda[(A_{22}^f+B_2^fG_f) + O(\varepsilon)] \end{cases} \tag{4.203}$$

and the corresponding responses satisfy

$$x_1^c(n,\varepsilon) = X_{10}(t) + O(\varepsilon) \tag{4.204.a}$$
$$x_2^c(t,\varepsilon) = X_{20}(t) + m_{f0}(n) + O(\varepsilon) \tag{4.204.b}$$

where
$$\begin{cases} dX_{10}/dt = (A_0+B_0G_s)X_{10}(t) \quad ; \quad X_{10}(0) = x_{10} \\ X_{20}(t) = -A_{22}^{-1}(A_{21}+B_2G_s)X_{10}(t) \end{cases} \tag{4.205}$$

$$m_{f0}(n+1) = (A_{22}^f+B_2^fG_f)m_{f0}(n) \quad ; \quad m_{f0}(0) = x_{20}+A_{22}^{-1}(A_{21}+B_2G_s)x_{10}. \tag{4.206}$$

Proof

Applying the fast transformation:

$$\begin{bmatrix} x_1^c(n,\varepsilon) \\ \gamma\ (n,\varepsilon) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{L} & I \end{bmatrix} \begin{bmatrix} x_1^c(n,\varepsilon) \\ x_2^c(n,\varepsilon) \end{bmatrix}. \tag{4.207}$$

where $\hat{L}$ satisfies the NARE:

$$R(\hat{L}) = -\hat{A}_{22}\hat{L} + \hat{L}(I+\varepsilon\hat{A}_{11}) - \varepsilon\hat{L}\hat{A}_{12}\hat{L} + \hat{A}_{21} = 0 \tag{4.208}$$

into eqn. (4.202) yields

$$\begin{bmatrix} x_1^c(n+1,\varepsilon) \\ \gamma\ (n+1,\varepsilon) \end{bmatrix} = \begin{bmatrix} \hat{A}_{slow}, & \varepsilon\hat{A}_{12} \\ R(\hat{L}), & \hat{A}_{fast} \end{bmatrix} \begin{bmatrix} x_1^c(n,\varepsilon) \\ \gamma\ (n,\varepsilon) \end{bmatrix} \tag{4.209}$$

where $\hat{A}_{slow} = I + \varepsilon(\hat{A}_{11}-\hat{A}_{12}\hat{L})$, $\hat{A}_{fast} = \hat{A}_{22} + \varepsilon\hat{L}\hat{A}_{12}$.

Due to the implicit function theorem, the solution $\hat{L}(\varepsilon)$ of the NARE (4.208) is analytic in ε at $\varepsilon = 0$, i.e.,

$$\hat{L}(\varepsilon) = \sum_{i=0}^{k}\hat{l}_i\varepsilon^i + O(\varepsilon^{k+1}). \tag{4.210}$$

It is easily shown (Kando et al. [Ka.83.2]) that the above approximation coincides with k-th iteration $\hat{L}_k$ of the Riccati iteration

$$\hat{L}_{k+1} = (\hat{A}_{22}-I)^{-1}[\varepsilon\hat{L}_k(\hat{A}_{11}-\hat{A}_{12}\hat{L}_k)+\hat{A}_{21}] \quad ; \quad \hat{L}_0 = (\hat{A}_{22}-I)^{-1}\hat{A}_{21}. \tag{4.211}$$

Then, there exists a positive scalar ε^* such that for $\forall\varepsilon\in[0,\varepsilon^*)$ the above iteration converges to a dichotomic solution $\hat{L}$ of (4.208) which satisfies the two-time-scale separation property

$$\max|\lambda_i(\hat{A}_{22}+\varepsilon\hat{L}\hat{A}_{12})| < \min|\lambda_j[I+\varepsilon(\hat{A}_{11}-\hat{A}_{12}\hat{L})]|.$$

Since the existence of $\hat{L}$ is guaranteed under the conditions (A-1) and (A-2) as we have shown in Section 4.6.1, we obtain from (4.197) and (4.200)

$$\begin{aligned}\hat{L}(\varepsilon) &= \hat{L}_0 + O(\varepsilon)\\ &= L_0^f + K_0^f + O(\varepsilon)\end{aligned} \tag{4.212}$$

where $\hat{L}_0 = (\hat{A}_{22}-I)^{-1}\hat{A}_{21}$

$L_0^f = (A_{22}^f-I)^{-1}A_{21}^f, \quad K_0^f = (A_{22}^f-I)^{-1}B_2^fG_s.$

Utilizing the above expression gives

$$\begin{aligned}\hat{A}_{slow} &= I + \varepsilon(\hat{A}_{11}-\hat{A}_{12}\hat{L})\\ &= I + \varepsilon[\hat{A}_{11}-\hat{A}_{12}(L_0^f+K_0^f)] + O(\varepsilon^2)\\ &= I + \varepsilon[(A_{11}^f+B_1^fG_1)-(A_{12}^f+B_1^fG_2)(A_{22}^f-I)^{-1}(A_{21}^f+B_2^fG_s)] + O(\varepsilon^2)\\ &= I + \varepsilon[(A_{11}^f-A_{12}^f(A_{22}^f-I)^{-1}A_{21}^f)+(B_1^f-A_{12}^f(A_{22}^f-I)^{-1}B_2^f)G_s] + O(\varepsilon^2)\\ &= I + \varepsilon(A_0+B_0G_s) + O(\varepsilon^2)\end{aligned} \tag{4.213.a}$$

$$\begin{aligned}\hat{A}_{fast} &= \hat{A}_{22} + \varepsilon\hat{L}\hat{A}_{12}\\ &= (A_{22}^f+B_2^fG_f) + O(\varepsilon).\end{aligned} \tag{4.213.b}$$

Secondly, we consider the transformation:

$$x_f^c(n,\varepsilon) = \hat{L}_0x_1^c(n,\varepsilon) + x_2^c(n,\varepsilon) \tag{4.214}$$

where

$$\hat{L}_0 = (\hat{A}_{22}-I)^{-1}\hat{A}_{21} = (A_{22}^f-I)^{-1}(A_{21}^f+B_2^fG_s).$$

Applying the above transformation to (4.202) leads to

$$\begin{bmatrix} x_1^c(n+1,\varepsilon) \\ x_f^c(n+1,\varepsilon) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(A_0+B_0G_s), & \varepsilon\hat{A}_{12} \\ \varepsilon\hat{L}_0(A_0+B_0G_s), & \hat{A}_{22}+\varepsilon\hat{L}_0\hat{A}_{12} \end{bmatrix}\begin{bmatrix} x_1^c(n,\varepsilon) \\ x_f^c(n,\varepsilon) \end{bmatrix} ; \begin{bmatrix} x_1(0,\varepsilon) \\ x_f(0,\varepsilon) \end{bmatrix} = \begin{bmatrix} x_{10} \\ \hat{L}_0x_{10}+x_{20} \end{bmatrix}. \tag{4.215}$$

Using the boundary layer method (Blankenship [Bl.81], Hoppensteadt et al. [Ho.77]) gives

$$\begin{cases} x_1^c(n,\varepsilon) = X_1(t,\varepsilon) + \varepsilon m_1(n,\varepsilon) \quad ; \quad t = nT_f = \varepsilon n \\ x_f^c(n,\varepsilon) = X_f(t,\varepsilon) + m_f(n,\varepsilon), \end{cases} \tag{4.216}$$

The outer-region terms $[X_1(t,\varepsilon),X_f(t,\varepsilon)]$ must satisfy eqn. (4.215). Setting $\varepsilon = 0$ leads to

$$\begin{cases} dX_{10}/dt = (A_0+B_0G_s)X_{10}(t) \quad ; \quad X_{10}(0) = x_{10} \\ X_{f0}(t) = 0. \end{cases} \tag{4.217}$$

On the other hand, by obtaining equations which the inner-region terms $[m_1(n,\varepsilon),\ m_f(n,\varepsilon)]$ satisfy and by setting $\varepsilon = 0$, we have

$$m_{f0}(n+1) = (A_{22}^f+B_2^fG_f)m_{f0}(n) \quad ; \quad m_{f0}(0) = \hat{L}_0x_{10} + x_{20}. \tag{4.218}$$

By using (4.214),(4.216),(4.217) and (4.218), the responses of the closed-loop system (4.202) can be expressed by

$$x_1^c(n,\varepsilon) = X_{10}(t) + O(\varepsilon) \tag{4.219.a}$$

$$\begin{aligned} x_2^c(n,\varepsilon) &= x_f^c(n,\varepsilon) - \hat{L}_0x_1^c(n,\varepsilon) \\ &= -\hat{L}_0X_{10}(t) + m_{f0}(n) + O(\varepsilon) \\ &= -A_{22}^{-1}(A_{21}+B_2G_s)X_{10}(t) + m_{f0}(n) + O(\varepsilon) \\ &= X_{20}(t) + m_{f0}(n) + O(\varepsilon) \end{aligned} \tag{4.219.b}$$

$$\text{where}\quad \begin{cases} dX_{10}/dt = (A_0+B_0G_s)X_{10}(t) \quad ; \quad X_{10}(0) = x_{10} \\ X_{20}(t) = -A_{22}^{-1}(A_{21}+B_2G_s)X_{10}(t) \end{cases} \tag{4.220}$$

$$m_{f0}(n+1) = (A_{22}^{f}+B_2^{f}G_f)m_{f0}(n) \quad ; \quad m_{f0}(0) = x_{20}+A_{22}^{-1}(A_{21}+B_2G_s)x_{10}. \tag{4.221}$$

Here, we consider the reduced system (4.10) and the fast subsystem (4.11). Then, the responses of (4.220) and (4.221) correspond to the responses when the slow control $U_0(t) = G_sX_{10}(t)$ and the fast control $v_0(n) = G_f m_{20}(n)$ are implemented to the systems (4.10) and (4.11).

Thus, the responses of the closed-loop system (4.202) are represented as a hybrid combination of the slow version in the continuous-time domain and the fast version in the discrete-time domain. Q.E.D.

[Remarks]

In (4.201), we will select the gains G_s and G_f as follows:

$$G_s = -\mathcal{R}_0^{-1}(B_0'\mathcal{P}_0-\mathcal{C}_0')$$

$$\text{where}\quad \mathcal{P}_0\mathcal{A}_0 + \mathcal{A}_0'\mathcal{P}_0 - \mathcal{P}_0\mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{P}_0 + \mathcal{Q}_0 = 0$$

$$G_f = -(R^f+B_2^{f\prime}P_{22}^{f}B_2^{f})^{-1}(B_2^{f\prime}P_{22}^{f}A_{22}^{f}+M_2^{f\prime})$$

$$\text{where}\quad P_{22}^{f} = Q_{22}^{f}+A_{22}^{f\prime}P_{22}^{f}A_{22}^{f}-(B_2^{f\prime}P_{22}^{f}A_{22}^{f}+M_2^{f\prime})(R^f+B_2^{f\prime}P_{22}^{f}B_2^{f})^{-1}(B_2^{f\prime}P_{22}^{f}A_{22}^{f}+M_2^{f\prime}).$$

Then, the state feedback controller (4.201) leads to the suboptimal controller of the singularly perturbed discrete-time regulator (Litkouhi et al. [Lit.84]). Thus, the results shown in this section include the results of the regulator problem.

4.6.3 Stabilizing feedback controller design

In this section, we assume that the matrix A_{22} of the continuous-time system (2.29) is a Hurwitz matrix, which is satisfied in many physical systems. Since it implies $|\lambda(A_{22}^{f})|<1$, the gain $G_f = 0$ in the controller (4.201) is an admissible choice in the stabilizing problem for the fast sampling model (4.6).

Thus, we obtain a reduced-order stabilizing controller:

$$u(n,\varepsilon) = G_s x_1(n,\varepsilon). \tag{4.222}$$

If $|\lambda(A_{22}^f)| < 1$ is not satisfied, by implementing a partial feedback $u(n,\varepsilon) = K_2 x_2(n,\varepsilon) + v(n,\varepsilon)$ to (4.6) beforehand, we have $|\lambda(A_{22}^f + B_2^f K_2)| < 1$.

Applying the controller (4.222) into (4.6) yields

$$\begin{bmatrix} x_1^c(n+1,\varepsilon) \\ x_2^c(n+1,\varepsilon) \end{bmatrix} = \begin{bmatrix} I+\varepsilon(A_{11}^f + B_1^f G_s), & \varepsilon A_{12}^f \\ A_{22}^f + B_2^f G_s, & A_{22}^f \end{bmatrix} \begin{bmatrix} x_1^c(n,\varepsilon) \\ x_2^c(n,\varepsilon) \end{bmatrix} = \begin{bmatrix} I+\varepsilon \hat{A}_{11}, & \varepsilon A_{12}^f \\ \hat{A}_{21}, & A_{22}^f \end{bmatrix} \begin{bmatrix} x_1^c(n,\varepsilon) \\ x_2^c(n,\varepsilon) \end{bmatrix}$$

$$; \ x_1^c(0,\varepsilon) = x_{10}, \quad x_2^c(0,\varepsilon) = x_{20}. \tag{4.223}$$

Then, we obtain the following theorem.

Theorem 4-4 Let the following conditions be satisfied:

(H-1) $0 \le \varepsilon < \varepsilon_1^*$.

(H-2) $\lambda(A_{22}^f) \cap \lambda[I+\varepsilon(A_0+B_0G_s)] = \emptyset$.

(H-3) (A_0, B_0) : stabilizable (continuous-time sense) and

A_{22} : Hurwitz matrix (continuous-time sense).

Then, there exist a gain matrix G_s and an upper bound ε^* such that for $\forall \varepsilon \in [0,\varepsilon^*)$, the closed-loop system (4.223) when the reduced-order controller (4.222) is implemented is asymptotically stable. Furthermore, the eigenvalues of (4.223) are approximated by

$$\begin{cases} \lambda(\hat{A}_{slow}) = \lambda[I + \varepsilon(A_0+B_0G_s) + O(\varepsilon^2)] \\ \lambda(\hat{A}_{fast}) = \lambda[A_{22}^f + O(\varepsilon)] \end{cases} \tag{4.224}$$

and the corresponding responses satisfy

$$\begin{cases} x_1^c(n,\varepsilon) = X_{10}(t) + O(\varepsilon) \quad ; \quad t = nT_f = \varepsilon n \\ x_2^c(n,\varepsilon) = -A_{22}^{-1}(A_{21}+B_2G_s)X_{10}(t) + m_{f0}(n) + O(\varepsilon) \end{cases} \tag{4.225}$$

where $dX_{10}/dt = (A_0+B_0G_s)X_{10}(t)$; $X_{10}(0) = x_{10}$ (4.226.a)

$$m_{f0}(n+1) = A^f_{22}m_{f0}(n) \quad ; \quad m_{f0}(0) = x_{20} + A^{-1}_{22}(A_{21}+B_2G_s)x_{10} . \tag{4.226.b}$$

Proof

In (4.223), we obtain

$$\begin{aligned} I + \varepsilon\hat{A}_0 &= I + \varepsilon[\hat{A}_{11}-A^f_{12}(A^f_{22}-I)^{-1}\hat{A}_{21}] \\ &= I + \varepsilon[(A^f_{11}+B^f_1G_s)-A^f_{12}(A^f_{22}-I)^{-1}(A^f_{21}+B^f_2G_s)] \\ &= I + \varepsilon(A_0+B_0G_s). \end{aligned}$$

Then, there exists a gain matrix G_s such that $|\lambda[I+\varepsilon(A_0+B_0G_s)]|<1$ under the condition (H-3).

By applying the lemma of the asymptotic stability for the fast sampling model (Kando et al. [Ka.86.1]), it can be proved that there exists an upper bound ε^* such that for $^\forall\varepsilon\in[0,\varepsilon^*)$, the closed-loop system (4.223) is asymptotically stable. The results (4.224) and (4.225) follows from Lemma 4-9 straightforwards. Q.E.D.

4.7 Multirate Digital Controller Design of Regulator Problems

It is well known that the the responses of the singularly perturbed systems exhibit slow and fast behaviors. From an intuitive point of view, the lower sampling rates can be tolerated for measurement and control of the slow state variable as compared to those of the fast state variable. This intuition leads to multirate control design naturally. So far, it has been pointed out (Sandell et al. [San.78]) that it would be useful to use the singular perturbation theory in order to lie down such multirate digital system design (Glasson [Gla.80]) in a theoretical framework.

In this section, on the basis of studies for the slow-time-scale and fast-time-scale discrete versions discussed in the preceding sections, such a multirate controller design of the regulator problem will be considered.

Firstly, it will be shown that the singularly perturbed continuous-time regulator leads to two different discrete-time versions via the slow and the fast sampling. Secondly, a multirate controller design will be developed in a framework of the decomposition-coordination principle from the temporal viewpoint, and some sufficient conditions under which such a design can be performed will be studied. Furthermore, the responses and the asymptotic stability of a closed-loop system will be investigated. In this process, the relationships between the original continuous-time regulator and the corresponding single rate/multirate regulators will be clarified.

4.7.1 Formulation of discrete-time regulators

We consider the following singularly perturbed optimal regulator.

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon\,\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{4.227.a}$$

$$J = \frac{1}{2}\int_0^{\infty} [(x_1', x_2')\; Q\; \binom{x_1}{x_2} + u' R u] dt \tag{4.227.b}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$

Q ; positive semidefinite matrix

R ; positive definite matrix.

Firstly, we shall formulate the discrete-time version of the above regulator.

In the same way as Section 2.3.2, applying the mode-decoupling transformation and discretizing with a sampling rate T yields

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} A_{11}^d(T,\varepsilon), A_{12}^d(T,\varepsilon) \\ A_{21}^d(T,\varepsilon), A_{22}^d(T,\varepsilon) \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} + \begin{bmatrix} B_1^d(T,\varepsilon) \\ B_2^d(T,\varepsilon) \end{bmatrix} u_i \tag{4.228.a}$$

where $x_{1,i} \triangleq x_1(iT,\varepsilon)$, $x_{2,i} \triangleq x_2(iT,\varepsilon)$, $u_i \triangleq u(iT,\varepsilon)$

and the matrices A^d_{ij} and B^d_i have been given already in Section 2.3.2. The performance index is given by

$$J = \frac{1}{2}\sum_{i=0}^{\infty}[(x'_{1,i},\ x'_{2,i})\ Q^d\binom{x_{1,i}}{x_{2,i}} + 2(x'_{1,i},\ x'_{2,i})\ M^d u_i + u'_i R^d u_i] \tag{4.228.b}$$

where $Q^d = \int_0^T \phi'(\tau,\varepsilon) Q\,\phi(\tau,\varepsilon)\,d\tau$, $M^d = \int_0^T \phi'(\tau,\varepsilon) Q\,\theta(\tau,\varepsilon)\,d\tau$

$$R^d = \int_0^T [R + \theta'(\tau,\varepsilon) Q\,\theta(\tau,\varepsilon)]d\tau$$

$$\phi(\tau,\varepsilon) = \begin{bmatrix} A^d_{11}(\tau,\varepsilon), & A^d_{12}(\tau,\varepsilon) \\ A^d_{21}(\tau,\varepsilon), & A^d_{22}(\tau,\varepsilon) \end{bmatrix} \qquad \theta(\tau,\varepsilon) = \begin{bmatrix} B^d_1(\tau,\varepsilon) \\ B^d_2(\tau,\varepsilon) \end{bmatrix}.$$

Since this discrete-time regulator depends on the sampling rate T critically, the selection of the fast sampling $T_f = \alpha_1\varepsilon$ and the slow sampling $T_s = \alpha_2[1/\varepsilon]T_f$ (where $[1/\varepsilon]$ is the largest integer $\le 1/\varepsilon$) leads to two different formulas, respectively.

For simplicity, by assuming $\alpha_1 = \alpha_2 = 1$, i.e., $T_f = \varepsilon$, $T_s = [1/\varepsilon]$ and neglecting $O(\varepsilon)$ errors, we obtain the following expressions, where a subscript n (k) represents the fast (slow) sampling point, and $n = k[1/\varepsilon]$.

[I] Fast sampling model

$$\begin{bmatrix} x_{1,n+1} \\ x_{2,n+1} \end{bmatrix} = \begin{bmatrix} I+\varepsilon A^f_{11}, & \varepsilon A^f_{12} \\ A^f_{21}, & A^f_{22} \end{bmatrix} \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix} + \begin{bmatrix} \varepsilon B^f_1 \\ B^f_2 \end{bmatrix} u_n \tag{4.229.a}$$

where $x_{i,n} \triangleq x_i(nT_f,\varepsilon)$, $u_n \triangleq u(nT_f,\varepsilon)$ and the matrices A^f_{ij}, B^f_i have been given already in (2.35).

$$J = \frac{\varepsilon}{2}\sum_{n=0}^{\infty}[(x'_{1,n},\ x'_{2,n})\ Q^f\binom{x_{1,n}}{x_{2,n}} + 2(x'_{1,n},\ x'_{2,n})M^f u_n + u'_n R^f u_n] \tag{4.229.b}$$

where
$$\begin{cases} Q^f = Q^f_0 + O(\varepsilon) = \int_0^1 \phi^{f\prime}_0(\tau) Q\,\phi^f_0(\tau)d\tau + O(\varepsilon) \\ M^f = M^f_0 + O(\varepsilon) = \int_0^1 \phi^{f\prime}_0(\tau) Q\,\theta^f_0(\varepsilon)d\tau + O(\varepsilon) \\ R^f = R^f_0 + O(\varepsilon) = \int_0^1 [R + \theta^{f\prime}_0(\varepsilon) Q\,\theta^f_0(\varepsilon)]d\tau + O(\varepsilon) \end{cases} \tag{4.230}$$

$$\begin{cases} \phi^f(\tau,\varepsilon) = \phi_0^f(\tau) + O(\varepsilon) = \begin{bmatrix} I & , 0 \\ (\exp(A_{22}\tau)-I)L_0, & \exp(A_{22}\tau) \end{bmatrix} + O(\varepsilon) \\ \theta^f(\tau,\varepsilon) = \theta_0^f(\tau) + O(\varepsilon) = \begin{bmatrix} 0 \\ (\exp(A_{22}\tau)-I)A_{22}^{-1}B_2 \end{bmatrix} + O(\varepsilon). \end{cases}$$

[II] Slow sampling model

We will discretize the continuous-time regulator (4.227.a) and (4.227.b) with the slow sampling rate $T_s = [1/\varepsilon]T_f \approx 1$. Then, there exist $k_f > 0$ and $\alpha_f > 0$ such that

$$\| \exp(\tfrac{A}{\varepsilon}fT_s) \| = k_f \exp(-\tfrac{\alpha}{\varepsilon}fT_s) = O(\varepsilon).$$

Setting $\exp(\frac{A}{\varepsilon}fT_s) = \varepsilon\bar{A}_f$

and neglecting $O(\varepsilon)$ errors leads to

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} A_{11}^s, \varepsilon A_{12}^s \\ A_{21}^s, \varepsilon A_{22}^s \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} B_1^s \\ B_2^s \end{bmatrix} u_k \tag{4.231.a}$$

where $x_{i,k} \triangleq x_i(kT_s,\varepsilon)$, $u_k \triangleq u(kT_s,\varepsilon)$ and the matrices A_{ij}^s, B_i^s have been given already in (2.37).

$$J = \frac{1}{2}\sum_{k=0}^{\infty}[(x_{1,k}', x_{2,k}')Q^s\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + 2(x_{1,k}', x_{2,k}')M^s u_k + u_k' R^s u_k] \tag{4.231.b}$$

where

$$\begin{cases} Q^s = \begin{bmatrix} Q_{11}^s, & \varepsilon Q_{12}^s \\ \varepsilon Q_{12}^{s\prime}, & \varepsilon^2 Q_{22}^s \end{bmatrix} = \int_0^1 \phi_0^{s\prime}(\tau) Q \phi_0^s(\tau)d\tau, \quad M^s = \begin{bmatrix} M_1 \\ \varepsilon M_2 \end{bmatrix} = \int_0^1 \phi_0^{s\prime}(\varepsilon) Q \theta_0^s(\varepsilon)d\tau \\ R^s = \int_0^1 [R + \theta_0^{s\prime}(\tau) Q \theta_0^s(\tau)]d\tau \\ \phi_0^s(\varepsilon) = \begin{bmatrix} \exp(A_0\tau), & \varepsilon\exp(A_0\tau)M_0 \\ -L_0\exp(A_0\tau), & -\varepsilon L_0\exp(A_0\tau)M_0 + \exp(\frac{A}{\varepsilon}22\tau) \end{bmatrix} \\ \theta_0^s(\tau) = \begin{bmatrix} (\exp(A_0\tau)-I)A_0^{-1}B_0 \\ -L_0(\exp(A_0\tau)-I)A_0^{-1}B_0 - A_{22}^{-1}B_2 \end{bmatrix}. \end{cases} \tag{4.232}$$

Thus, the penalty to the fast variable x_2 is $O(\varepsilon)$ in comparison with one to the slow variable x_1.

4.7.2 Multirate controller design

After a global problem is decomposed into some local subproblems defined at respective time-scales, they are solved independently. Then, they are combined together so as to approximate the global solution asymptotically.

On the basis of such the decomposition-coordination principle from the temporal viewpoint, a design procedure of the multirate digital controller is developed and some sufficient conditions are studied in this section.

(a) Decomposition into slow and fast subsystems

[I] Fast sampling regulator

Firstly, we consider the fast sampling regulator. Similarly as Section 4.6, by using the boundary layer method (Blankenship [Bl.81], Hoppensteadt et al. [Ho.77]), and after considerable algebraic manipulations, the discrete-time regulator (4.229.a),(4.229.b) can be decomposed into the following subproblems:

[Slow subproblem]

$$\begin{cases} dX_{10}/dt = \mathcal{A}_0 X_{10}(t) + B_0 V_0(t) & (4.233.a) \\ J = 1/2 \int_0^\infty [X_{10}'(t)\mathcal{Q}_0 X_{10}(t) + V_0'(t)\mathcal{R}_0 V_0(t)]dt & (4.233.b) \end{cases}$$

$$X_{20}(t) = -A_{22}^{-1}(A_{21}X_{10}(t)+B_0U_0(t)) \qquad (4.233.c)$$

$$U_0(t) = V_0(t) + \mathcal{R}_0^{-1}\mathcal{C}_0' X_{10}(t) \qquad (4.233.d)$$

where $\mathcal{A}_0 = A_0 + B_0\mathcal{R}_0^{-1}\mathcal{C}_0'$, $\qquad A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$

$\mathcal{C}_0 = (Q_{12}-A_{21}'A_{22}'^{-1}Q_{22})A_{22}^{-1}B_2$, $\qquad B_0 = B_1 - A_{12}A_{22}^{-1}B_2$

$\mathcal{Q}_0 = (I,-A_{21}'A_{22}'^{-1})\ Q \begin{pmatrix} I \\ -A_{22}^{-1}A_{21} \end{pmatrix} - \mathcal{C}_0\mathcal{R}_0^{-1}\mathcal{C}_0'$

$$\mathcal{R}_0 = R + B_2' A_{22}'^{-1} Q_{22} A_{22}^{-1} B_2 .$$

[Fast subproblem]

$$\begin{cases} m_{20}(n+1) = \bar{A}_{22}^f m_{20}(n) + B_2^f w_0(n) & (4.234.a) \\ J = 1/2 \sum_{n=0}^{\infty} [m_{20}'(n) \bar{Q}_{22}^f m_{20}(n) + w_0'(n) R^f w_0(n)] & (4.234.b) \end{cases}$$

$$v_0(n) = w_0(n) - R^{f-1} M_2^{f'} m_{20}(n) \qquad (4.234.c)$$

where $\bar{A}_{22}^f = A_{22}^f - B_2^f R^{f-1} M_2^{f'}$, $\bar{Q}_{22}^f = Q_{22}^f - M_2^f R^{f-1} M^{f'}$.

Then, the responses of the regulator (4.228.a), (4.228.b) at the fast sampling point n can be represented as

$$\begin{cases} x_1(nT_f, \varepsilon) = X_{10}(t) + O(\varepsilon) \quad ; \quad t = \varepsilon n = nT_f \\ x_2(nT_f, \varepsilon) = X_{20}(t) + m_{20}(n) + O(\varepsilon) \end{cases} \qquad (4.235)$$

where $[X_{10}(t), X_{20}(t)]$ $(m_{20}(n))$ represent the responses of the slow subproblem (4.233.a)-(4.233.d) (fast subproblem (4.234.a)-(4.234.c)). It is noted that the slow subsystem (4.233) corresponds to the natural reduced problem (O'Malley et al. [OM.75]) of the singularly perturbed regulator (4.227.a),(4.227.b).

[II] Slow sampling regulator

In (4.231.a) and (4.231.b), after we introduce the Hamiltonian and derive the canonical equations, we apply the boundary layer method. Then, we obtain the following subproblems:

[Slow subproblem]

$$\begin{cases} X_{10}(k+1) = \bar{A}_{11}^s X_{10}(k) + B_1^s V_0(k) & (4.236.a) \\ J = 1/2 \sum_{k=0}^{\infty} [X_{10}'(k) \bar{Q}_{11}^s X_{10}(k) + V_0'(k) R^s V_0(k)] & (4.236.b) \end{cases}$$

$$X_{20}(k) = A_{21}^s A_{11}^{s-1} X_{10}(k) + (B_2^s - A_{21}^s A_{11}^{s-1} B_1^s) U_0(k-1) \qquad (4.236.c)$$

$$U_0(k) = V_0(k) - R^{s-1} M_1^{s'} X_{10}(k) \qquad (4.236.d)$$

where $\bar{A}^s_{11} = A^s_{11} - B^s_1 R^{s-1} M^{s'}_1$, $\bar{Q}^s_{11} = Q^s_{11} - M^s_1 R^{s-1} M^{s'}_1$

$$Q^s_{11} = \int_0^1 (A^{s'}_{11}(\tau), A^{s'}_{21}(\tau))\, Q \begin{pmatrix} A^s_{11}(\tau) \\ A^s_{21}(\tau) \end{pmatrix} d\tau$$

$$M^s_{11} = \int_0^1 (A^{s'}_{11}(\tau), A^{s'}_{21}(\tau))\, Q \begin{pmatrix} B^s_1(\tau) \\ B^s_2(\tau) \end{pmatrix} d\tau$$

$$R^s = \int_0^1 [(B^{s'}_1(\tau), B^{s'}_2(\tau))\, Q \begin{pmatrix} B^s_1(\tau) \\ B^s_2(\tau) \end{pmatrix} + R] d\tau$$

$$A^s_{11}(\tau) = \exp(A_0\tau), \quad A^s_{21}(\tau) = -A^{-1}_{22}A_{21}\exp(A_0\tau)$$

$$B^s_1(\tau) = (\exp(A_0\tau)-I)A^{-1}_0 B_0, \quad B^s_2(\tau) = -A^{-1}_{22}A_{21}(\exp(A_0\tau)-I)A^{-1}_0 B_0 - A^{-1}_{22}B_2 .$$

[Fast subproblem]

$$m_{20}(k+1) = (A^s_{22} - A^s_{21}\bar{A}^{s-1}_{11}A^s_{12})m_{20}(k) \quad ; \quad m_{20}(0) = x_{20} - A^s_{21}\bar{A}^{s-1}_{11}x_{10}. \tag{4.237}$$

The above subproblems have been already defined in Section 4.4. Then, the responses of the regulator (4.231.a), (4.231.b) at the slow sampling point k can be represented as

$$\begin{cases} x_1(kT_s, \varepsilon) = X_{10}(k) + O(\varepsilon) \\ x_2(kT_s, \varepsilon) = X_{20}(k) + \varepsilon^k m_{20}(k) + O(\varepsilon). \end{cases} \tag{4.238}$$

Thus, the fast mode exhibits the dead-beat behavior.

[Remarks]

By applying the boundary layer method, the original continuous-time regulator (4.227.a),(4.227.b) can be decomposed into the following local problems.

[Slow subproblem]

$$\begin{cases} dX_{10}/dt = \mathcal{A}_0 X_{10}(t) + B_0 V_0(t) \quad ; \quad X_{10}(0) = x_{10} & (4.233.a) \\ J = 1/2\int_0^\infty [X'_{10}(t)\mathcal{Q}_0 X_{10}(t) + V'_0(t)\mathcal{R}_0 V_0(t)]dt & (4.233.b) \end{cases}$$

$$U_0(t) = V_0(t) + \mathcal{R}^{-1}\mathcal{C}'_0 X_{10}(t). \tag{4.233.c}$$

[Fast subproblem]

$$\begin{cases} dm_{20}/d\tau = A_{22}m_{20}(\tau) + B_2v_0(\tau) \quad ; \quad m_{20}(0) = x_{20} - X_{20}(0) & (4.239.a) \\ J = 1/2\int_0^{\infty}[m_{20}'(\varepsilon)Q_{22}m_{20}(\tau) + v_0'(\tau)Rv_0(\tau)]d\tau. & (4.239.b) \end{cases}$$

Then, after algebraic manipulation, it can be shown that a discretization of the above slow subproblem (4.233.a),(4.233.b) (the fast subproblem (4.239.a),(4.239.b)) with the slow sampling rate $T_s = 1$ (the fast sampling rate $T_f = \varepsilon$, i.e., $\tau_f = 1$ at τ-time-scale) leads to the slow subproblem (4.236.a)-(4.236.d) (the fast subproblem (4.234.a)-(4.234.c)).

(b) Near optimum controller design of the fast sampling regulator

In this section, we consider the fast sampling regulator (4.229.a),(4.229.b).

First, we assume the following conditions:

(A-1) (C_0, A_0, B_0) : completely controllable and observable
where $C_0 = C_1 - C_2A_{22}^{-1}A_{21}$, $C_i'C_j = Q_{ij}$.

(A-2) (C_2^f, A_{22}^f, B_2^f) : completely controllable and observable
where $C_2^{f\prime}C_2^f = Q_{22}^f$.

Under the above conditions, the control $U_0(t)$ and $v_0(n)$ of the slow subproblem (4.233.a)-(4.233.d) and the fast subproblem (4.234.a)-(4.234.c) are given by

$$U_0(t) = -\mathcal{R}_0^{-1}(B_0'\mathcal{P}_0 - \mathcal{C}_0')X_{10}(t)$$

$$= G_0X_{10}(t) \tag{4.240.a}$$

where $\quad \mathcal{P}_0\mathcal{A}_0 + \mathcal{A}_0'\mathcal{P}_0 - \mathcal{P}_0B_0\mathcal{R}_0^{-1}B_0'\mathcal{P}_0 - \mathcal{Q}_0 = 0$

$$v_0(n) = -(R^f + B_2^{f\prime}P_{22}^fB_2^f)^{-1}(B_2^{f\prime}P_{22}^fA_{22}^f + M_2^{f\prime})m_{20}(n)$$

$$= G_fm_{20}(n) \tag{4.240.b}$$

where $P_{22}^{f} = Q_{22}^{f}+A_{22}^{f'}P_{22}^{f}A_{22}^{f}-(B_2^{f'}P_{22}^{f}A_{22}^{f}+M_2^{f'})'(R^{f}+B_2^{f'}P_{22}^{f}B_2^{f})^{-1}(B_2^{f'}P_{22}^{f}A_{22}^{f}+M_2^{f'})$.

Then, by following the state feedback controller design for the fast sampling model shown in Section 4.6, the suboptimal control is given by

$$\begin{aligned} u_0^{s}(n,\varepsilon) &= U_0(t) + v_0(n) \\ &= [G_0+G_f(A_{22}^{f}-I)^{-1}(A_{21}^{f}+B_2^{f}G_0)]x_1(n,\varepsilon) + G_f x_2(n,\varepsilon). \quad (4.241) \end{aligned}$$

The responses of a closed-loop system when the above control is implemented are given by

$$\begin{cases} x_1^{s}(n,\varepsilon) = X_{10}(t) + O(\varepsilon) \quad ; \quad t = nT_f \\ x_2^{s}(n,\varepsilon) = X_{20}(t) + m_{20}(n) + O(\varepsilon) \end{cases} \quad (4.242)$$

where $[X_{10}(t),X_{20}(t)]$ $(m_{20}(n))$ represent the responses of (4.233.a)-(4.233.d) ((4.234.a)-(4.234.c)), i.e.,

$$\begin{cases} dX_{10}/dt = (A_0+B_0G_0)X_{10}(t) \quad ; \quad X_{10}(0) = x_{10} \\ X_{20}(t) = -A_{22}^{-1}(A_{21}+B_2G_0)X_{10}(t) \end{cases} \quad (4.243)$$

$$m_{20}(n+1) = (A_{22}^{f}+B_2^{f}G_f)m_{20}(n) \quad ; \quad m_{20}(0) = x_{20}+A_{22}^{-1}(A_{21}+B_2G_0)x_{10}. \quad (4.244)$$

Thus, we have the relation:

$$x_1^{o}(n,\varepsilon) - x_1^{s}(n,\varepsilon) = O(\varepsilon), \quad x_2^{o}(n,\varepsilon) - x_2^{s}(n,\varepsilon) = O(\varepsilon)$$

where $x_1^{o}(n,\varepsilon)$ and $x_2^{o}(n,\varepsilon)$ represent optimal responses.

However, as the responses of the slow subproblem exhibit the slow behavior, it seems that they need not be necessarily controlled at the fast sampling rate T_f. For this reason, we assume that the control for the slow response is kept constant over the slow interval T_s, i.e.,

$$U_0(t) = U_0(k) = \text{constant for } kT_s \le t < (k+1)T_s.$$

Here, as described in [Remarks], it is noted that under this condition, discretizing the slow subproblem (4.233.a)-(4.233.d) leads to the slow subproblem (4.236.a)-(4.236.d) of the slow sampling regulator.

(c) Control of the slow sampling regulator

Thus, under the condition (A-2), the control $v_0(n)$ of the fast subproblem (4.234.a)-(4.234.c) has been given by (4.240.b).

On the other hand, under the following condition:

(A-3) $(\bar{C}_1^s, \bar{A}_{11}^s, \bar{B}_1^s)$: completely controllable and observable
where $\bar{C}_1^{s'}\bar{C}_1^s = \bar{Q}_{11}^s$,

the control $U_0(k)$ of the slow subproblem (4.236.a)-(4.236.d) is given by

$$\begin{aligned} U_0(k) &= -(R^s + B_1^{s'}P_{11}^s B_1^s)^{-1}(B_1^{s'}P_{11}^s A_{11}^s + M_1^{s'})X_{10}(k) \\ &= G_s X_{10}(k) \end{aligned} \tag{4.245}$$

where $P_{11}^s = Q_{11}^s + A_{11}^{s'}P_{11}^s A_{11}^s - (B_1^{s'}P_{11}^s A_{11}^s + M_1^{s'})(R^s + B_1^{s'}P_{11}^s B_1^s)^{-1}(B_1^{s'}P_{11}^s A_{11}^s + M_1^{s'})$.

[Remarks]

The slow subproblem (4.236.a)-(4.236.d) (the fast subproblem (4.234.a)-(4.234.c)) corresponds to the discrete-time version of the slow subproblem (4.233.a)-(4.233.d) (the fast subproblem (4.239.a), (4.239.b)) of the original continuous-time regulator.

In case of $\lambda_i - \lambda_j \neq j\frac{2\pi\ell}{T_s}$ $(\mu_i - \mu_j \neq j\frac{2\pi\ell}{T_f})$ where λ_i, λ_j (μ_i, μ_j) are eigenvalues of A_0 (A_{22}/ε), the controllabilty and the observability of the continuous-time version are preserved in the discrete-time version.

Then, the above conditions (A-2) and (A-3) can be rewritten as

(A-2)' (C_2, A_{22}, B_2) : completely controllable and observable
where $C_2'C_2 = Q_{22}$.

(A-3)' (C_0, A_0, B_0) : completely controllable and observable
where $C_0 = C_1 - C_2 A_{22}^{-1} A_{21}$, $C_i'C_j = Q_{ij}$.

The relation $\lambda_i - \lambda_j \neq j\frac{2\pi\ell}{T_s}(\mu_i - \mu_j \neq j\frac{2\pi\ell}{T_f})$ implies that the slow eigenvalues (A_0) (the fast eigenvalues (A_{22}/ε)) of the continuous-time system (4.227.a) lie in the primary strip of s-plane, i.e., the well-damped system.

In the next section, it will be shown that a composite control of these slow and fast controls leads to a multirate control naturally.

(d) Multirate controller design

In the singularly perturbed system, $x_1(t,\varepsilon)$ and $x_2(t,\varepsilon)$ exhibit the slow behavior and the fast behavior, respectively. Thus, the slow sampling rate (T_s) can be tolerated for measurement of the slow variable $x_1(t,\varepsilon)$, i.e., $x_1(t,\varepsilon)$ can be measured at the slow rate $t = kT_s$ $(k = 1,2,..)$. On the other hand, $x_2(t,\varepsilon)$ is measured at the fast rate $t = nT_f = k[1/\varepsilon]T_f$.

In this section, such multirate measurement is assumed too.

By combining the controls of the slow and fast subproblems defined in (b), (c), and by using the expressions (4.235) and (4.238), we obtain

$$\begin{aligned} u_n^c &= U_0(k) + v_0(n) \\ &= G_s X_{10}(k) + G_f m_{20}(n) \\ &= (I+G_f A_{22}^{-1} B_2) G_s x_1(kT_s,\varepsilon) + G_f A_{22}^{-1} A_{21} x_1(nT_s,\varepsilon) + G_f x_2(nT_f,\varepsilon) \end{aligned} \tag{4.246}$$

where $u_n^c \triangleq u^c(nT_f,\varepsilon)$.

Here, the states $x_1(kT_s,\varepsilon)$ and $x_2(nT_f,\varepsilon)$ are measurable. But since the state $x_1(nT_f,\varepsilon)$ is unmeasurable between $k[1/\varepsilon]T_f \le nT_f < (k+1)[1/\varepsilon]T_f$, the above state feedback control cannot be implemented. In order to get around this difficulty, we use the estimate of $x_1(nT_f,\varepsilon)$. When the control

$$U_0(t) = U_0(k) = G_s X_{10}(k) = \text{constant for } k[1/\varepsilon]T_f \le t < (k+1)[1/\varepsilon]T_f$$

is applied to the system (4.233.a), we have

$$dX_{10}/dt = A_0 X_{10}(t) + B_0 G_s X_{10}(kT_s) \tag{4.247}$$

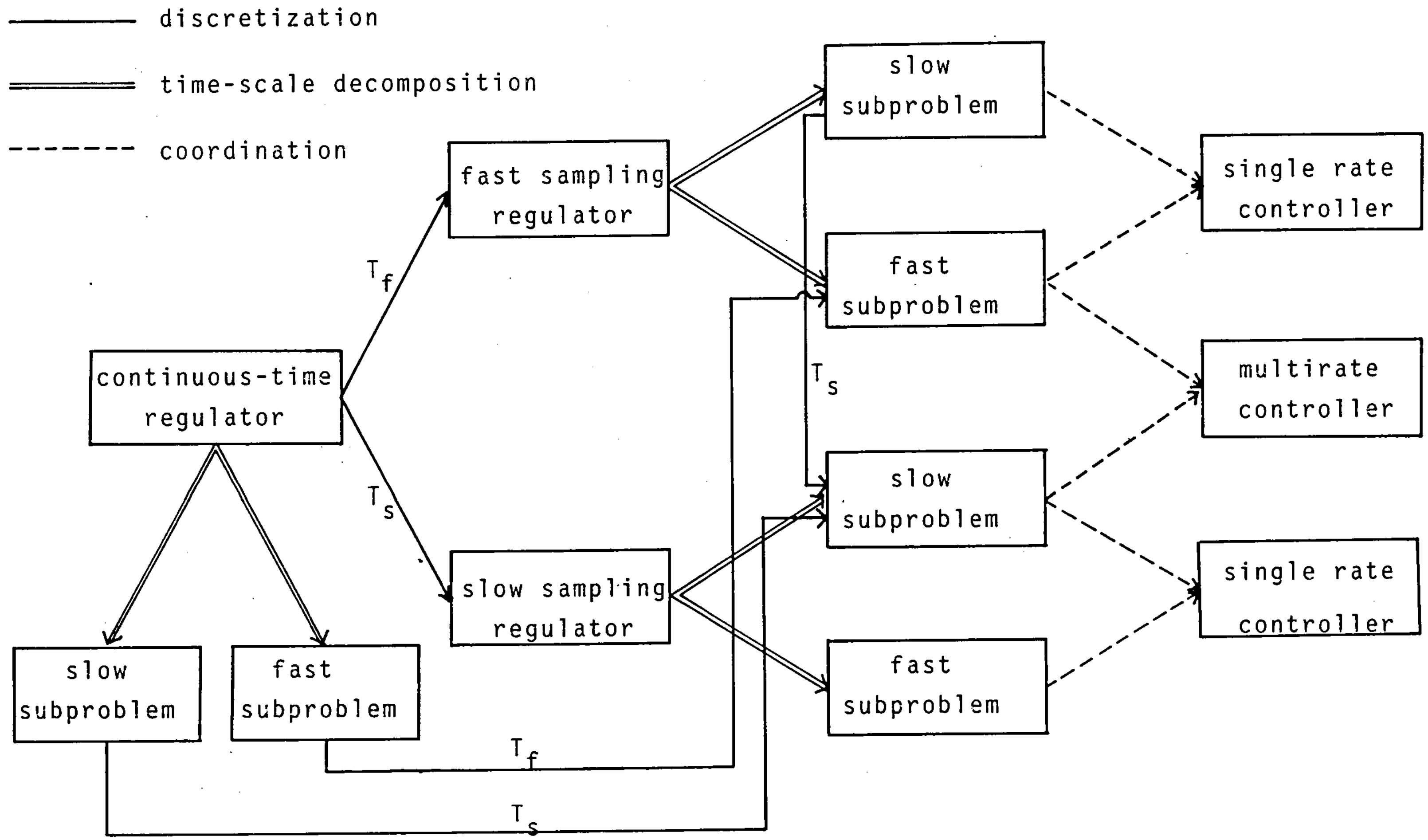

Fig. 4.6. Basic concept of the multirate controller design.

Then, we use the above response $X_{10}(t)$ as the estimate of $x_1(nT_f,\varepsilon)$, that is,

$$x_1(nT_f,\varepsilon) = X_{10}(t) + O(\varepsilon)$$
$$\approx \Phi(t)X_{10}(kT_s) \quad \text{for} \quad kT_s \le t < (k+1)T_s \tag{4.248.a}$$

where $d\Phi/dt = A_0\Phi(t) + B_0G_s \quad ; \quad \Phi(kT_s) = I.$ (4.248.b)

Or, by using the equivalent difference formula, we have

$$\Phi(n+1) = (I+\varepsilon A_0)\Phi(n) + \varepsilon B_0G_s \quad ; \quad \Phi(k[1/\varepsilon]) = I \tag{4.248.c}$$

where $\Phi(n) \triangleq \Phi(nT_f)$.

By utilizing the above estimate and after algebraic manipulation, we obtain the multirate control:

$$u_n^c = [G_s+G_fA_{22}^{-1}(A_{21}\Phi(t)+B_2G_s)]x_1(kT_s,\varepsilon) + G_fx_2(nT_f,\varepsilon)$$
$$= G_1x_1(kT_s,\varepsilon) + G_2x_2(nT_f,\varepsilon). \tag{4.249}$$

Thus, we notice that the multirate control design can be also treated within a framework of the decomposition-coordination principle which has common ground with other methods for large-scale systems, e.g., model reduction methods and hierarchical control methods (Sandell et al. [San.78]).

Fig. 4.6 shows basic concepts of such a multirate control design, where the relationships between the original continuous-time regulator and the corresponding single rate/multirate digital regulators defined up to this point are clarified.

4.7.3 Responses of a closed-loop system

When the multirate control (4.249) is implemented to the system (4.229.a), a closed-loop system is given by

$$\begin{cases} x_1^c(n+1,\varepsilon) = (I+\varepsilon A_{11}^f)x_1^c(n,\varepsilon) + \varepsilon(A_{12}^f+B_1^fG_f)x_2^c(n,\varepsilon) + \varepsilon B_1^fG_1x_1^c(kT_s,\varepsilon) \\ x_2^c(n+1,\varepsilon) = A_{21}^f x_1^c(n,\varepsilon) + (A_{22}^f+B_2^fG_f)x_2^c(n,\varepsilon) + B_2^fG_1x_1^c(kT_s,\varepsilon) \end{cases} \tag{4.250}$$

where $x_i^c(n,\varepsilon) \triangleq x_i^c(nT_f,\varepsilon)$; $x_1^c(0,\varepsilon) = x_{10}$, $x_2^c(0,\varepsilon) = x_{20}$.

Here, we have an interest in predicting the behavior of the closed-loop system (4.250) by using the responses of the slow subproblem (4.236.a)-(4.236.d) and the fast subproblem (4.234.a)-(4.234.c). By following the boundary layer method (Blankenship [Bl.81]), we set

$$\begin{cases} x_1^c(n,\varepsilon) = X_1^c(t,\varepsilon) + \varepsilon m_1^c(n,\varepsilon) \;; \quad t = nT_f \\ x_2^c(n,\varepsilon) = X_2^c(t,\varepsilon) + m_2^c(n,\varepsilon). \end{cases} \tag{4.251}$$

(a) Slow terms $[X_{10}^c(t),X_{20}^c(t)]$

The slow terms $[X_1^c(t,\varepsilon),X_2^c(t,\varepsilon)]$ must satisfy (4.250). Setting $\varepsilon = 0$ yields

$$dX_{10}^c dt = A^f X_{10}^c(t) + B^f G_1 X_{10}^c(kT_s) \tag{4.252.a}$$

$$X_{20}^c(t) = (I-A_{22}^f-B_2^fG_f)^{-1}(A_{21}^f X_{10}^c(t)+B_2^fG_1X_{10}^c(kT_s)) \tag{4.252.b}$$

where
$$A^f = A_{11}^f + (A_{12}^f+B_1^fG_f)(I-A_{22}^f-B_2^fG_f)^{-1}A_{21}^f$$
$$B^f = B_1^f + (A_{12}^f+B_1^fG_f)(I-A_{22}^f-B_2^fG_f)^{-1}B_2^f$$
$$G_1 = G_s+G_fA_{22}^{-1}(A_{21}\Phi(t)+B_2G_s) = G_s-G_f(I-A_{22}^f)^{-1}(A_{21}^f\Phi(n)+B_2^fG_s).$$

Applying the following formula (Kando et al. [Ka.86.2])

$$A^f\Phi(t) + B^fG_1 = A_0\Phi(t) + B_0G_s \tag{4.253}$$

to (4.252.a) gives

$$\begin{aligned} X_{10}^c(t) &= \{\exp(A^f(t-kT_s)+\int_{kT_s}^t \exp(A^f(t-\tau)[A_0\Phi(\tau)+B_0G_s-A^f\Phi(\tau)]d\tau\}X_{10}^c(kT_s) \\ &= \Phi(t)X_{10}^c(kT_s). \end{aligned} \tag{4.254}$$

The above response corresponds to the response of (4.247). At slow sampling points $t = kT_s$ $(k = 1,2,..)$, we have

$$X_{10}^{c}(k+1) = \Phi(t)\big|_{t=(k+1)T_s} X_{10}^{c}(k)$$

$$= [\exp(A_0 T_s) + \int_0^{T_s} \exp(A_0 \tau) d\tau B_0 G_s] X_{10}^{c}(k)$$

$$= (A_{11}^{s} + B_1^{s} G_s) X_{10}^{c}(k) \tag{4.255}$$

where $X_{10}^{c}(k) = X_{10}^{c}(kT_s)$, $A_{11}^{s} = \exp(A_0 T_s)$, $B_1^{s} = \int_0^{T_s} \exp(A_0 \tau) d\tau B_0$, which corresponds to the response of the closed-loop system of (4.236.a)-(4.236.d), i.e.,

$$X_{10}(k+1) = (A_{11}^{s} + B_1^{s} G_s) X_{10}(k). \tag{4.256}$$

On the other hand, applying (4.254) to (4.252.b) yields

$$X_{20}^{c}(t) = (I-A_{22}^{f}-B_2^{f}G_f)^{-1}\{A_{21}^{f}\Phi(t)+B_2^{f}[G_s+G_f A_{22}^{-1}(A_{21}\Phi(t)+B_2G_s)]\}X_{10}^{c}(kT_s)$$

$$= -A_{22}^{-1}(A_{21}\Phi(t)+B_2G_s)X_{10}^{c}(kT_s)$$

$$= -A_{22}^{-1}(A_{21}X_{10}^{c}(t)+B_2G_sX_{10}^{c}(kT_s)) \tag{4.257}$$

which corresponds to the response when the control $U_0(t) = G_s X_{10}(kT_s)$ is implemented to (4.233.c).

(b) Correction terms $[m_{10}^{c}(n),\ m_{20}^{c}(n)]$

From (4.250) and (4.251), we have

$$\begin{cases} m_1^{c}(n+1,\varepsilon) = (I+\varepsilon A_{11}^{f})m_1^{c}(n,\varepsilon) + \varepsilon(A_{12}^{f}+B_1^{f}G_f)m_2^{c}(n,\varepsilon) & (4.258.a) \\ m_2^{c}(n+1,\varepsilon) = A_{21}^{f}m_1^{c}(n,\varepsilon) + (A_{22}^{f}+B_2^{f}G_f)m_2^{c}(n,\varepsilon). & (4.258.b) \end{cases}$$

Setting $\varepsilon = 0$ in the above yields

$$\begin{cases} m_{10}^{c}(n+1) = m_{10}^{c}(n) + (A_{12}^{f}+B_1^{f}G_f)m_{20}^{c}(n) & (4.259.a) \\ m_{20}^{c}(n+1) = (A_{22}^{f}+B_2^{f}G_f)m_{20}^{c}(n). & (4.259.b) \end{cases}$$

It is noted that (4.259.b) corresponds to the response of the fast subsystem (4.234.a)-(4.234.c).

From (4.259.b), we obtain

$$m_{20}^{c}(n) = (A_{22}^{f}+B_2^{f}G_f)^{n-k[1/\varepsilon]}m_{20}^{c}(k[1/\varepsilon]). \tag{4.260.a}$$

At the points $n = (k+1)[1/\varepsilon]$, we obtain

$$m_{20}^c((k+1)[1/\varepsilon]) = (A_{22}^f+B_2^fG_f)^{[1/\varepsilon]}m_{20}^c(k[1/\varepsilon]). \tag{4.260.b}$$

Then, since $|\lambda(A_{22}^f+B_2^fG_f)| < 1$ under the condition (A-2), we have the relation:

$$(A_{22}^f+B_2^fG_f)^{[1/\varepsilon]} = O(\varepsilon).$$

Thus, the fast response $m_{20}^c(n)$ exhibits the dead-beat behavior between one cycle of the slow sampling period T_s.

(c) Initial condition of the correction $m_{20}^c(n)$

From the above discussions, the responses of the closed-loop system (4.250) for $kT_s \leq t < (k+1)T_s$ are approximated by

$$\begin{cases} x_1^c(t,\varepsilon) = X_{10}(t) + O(\varepsilon) & (4.261.a) \\ x_2^c(t,\varepsilon) = -A_{22}^{-1}(A_{21}X_{10}(t)+B_2G_sX_{10}(k)) + m_{20}(n) + O(\varepsilon) \\ \quad = -A_{22}^{-1}(A_{21}\Phi(t)+B_2G_s)X_{10}(k)+(A_{22}^f+B_2^fG_f)^{n-k[1/\varepsilon]}m_{20}(k[1/\varepsilon])+O(\varepsilon) & (4.261.b) \end{cases}$$

where $X_{10}(t)$ represents the response of (4.247) and $m_{20}(n)$ represents the response of the closed-loop system (4.259.b) of the fast subproblem (4.234.a)-(4.234.c), respectively. At $t = kT_s$, we have

$$\begin{cases} x_1^c(kT_s,\varepsilon) = X_{10}(k) + O(\varepsilon) & (4.262.a) \\ x_2^c(kT_s,\varepsilon) = -A_{22}^{-1}(A_{21}+B_2G_s)X_{10}(k) + m_{20}(k[1/\varepsilon]) + O(\varepsilon) & (4.262.b) \end{cases}$$

where $X_{10}(k)$ represents the response of the closed-loop system (4.256) of the slow subproblem (4.236.a)-(4.236.d).

On the other hand, when we consider the responses for $(k-1)T_s \leq t < kT_s$, we have

$$x_2^c(t,\varepsilon) = -A_{22}^{-1}(A_{21}X_{10}(t)+B_2G_sX_{10}(k-1)) + (A_{22}^f+B_2^fG_f)^{n-(k-1)[1/\varepsilon]}m_{20}((k-1)[1/\varepsilon]) + O(\varepsilon). \tag{4.263}$$

At $t = kT_s$, by taking the relation $\lambda(A_{22}^f+B_2^fG_f)^{[1/\varepsilon]}m_{20}((k-1)[1/\varepsilon]) = O(\varepsilon)$

into account, we have

$$x_2^c(kT_s,\varepsilon) = -A_{22}^{-1}(A_{21}X_{10}(k)+B_2G_sX_{10}(k-1)) + O(\varepsilon)$$
$$= A_{21}^sA_{11}^{s-1}X_{10}(k) + (B_2^s-A_{21}^sA_{11}^{s-1}B_1^s)G_sX_{10}(k-1) + O(\varepsilon)$$
$$= X_{20}(k) + O(\varepsilon) \qquad (4.264)$$

where $X_{20}(k)$ corresponds to (4.236.c).
The above results imply that the responses $[X_{10}(k),\ X_{20}(k)]$ of the slow subproblems (4.236.a)-(4.236.d) correspond to the quasi-steady-state responses after the fast mode $m_{20}(n)$ is dead-beat.

Thus, from (4.262.b) and (4.264), the initial condition at $t = kT_s$ is given by

$$m_{20}^c(k[1/\varepsilon]) = A_{22}^{-1}B_2G_s(x_1^c(kT_s,\varepsilon)-x_1^c((k-1)T_s,\varepsilon)). \qquad (4.265)$$

In the case of the single rate regulator, i.e., the fast sampling regulator (4.229.a),(4.229.b) (or the slow sampling regulator (4.231.a), (4.231.b)), the jump phenomenon between the closed-loop response $x_2^c(t,\varepsilon)$ and the slow response $X_{20}(t)$ (or $X_{20}(k)$) appears only near the initial time. It is well known that the term $m_{20}(n)$ which decays to zero in the outer-region corrects this boundary-layer jump. However, in the multirate case, this jump phenomenon shown in (4.265) appears at every slow sampling point $t = kT_s$ (k=1,2,..), which is caused by an implementation of the slow control $U_0(k) = G_sX_{10}(k)$ stepwise at this point. The fast response $m_{20}(n)$ which decays to zero in one cycle of the slow sampling corrects these jump phenomena too.
It is noted that this is a main point of difference between the single rate case and the multirate case. Fig. 4.7 shows the behavior of the responses of the multirate controlled system.

4.7.4 Asymptotic stability of a closed-loop system

Applying the fast transformation:

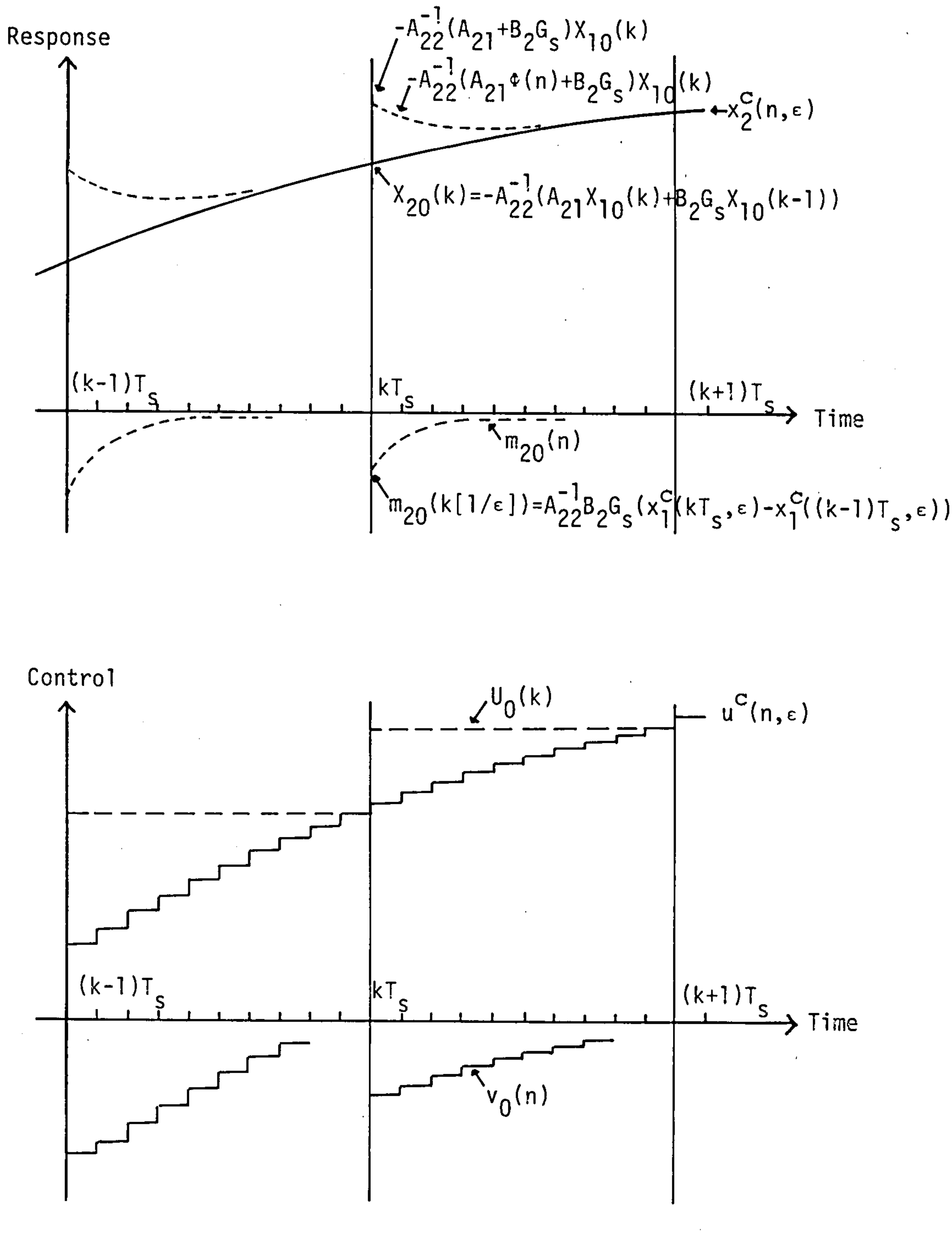

Fig. 4.7. Response of the multirate controlled system.

$$\begin{bmatrix} x_1^c(n,\varepsilon) \\ x_2^c(n,\varepsilon) \end{bmatrix} = \begin{bmatrix} I, & -\varepsilon\hat{H}_0 \\ -\hat{L}_0, & I+\varepsilon\hat{L}_0\hat{H}_0 \end{bmatrix} \begin{bmatrix} \xi(n,\varepsilon) \\ \eta(n,\varepsilon) \end{bmatrix} \quad ; \quad \begin{aligned} \hat{L}_0 &= -(I-A_{22}^f-B_2^fG_f)^{-1}A_{21}^f \\ \hat{H}_0 &= (A_{12}^f+B_1^fG_f)(I-A_{22}^f-B_2^fG_f)^{-1} \end{aligned} \tag{4.266}$$

where $\hat{L}(\varepsilon) = \hat{L}_0 + O(\varepsilon)$, $\hat{H}(\varepsilon) = \hat{H}_0 + O(\varepsilon)$ and $\hat{L}(\varepsilon)$, $\hat{H}(\varepsilon)$ satisfy

$$R(\hat{L}) = -(A_{22}^f+B_2^fG_f)\hat{L} + \hat{L}(I+\varepsilon A_{11}^f) - \varepsilon\hat{L}(A_{12}^f+B_1^fG_f)\hat{L} + A_{21}^f = 0$$

$$R(\hat{H}) = \hat{H}[(A_{22}^f+B_2^fG_f)+\varepsilon\hat{L}(A_{12}^f+B_1^fG_f)] - [I+\varepsilon(A_{11}^f-(A_{12}^f+B_1^fG_f)\hat{L}]\hat{H} + (A_{12}^fB_1^fG_f) = 0$$

into the closed-loop system (4.250) yields

$$\begin{cases} \xi(n+1,\varepsilon) = [I+\varepsilon A^f+\varepsilon^2\hat{H}_0\hat{L}_0A^f]\xi(n,\varepsilon)+\varepsilon^2[\hat{H}_0\hat{L}_0(A_{12}^f+B_1^fG_f)-A^f\hat{H}_0-\varepsilon\hat{H}_0\hat{L}_0A^f\hat{H}_0]\eta(n,\varepsilon) \\ \qquad +\varepsilon[B^f+\varepsilon\hat{H}_0\hat{L}_0B_1^f]G_1x_1^c(kT_s,\varepsilon) \qquad (4.267.a) \\ \eta(n+1,\varepsilon) = \varepsilon\hat{L}_0A^f\xi(n,\varepsilon)+[(A_{22}^f+B_2^fG_f)+\varepsilon\hat{L}_0(A_{12}^f+B_1^fG_f)-\varepsilon^2\hat{L}_0A^f\hat{H}_0]\eta(n,\varepsilon) \\ \qquad +(B_2^f+\varepsilon\hat{L}_0B_1^f)G_1x_1^c(kT_s,\varepsilon) \qquad (4.267.b) \end{cases}$$

where $A^f = A_{11}^f + (A_{12}^f+B_1^fG_f)(I-A_{22}^f-B_2^fG_f)^{-1}A_{21}^f$

$$B^f = B_1^f + (A_{12}^f+B_1^fG_f)(I-A_{22}^f-B_2^fG_f)^{-1}B_2^f$$

$$G_1(n) = G_s+G_fA_{22}^{-1}(A_{21}\Phi(n)+B_2G_s) = G_s-G_f(I-A_{22}^f)^{-1}(A_{21}^f\Phi(n)+B_2^fG_s).$$

From the above expression, we have for $k[1/\varepsilon] \le n < (k+1)[1/\varepsilon]$

$$\xi(n,\varepsilon) = [I+\varepsilon A^f+O(\varepsilon^2)]^{n-k[1/\varepsilon]}\xi(k[1/\varepsilon])+\sum_{j=k[1/\varepsilon]}^{n-1}(I+\varepsilon A^f+O(\varepsilon^2))^{n-1-j} \times \{[O(\varepsilon^2)]\eta(j,\varepsilon)+\varepsilon[B^f+O(\varepsilon)]G_1(j)x_1^c(kT_s,\varepsilon)\}.$$

Using the expression

$$x_1^c(kT_s,\varepsilon) = \xi(k[1/\varepsilon],\varepsilon) + [O(\varepsilon)]\eta(k[1/\varepsilon],\varepsilon)$$

and neglecting $O(\varepsilon^2)$ errors for simplicity yields

$$\xi(n,\varepsilon) = (I+\varepsilon A^f)^{n-k[1/\varepsilon]}\xi(k[1/\varepsilon])+\sum_{j=k[1/\varepsilon]}^{n-1}(I+\varepsilon A^f)^{n-1-j}\varepsilon B^fG_1(j)\xi(k[1/\varepsilon],\varepsilon) + [O(\varepsilon^2)]\eta(k[1/\varepsilon],\varepsilon). \tag{4.268}$$

Applying the following expression

$$A^f\Phi(n) + B^fG_1(n) = A_0\Phi(n) + B_0G_s \tag{4.253}$$

where $\Phi(n+1) = (I+\varepsilon A_0)\Phi(n)+\varepsilon B_0G_s$; $\Phi(k[1/\varepsilon]) = I$,

into (4.268) and utilizing the expression

$$\sum_{j=k[1/\varepsilon]}^{n-1}(I+\varepsilon A^f)^{n-1-j}(-A^f)\Phi(j) = [(I+\varepsilon A^f)^{n-j}\Phi(j)]_{j=k[1/\varepsilon]}^{n} - \sum_{j=k[1/\varepsilon]}^{n-1}(I+\varepsilon A^f)^{n-1-j}[\Phi(j+1)-\Phi(j)]$$

leads to

$$\xi(n,\varepsilon) = [\Phi(n)+O(\varepsilon)]\xi(k[1/\varepsilon],\varepsilon) + [O(\varepsilon^2)]\eta(k[1/\varepsilon],\varepsilon). \tag{4.269}$$

In the similar way, from (4.267.b), we have

$$\begin{aligned}\eta(n,\varepsilon) &= [(A_{22}^f+B_2^fG_f)^{n-k[1/\varepsilon]}+O(\varepsilon)]\eta(k[1/\varepsilon],\varepsilon) + \\ &\quad [\sum_{j=k[1/\varepsilon]}^{n-1}(A_{22}^f+B_2^fG_f)^{n-1-j}B_2^fG_1(j)+O(\varepsilon)]\xi(k[1/\varepsilon],\varepsilon) \\ &= [(A_{22}^f+B_2^fG_f)^{n-k[1/\varepsilon]}+O(\varepsilon)]\eta(k[1/\varepsilon],\varepsilon) + \\ &\quad \{\sum_{j=k[1/\varepsilon]}^{n-1}(A_{22}^f+B_2^fG_f)^{n-1-j}B_2^f[G_s+G_fA_{22}^{-1}(A_{21}\Phi(n)+B_2G_s)]+O(\varepsilon)\}\xi(k[1/\varepsilon],\varepsilon).\end{aligned}$$

Applying the following expression (Kando et al. [Ka.86.2])

$$\begin{aligned}\sum_{j=k[1/\varepsilon]}^{n-1}(A_{22}^f+B_2^fG_f)^{n-1-j}B_2^fG_1(j) &= (I-A_{22}^f-B_2^fG_f)^{-1}B_2^fG_1(n) \\ &\quad -(A_{22}^f+B_2^fG_f)^{n-k[1/\varepsilon]}(I-A_{22}^f-B_2^fG_f)^{-1}B_2^fG_1(k[1/\varepsilon])+O(\varepsilon)\end{aligned} \tag{4.270}$$

where $G_1(n) = G_s-G_f(I-A_{22}^f)^{-1}(A_{21}^f\Phi(n)+B_2^fG_s)$; $\Phi(k[1/\varepsilon]) = I$,

into the above equation leads to

$$\begin{aligned}\eta(n,\varepsilon) &= [(A_{22}^f+B_2^fG_f)^{n-k[1/\varepsilon]}+O(\varepsilon)]\eta(k[1/\varepsilon],\varepsilon)+[(I-A_{22}^f-B_2^fG_f)^{-1}B_2^fG_1(n) \\ &\quad -(A_{22}^f+B_2^fG_f)^{n-k[1/\varepsilon]}(I-A_{22}^f-B_2^fG_f)^{-1}B_2^fG_1(k[1/\varepsilon])+O(\varepsilon)]\xi(k[1/\varepsilon],\varepsilon).\end{aligned}$$

Furthermore, utilizing the expression

$$(I-A_{22}^f-B_2^fG_f)^{-1}(A_{21}^f+B_2^fG_1(n)) = -A_{22}^{-1}(A_{21}\Phi(n)+B_2G_s) \tag{4.271}$$

yields

$$\eta(n,\varepsilon) = \{-(I-A_{22}^{f}-B_{2}^{f}G_{f})^{-1}A_{21}^{f}\Phi(n)-A_{22}^{-1}(A_{21}\Phi(n)+B_{2}G_{s})-(A_{22}^{f}+B_{2}^{f}G_{f})^{n-k[1/\varepsilon]}\times$$

$$[-(I-A_{22}^{f}-B_{2}^{f}G_{f})^{-1}A_{21}^{f}-A_{22}^{-1}(A_{21}+B_{2}G_{s})]+O(\varepsilon)\}\xi(k[1/\varepsilon],\varepsilon) +$$

$$[(A_{22}^{f}+B_{2}^{f}G_{f})^{n-k[1/\varepsilon]}+O(\varepsilon)]\eta(k[1/\varepsilon],\varepsilon). \qquad (4.272)$$

Setting $n = (k+1)[1/\varepsilon]$ in (4.269) and (4.272), and using the relations

$$\Phi(n)\big|_{n=(k+1)[\frac{1}{\varepsilon}]} = A_{11}^{s} + B_{1}^{s}G_{s}, \quad (A_{22}^{f}+B_{2}^{f}G_{f})^{[1/\varepsilon]} = O(\varepsilon)$$

gives

$$\begin{cases} \xi((k+1)[1/\varepsilon]) = [(A_{11}^{s}+B_{1}^{s}G_{1})+O(\varepsilon)]\xi(k[1/\varepsilon]) + [O(\varepsilon^{2})]\eta(k[1/\varepsilon]) & (4.273.a) \\ \eta((k+1)[1/\varepsilon]) = \{(A_{22}+B_{2}G_{f})^{-1}A_{21}(A_{11}^{s}+B_{1}^{s}G_{1})-A_{22}^{-1}[A_{21}(A_{11}^{s}+B_{1}^{s}G_{1}) & \\ \qquad +B_{2}G_{s}]+O(\varepsilon)\}\xi(k[1/\varepsilon]) + [O(\varepsilon)]\eta(k[1/\varepsilon]) & (4.273.b) \end{cases}$$

where $A_{11}^{s} = \exp(A_0)$, $B_{1}^{s} = \int_0^1 \exp(A_0\tau)d\tau B_0$.

Then, by applying Lemma 4-2, it can be shown that there exists $\varepsilon^{*}>0$ such that for $\forall\varepsilon\in[0,\varepsilon^{*})$, (4.273) is asymptotically stable.

From eqn.(4.269) and (4.272), $\xi(n,\varepsilon)$ and $\eta(n,\varepsilon)$ is related to $\xi(k[1/\varepsilon],\varepsilon)$ and $\eta(k[1/\varepsilon],\varepsilon)$ by a nonsingular transformation.

Thus, the system (4.267), i.e., the multirate controlled system (4.250) is asymptotically stable as $n\to\infty$.

4.7.5 Main results

Summarizing the above results, we obtain the following theorem.

Theorem 4-5 Let the following conditions be satisfied:

(C-1) (C_0, A_0, B_0) : completely controllable and observable.

(C-2) (C_2, A_{22}, B_2) : completely controllable and observable.

Then, there exists $\varepsilon^{*}>0$ such that for $\forall\varepsilon\in[0,\varepsilon^{*})$ the closed-loop system (4.250) when the multirate control (4.249) is implemented is asymptotically stable and the corresponding trajectories satisfy

$$\begin{cases} x_1^c(kT_s,\varepsilon) = X_{10}(k) + O(\varepsilon) \quad ; \quad X_{10}(0) = x_{10} \\ x_2^c(nT_f,\varepsilon) = -A_{22}^{-1}(A_{21}\Phi(t)+B_2G_s)X_{10}(k)+(A_{22}+B_2G_f)^{n-k[1/\varepsilon]}m_{20}(k[1/\varepsilon],\varepsilon)+O(\varepsilon) \end{cases}$$

$$\begin{cases} m_{20}(0) = x_{20} + A_{22}^{-1}(A_{21}+B_2G_s)x_{10} \\ m_{20}(k[1/\varepsilon]) = A_{22}^{-1}B_2G_s(x_1^c(kT_s,\varepsilon)-x_1^c((k-1)T_s,\varepsilon)) \quad (k \geq 1) \end{cases}$$

$$: \quad d\Phi/dt = A_0\Phi(t) + B_0G_s \;;\; \Phi(kT_s) = I$$

$$kT_s \leq t = nT_f < (k+1)T_s \quad (T_s = [1/\varepsilon]T_f)$$

where $X_{10}(k)$ represents the response of the slow subproblem (4.236.a)-(4.236.d) and $m_{20}(n)$ represents the response of the fast subproblem (4.234.a)-(4.234.c).

4.7.6 Discussions

As discussed in Section 4.7.2, the optimal responses for the fast sampling regulator (4.229.a),(4.229.b) are represented by

$$\begin{cases} x_1^o(t,\varepsilon) = X_{10}(t) + O(\varepsilon) \quad ; \quad t = nT_f \\ x_2^o(t,\varepsilon) = X_{20}(t) + m_{20}(n) + O(\varepsilon) \end{cases} \tag{4.274}$$

where $[X_{10}(t),\ X_{20}(t)]$ $(m_{20}(n))$ satisfy (4.243) ((4.244)) which corresponds to the closed-loop system of the slow subproblem (4.233.a)-(4.233.d) (the fast subproblem (4.234.a)-(4.234.c)). Especially, at the points $t = k[1/\varepsilon]T_f$ $(k \geq 1)$, we have

$$\begin{cases} x_1^o(t,\varepsilon) = X_{10}(t) + O(\varepsilon) \quad ; \quad t = k[1/\varepsilon]T_f \\ x_2^o(t,\varepsilon) = X_{20}(t) + O(\varepsilon) \end{cases} \tag{4.275}$$

because $m_{20}(n)$ decays to zero in the above region, i.e., the outer-region.

On the other hand, from (4.262.a) and (4.264), the responses of the multirate controlled system (4.250) at the points $kT_s = k[1/\varepsilon]T_f$ ($k \geq 1$) are given by

$$\begin{cases} x_1^c(kT_s,\varepsilon) = X_{10}(k) + O(\varepsilon) \\ x_2^c(kT_s,\varepsilon) = X_{20}(k) + O(\varepsilon) \end{cases} \tag{4.276}$$

where $[X_{10}(k),\ X_{20}(k)]$ satisfy (4.256) and (4.264) which coincide with the responses of the closed-loop system of the slow subproblem (4.236.a)-(4.236.d) defined in the slow sampling regulator (4.231.a), (4.231.b).

Noting that this slow subproblem corresponds to the discrete-time version of the continuous-time slow subproblem (4.233.a)-(4.233.d) via the slow sampling T_s, we see that the errors between the responses of the multirate controlled system and them of the fast sampling regulator (4.229.a),(4.229.b) are composed of the error estimated from the viewpoint of digital redesign (Kuo [Kuo.81]), i.e., the performance deterioration caused by the discretization of the continuous-time subproblem (4.233.a)-(4.233.d) via the slow sampling rate $T_s = 1$, and the $O(\varepsilon)$ error caused by time-scale decomposition approach (see Fig. 4.6). However, since the responses of (4.233.a)-(4.233.d) exhibit the slow behavior originally, the former error is not so large even if we select the sampling rate as $T_s = 1$.

On the other hand, it can be easily shown that the differences of the responses between the fast sampling regulator (4.229.a),(4.229.b) and the original continuous-time regulator (4.227.a),(4.227.b) can be estimated as $O(\varepsilon)$ by using the boundary layer method.

Thus, we can obtain the multirate control which gives the satisfactory responses which are close to the responses of the original continuous-time regulator.

These discussions provide us a suggestion for the determination of sampling rates. That is, first, we choose $T_f = \varepsilon$ as the fast sampling rate. Second, we determine the slow sampling rate T_s so that the degree of the performance degradation between the continuous-time slow subproblem (4.233.a)-(4.233.d) and the corresponding discrete-time version (4.236.a)-(4.236.d) is kept within an acceptable range via a numerical simulation or the sampling period sensitivity approach (Kuo [Kuo.81]). It is noted that the determination of these sampling rates can be performed on the basis of reduced-order models in comparison with the global model.

4.8 Concluding Remarks

In this section, firstly, it has been shown that the singularly perturbed continuous-time system leads to two different discrete-time versions via the selection of the sampling rates, i.e., the slow-time-scale version and the fast-time-scale version, and the properties of these versions have been briefly discussed.

Secondly, a variety of the state feedback controller design problems, i.e., the stabilizing control problem, the observer design problem and the regulator problem have been studied for two time-scale versions. However, in these single rate cases, the problems such as a performance degradation between the continuous-time system and the corresponding sampled system, and the time delay due to the computation time of the control law cannot be avoidable.

In order to alleviate these difficulties and to reduce the computational labors in the controller design, the multirate digital controller design problem has been studied. The relationships between the original continuous-time problem and the corresponding two single rate sampled problems have been clarified, and then, on the basis of these results, the multirate controller design procedure has been developed from a viewpoint of the temporal decomposition-coordination principle. It has been shown that it is very useful to use the singular perturbation method to put such a multirate controller design in a theoretical framework.

CHAPTER 5 NEAR OPTIMUM CONTROLLER DESIGN OF CONTINUOUS-TIME SYSTEMS

5.1 Introduction

In the fixed-endpoint problems of singularly pertuirbed systems, the boundary layer phenomena often appear near the final time as well as the initial time. However, these phenomena are undesirable generally in the controller design of some physical systems, where the fast behaviors play an important role, e.g., a controller design of a nuclear reactor model (Kando et al. [Ka.78.2]).

In Section 5.2, a near optimum controller design of such systems is considered, which transfers a fast state variable in the steady state to a desired value with zero slope. By applying the boundary layer method, some sufficient conditions under which such a near optimum controller design can be performed are investigated, and systematic procedures to calculate them are discussed. It will become clear that the singular perturbation theory is useful to alleviate difficulties caused by the high dimensionality and the ill-conditioning appeared in two point boundary value problems (T.P.B.V.P) with special boundary conditions.

In general, large-scale systems often exhibit multi-time-scale behaviors due to the presence of physical small parastics and high gain feedback and so on. These multi-time-scale hierarchy can be identified on the use of the modelling concepts and algorithms shown in Chapter 2 repeatedly.

In Section 5.3, the fixed-endpoint problem of such multi-time-scale systems is dealt with. On the basis of the inverse Riccati approach (Mufti et al. [Muf.69], Kwon et al. [Kw.77]), near optimum feedback controller designs have been studied so far (Asatani [As.74]), but in these results the effects of boundary-layer jump phenomena at the final time have not been taken into account. These jump phenomena have been improved by considering the boundary-layer corrections (Kando et al. [Ka.78.1]), where the sufficient conditions have been compared with the earlier ones. Futhermore, the matched asymptotic series expansions (Wasow [Wa.76], Vasileva [Va.63]) have been given in order to improve the suboptimality (Rajagopalan et al. [Raj.80.2]). On the other hand, an open-loop control have been studied (Wilde et al [Wi.73]) by the use of the dichotomy transformation (Wilde et al. [Wi.72]), which corresponds to a special case of the combined transformation shown in Chapter 2.

In view of this situation, in Section 5.3, the near optimum controller designs of multi-time-scale systems are developed on the basis of the docomposition-coordination principle with respect to time-scales. A mode-decoupled open-loop and a partial feedback controllers are given, and the relationships among them are discussed. The results obtained are different from the earlier ones (Asatani [As.74], Kando et al. [Ka.78.1], Rajagopalan et al. [Raj.80.2]), where the controller gains have been mutually interconnected and have been serially computed. The computational procedure shown here will be suitable for pararell processing and will be useful for large-scale system design and analysis.

In Section 5.4, the regulator problem with the general terminal penalty of multi-time-scale systems is investigated. The special terminal penalty problems have been extensively developed so far (Kokotovic et al. [Ko.76]): e.g., the open-loop approach (O'Malley [OM.72], [OM.75.1]), the feedback approach (Yackel et al. [Ya.73], Kokotovic et al. [Ko.72.3], Chow et al. [Cho.76.2], Naidu et al. [Na.80]), and the Chandrasekhar approach (Syrcos et al. [Sy.82],[Sy.84]).

However, when the terminal penalty is represented as a general function of the fast state $x_2(T)$, the problem is more complex (Vasileva et al. [Va.78]). Such a general terminal problem has been considered via the feedback approach by Glizer et al. [Gl.75.1],[Gl.78]. It has been shown that due to the nonlinearity and the singularity of terminal values of the Riccati equation, the conventional forms of Riccati solutions, i.e.,

$$\tilde{K}_{ij}(t,\varepsilon) = \tilde{H}_{ij}(t,\varepsilon) + \tilde{P}_{ij}(\tau,\varepsilon) \quad ; \quad \tau = (t_f - t)/\varepsilon$$

cannot be applied in this case, and the boundary-layer jump phenomena appear also in the reduced Riccati solution $H_{11}^{0}(t)$. Furthermore, the special cases of the general terminal penalty problems have been treated by Glizer and Dmitriev [Gl.75.2],[Gl.76],[Dm.78].

In Section 5.4, such a general terminal penalty problem is studied on the basis of the decomposition-coordination principle from the temporal viewpoint similarly as Section 5.3. A mode-decoupled partial feedback controller is given, where a open-loop part works to correct the final jump phenomenon. In the infinite-time control case, it is shown that this controller leads to a pure feedback one.

5.2 Near Optimum Controller Design of a System Having the Terminal Condition With Zero Slope

5.2.1 Problem formulation

We consider a linear time-invariant system:

$$\begin{cases} dx_1/dt = \tilde{A}_{11}x_1 + \tilde{A}_{12}x_2 + \tilde{B}_1 u \\ dx_2/dt = \tilde{A}_{21}x_1 + \tilde{A}_{22}x_2 + \tilde{B}_2 u \end{cases} ; \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\varepsilon & A_{22}/\varepsilon \end{bmatrix}, \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2/\varepsilon \end{bmatrix}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $u \in R^r$ $(n_1 \geq n_2)$. (5.1)

The boundary conditions are

$$\begin{cases} x_1(t_0,\varepsilon) = x_{10}, & dx_2/dt(t_f,\varepsilon) = 0 \\ x_2(t_0,\varepsilon) = x_{20}, & x_2(t_f,\varepsilon) = 0. \end{cases} \tag{5.2}$$

The performance index to be minimized is

$$J = \frac{1}{2}\int_{t_0}^{t_f}[(x_1', x_2')Q\binom{x_1}{x_2} + u'Ru]dt \tag{5.3}$$

where $\quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix}$: $(n_1+n_2)\times(n_1+n_2)$ positive semidefinite matrix

R ; $r\times r$ positive definite matrix.

The aim of this problem is to transfer the fast state variable x_2 to the desired value with zero slope at the specified time t_f, e.g., a power level change problem of a nuclear reactor system (Weaver [We.68]).

By introducing the Hamiltonian and by applying the variational calculus, the following canonical equations are obtained:

$$\begin{cases} dx_1/dt = A_{11}x_1 + A_{12}x_2 - S_{11}p_1 - S_{12}p_2 \\ \varepsilon dx_2/dt = A_{21}x_1 + A_{22}x_2 - S_{12}'p_1 - S_{22}p_2 \\ dp_1/dt = -Q_{11}x_1 - Q_{12}x_2 - A_{11}'p_1 - A_{21}'p_2 \\ \varepsilon dp_2/dt = -Q_{12}'x_1 - Q_{22}x_2 - A_{12}'p_1 - A_{22}'p_2 \end{cases} \tag{5.4}$$

where $S_{11} = B_1R^{-1}B_1'$, $S_{12} = B_1R^{-1}B_2'$, $S_{22} = B_2R^{-1}B_2'$.

Then, an optimal control is given by

$$u^o(t,\varepsilon) = -R^{-1}B_1'p_1(t,\varepsilon) - R^{-1}B_2'p_2(t,\varepsilon). \tag{5.5}$$

The transversality condition is given by

$$\delta x_1'(t_f,\varepsilon)p_1(t_f,\varepsilon) = 0. \tag{5.6}$$

By taking the condition $dx_2/dt(t_f,\varepsilon) = 0$ into account, δx_1 satisfies the following constraints at the terminal time $t = t_f$:

$$A_{21}\delta x_1(t_f,\varepsilon) + B_2\delta u(t_f,\varepsilon) = 0. \tag{5.7}$$

From (5.6) and (5.7), we obtain

$$p^{(i)}(t_f,\varepsilon) = 0 \quad ; \quad i = 1,\dots,(n_1-n_2) \tag{5.8}$$

where $p^{(i)}$ represents an i-th element.

In this way, the optimal control problem (5.1)-(5.3) have led to the T.P.B.V.P. (5.4) with the conditions (5.2) and (5.8).

5.2.2 Near optimum controller design and existence conditions

Since the boundary-layer jump phenomenon does not appear near $t = t_f$ due to the special conditions (5.2), the solutions of (5.4) can be represented as follows:

$$\begin{cases} x_1(t,\varepsilon) = X_1(t,\varepsilon) + \varepsilon m_1(\tau,\varepsilon) + \varepsilon n_1(\sigma,\varepsilon) \\ x_2(t,\varepsilon) = X_2(t,\varepsilon) + m_2(\tau,\varepsilon) + \varepsilon n_2(\sigma,\varepsilon) \\ p_1(t,\varepsilon) = P_1(t,\varepsilon) + \varepsilon \rho_1(\tau,\varepsilon) + \varepsilon \nu_1(\sigma,\varepsilon) \\ p_2(t,\varepsilon) = P_2(t,\varepsilon) + \rho_2(\tau,\varepsilon) + \varepsilon \nu_2(\sigma,\varepsilon) \;;\; \tau=(t-t_0)/\varepsilon,\; \sigma=(t-t_f)/\varepsilon \end{cases} \tag{5.9}$$

where the outer-region terms $[X_i(t,\varepsilon),P_i(t,\varepsilon)]$ satisfy (5.4) and have asymptotic series expansions in ε and the initial and terminal boundary-layer correction terms $[m_i(\tau,\varepsilon),\rho_i(\tau,\varepsilon)]$ and $[n_i(\sigma,\varepsilon),\nu_i(\sigma,\varepsilon)]$ have asymptotic series expansions in ε whose terms tend to zero as $\tau\to\infty$ and $\sigma\to-\infty$, respectively.

The initial and terminal boundary layer correction terms satisfy the following equations:

$$\begin{cases} dm_1/d\tau = \varepsilon A_{11}m_1 + A_{12}m_2 - \varepsilon S_{11}\rho_1 - S_{12}\rho_2 \\ d\rho_1/d\tau = -\varepsilon Q_{11}m_1 - Q_{12}m_2 - \varepsilon A'_{11}\rho_1 - A'_{21}\rho_2 \\ dm_2/d\tau = \varepsilon A_{21}m_1 + A_{22}m_2 - \varepsilon S'_{12}\rho_1 - S_{22}\rho_2 \\ d\rho_2/d = -\varepsilon Q'_{12}m_1 - Q_{22}m_2 - \varepsilon A'_{12}\rho_1 - A'_{22}\rho_2 \quad ; \quad \tau = (t-t_0)/\varepsilon \end{cases} \tag{5.10}$$

$$\begin{cases} dn_1/d\sigma = \varepsilon A_{11}n_1 + \varepsilon A_{12}n_2 - \varepsilon S_{11}\nu_1 - \varepsilon S_{12}\nu_2 \\ d\nu_1/d\sigma = -\varepsilon Q_{11}n_1 - \varepsilon Q_{12}n_2 - \varepsilon A'_{11}\nu_1 - \varepsilon A'_{21}\nu_2 \\ dn_2/d\sigma = A_{21}n_1 + A_{22}n_2 - S'_{12}\nu_1 - S_{22}\nu_2 \\ d\nu_2/d\sigma = -Q'_{12}n_1 - Q_{22}n_2 - A'_{12}\nu_1 - A'_{22}\nu_2 \quad ; \quad \sigma = (t-t_f)/\varepsilon . \end{cases} \tag{5.11}$$

The boundary conditions are given by

$$\begin{cases} X_1(t_0,\varepsilon) + \varepsilon m_1(0,\varepsilon) = x_{10} & (5.12.a) \\ X_2(t_0,\varepsilon) + m_2(0,\varepsilon) = x_{20} & (5.12.b) \\ X_2(t_f,\varepsilon) + \varepsilon n_2(0,\varepsilon) = 0 & (5.12.c) \\ dX_2/dt(t_f,\varepsilon) + dn_2/d\sigma(0,\varepsilon) = 0 & (5.12.d) \end{cases}$$

$$P_1^{(i)}(t_f,\varepsilon) + \varepsilon \nu_1^{(i)}(0,\varepsilon) = 0 \quad ; \quad i = 1,\ldots,(n_1 - n_2). \qquad (5.12.e)$$

Now, we are interested in existence conditions under which these asymptotic series solutions exist.

(a) The outer-region terms

The outer-region terms must satisfy (5.4). Setting $\varepsilon = 0$ in (5.4) leads to

$$\begin{cases} dX_{10}/dt = A_{11}X_{10} + A_{12}X_{20} - S_{11}P_{10} - S_{12}P_2 \\ dP_{10}/dt = -Q_{11}X_{10} - Q_{12}X_{20} - A_{11}'P_{10} - A_{21}'P_2 \\ \qquad 0 = A_{21}X_{10} + A_{22}X_{20} - S_{12}'P_{10} - S_{22}P_{20} \\ \qquad 0 = -Q_{12}'X_{10} - Q_{22}X_{20} - A_{12}'P_{10} - A_{22}'P_{20}. \end{cases} \qquad (5.13)$$

Here, we assume the following conditions:

(H-1) A_{22} : Hurwitz matrix.

(H-2) (C_2, A_{22}, B_2) : stabilizable and detectable ; $Q_{22} = C_2'C_2$.

Since the Hamiltonian matrix G of dimension $2n_2 \times 2n_2$ defined by

$$G = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}' \end{bmatrix}$$

has no imaginary axis eigenvalues (Kučera [Kuc.72], Callier et al. [Ca.81]), i.e., invertible under these conditions, we obtain from (5.13),

$$\begin{bmatrix} X_{20} \\ P_{20} \end{bmatrix} = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}' \end{bmatrix}^{-1} \begin{bmatrix} -A_{21} & S_{12}' \\ Q_{12}' & A_{12}' \end{bmatrix} \begin{bmatrix} X_{10} \\ P_{10} \end{bmatrix}. \tag{5.14}$$

Applying a matrix inversion formula (Henderson et al. [He.81]) yields

$$\begin{cases} dX_{10}/dt = \mathcal{A}_0 X_{10} - \mathcal{B}_0 \mathcal{R}_0^{-1} \mathcal{B}_0' P_{10} & (5.15.a) \\ dP_{10}/\, dt = -\mathcal{Q}_0 X_{10} - \mathcal{A}_0' P_{10} & (5.15.b) \\ X_{20}(t) = -A_{22}^{-1}[A_{21} + S_{22}\theta^{-1}(Q_{12}' - Q_{22}A_{22}^{-1}A_{21})]X_{10}(t) + & \\ \qquad\qquad A_{22}^{-1}[S_{12}' - S_{22}\theta^{-1}(A_{12}' + Q_{22}A_{22}^{-1}S_{12}')]P_{10}(t) & (5.15.c) \\ P_{20}(t) = -\theta^{-1}(Q_{12}' - Q_{22}A_{22}^{-1}A_{21})X_{10}(t) - \theta^{-1}(A_{12}' + Q_{22}A_{22}^{-1}S_{12}')P_{10}(t) & \\ & (5.15.d) \end{cases}$$

where $\mathcal{A}_0 = A_0 + \mathcal{B}_0 \mathcal{R}_0^{-1} \mathcal{C}_0'$, $\qquad \mathcal{B}_0 = B_0$

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_0 = B_1 - A_{22}^{-1}A_{21}B_2$$

$$\mathcal{C}_0 = (Q_{12} - A_{21}'A_{22}'^{-1}Q_{22})A_{22}^{-1}B_2$$

$$\mathcal{R}_0 = R + B_2'A_{22}'^{-1}Q_{22}A_{22}^{-1}B_2, \quad \theta = A_{22}' + Q_{22}A_{22}^{-1}S_{22}$$

$$\mathcal{Q}_0 = [I, -A_{21}'A_{22}'^{-1}]\, Q \begin{bmatrix} I \\ -A_{22}^{-1}A_{21} \end{bmatrix} - \mathcal{C}_0 \mathcal{R}_0^{-1} \mathcal{C}_0'.$$

From (5.12.a), (5.12.b) and (5.12.e), we have

$$\begin{cases} X_{10}(t_0) = x_1 & (5.16.a) \\ X_{20}(t_f) = 0 & (5.16.b) \\ P_{10}^{(i)}(t_f) = 0 \quad ; \quad i = 1,\ldots,(n_1 - n_2). & (5.16.c) \end{cases}$$

Using (5.14) and (5.16.b) gives

$$B_2 \mathcal{R}_0^{-1} B_0' P_{10}(t_f) = (A_{21} + B_2 \mathcal{R}_0^{-1} \mathcal{C}_0') X_{10}(t_f). \tag{5.17}$$

From (5.16.c) and (5.17), we obtain

$$\Gamma\, P_{10}(t_f) = \begin{bmatrix} 0 \\ \cdots\cdots\cdots\cdots \\ A_{21} + B_2 \mathcal{R}_0^{-1} \mathcal{C}_0' \end{bmatrix} X_{10}(t_f) \tag{5.18}$$

where

$$\Gamma = \begin{bmatrix} I & 0 \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \begin{array}{c} \begin{array}{cc} \underbrace{\quad}_{n_1-n_2} & \underbrace{\quad}_{n_2} \end{array} \\ \left[\begin{array}{c|c} I & 0 \\ \hline B_2\mathcal{R}_0^{-1}B_0' & \end{array}\right] \end{array} \begin{array}{l} \}n_1-n_2 \\ \}n_2. \end{array}$$

Then, under the condition:

(H-3) Γ_{22} ; nonsingular

we have

$$P_{10}(t_f) = \begin{bmatrix} 0 \\ \Gamma_{22}^{-1}(A_{21} + B_2\mathcal{R}_0^{-1}C_0') \end{bmatrix} X_{10}(t_f)$$
$$= \Pi \cdot X_{10}(t_f). \qquad (5.19)$$

Furthermore, we assume the following conditions:

(H-4) Π : positive semidefinite matrix.

(H-5) Q_0 : positive semidefinite matrix.

(H-6) $\mathcal{R}_0$: positive definite matrix.

Here, since $\mathcal{R}_0 = R + B_2'A_{22}'^{-1}Q_{22}A_{22}^{-1}B_2$, (H-6) is self-evident. Thus, under the conditions (H-1)-(H-5), the T.P.V.B.P. (5.15) with the boundary conditions (5.16.a), (5.19) has a unique solution (Bucy [Bu.67]).

Applying the conventional Riccati transformation:

$$P_{10}(t) = K_0(t)X_{10}(t) \qquad (5.20)$$

where $dK_0/dt = -A_0K_0 - K_0A_0' + K_0B_0\mathcal{R}_0^{-1}B_0'K_0 - Q_0 \quad ; \quad K_0(t_f) = \Pi$ (5.21)

yields

$$dX_{10}/dt = (A_0 - B_0\mathcal{R}_0^{-1}B_0'K_0)X_{10}(t) \quad ; \quad X_{10}(t_0) = x_{10}. \qquad (5.22)$$

That is, the slow part of the original problem leads to the initial value problem. Thus, numerical difficulties caused by the T.P.B.V.P. with the split special boundary conditions can be alleviated.

[Remarks]

Under (H-1) and (H-2), after algebraic manipulations, we obtain the following expression:

$$Q_0 = (I, -A_{21}'A_{22}'^{-1})\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix}\begin{pmatrix} I \\ -A_{22}^{-1}A_{21} \end{pmatrix} - C_0 R_0^{-1} C_0'$$

$$= C_r' H_r C_r$$

where $C_r = C_1 - C_2 A_{22}^{-1} A_{21}$

$H_r = [I + (C_2 A_{22}^{-1} B_2) R^{-1} (C_2 A_{22}^{-1} B_2)']^{-1}$

$Q_{ij} = C_i' C_j$.

That is, the condition (H-5) is also self-evident.

(b) The boundary-layer correction terms

[I] The correction terms near $t = t_0$

Setting $\varepsilon = 0$ in (5.10) gives

$$\begin{cases} dm_{10}/d\tau = A_{12} m_{20} - S_{12} \rho_{20} & (5.24.a) \\ d\rho_{10}/d\tau = -Q_{12} m_{20} - A_{21}' \rho_{20} & (5.24.b) \\ dm_{20}/d\tau = A_{22} m_{20} - S_{22} \rho_{20} & (5.24.c) \\ d\rho_{20}/d\tau = -Q_{22} m_{20} - A_{22}' \rho_{20}\,. & (5.24.d) \end{cases}$$

From (5.24.c) and (5.24d), we have

$$\frac{d}{d\tau}\begin{bmatrix} m_{20} \\ \rho_{20} \end{bmatrix} = G \begin{bmatrix} m_{20} \\ \rho_{20} \end{bmatrix}. \qquad (5.25)$$

The boundary condition is given by

$$m_{20}(0) = x_{20} - X_{20}(t_0). \qquad (5.26)$$

By noting the condition (H-2) that G has no imaginary axis eigenvalue (Kucera [Kuc.72], Callier [Ca.81]), we can find a nonsingular transformation T (or U) of dimension $R^{2n_2 \times 2n_2}$ which put the Hamiltonian

G in real Jordan form (or in real Schur form (Laub [Lau.79])):

$$T^{-1}GT = \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \quad \left(U'GU = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \; ; \; \begin{matrix} Re[\lambda_i(S_{11})]<0 \\ Re[\lambda_i(S_{22})]>0 \end{matrix} \right). \tag{5.27}$$

Applying the above relation to (5.25) yields

$$\begin{bmatrix} m_{20}(\tau) \\ \rho_{20}(\tau) \end{bmatrix} = T \begin{bmatrix} e^{-\Lambda\tau} & 0 \\ & e^{\Lambda\tau} \end{bmatrix} T^{-1} \begin{bmatrix} m_{20}(0) \\ \rho_{20}(0) \end{bmatrix}. \tag{5.28}$$

Using the inverse of T (O'Donnel [OD.66]):

$$T^{-1} = \begin{bmatrix} T_{22}' & -T_{12}' \\ -T_{21}' & T_{11}' \end{bmatrix} \tag{5.29}$$

gives

$$\begin{bmatrix} m_{20}(\tau) \\ \rho_{20}(\tau) \end{bmatrix} = \begin{bmatrix} T_{11}e^{-\Lambda\tau} & T_{12}e^{\Lambda\tau} \\ T_{21}e^{-\Lambda\tau} & T_{22}e^{\Lambda\tau} \end{bmatrix} \begin{bmatrix} T_{22}'m_{20}(0) - T_{12}'\rho_{20}(0) \\ -T_{21}'m_{20}(0) + T_{11}'\rho_{20}(0) \end{bmatrix}. \tag{5.30}$$

Here, by choosing the initial condition as

$$\rho_2(0,\varepsilon) = T_{21}T_{11}^{-1}m_2(0,\varepsilon) \tag{5.31}$$

we have

$$\begin{bmatrix} m_{20}(\tau) \\ \rho_{20}(\tau) \end{bmatrix} = \begin{bmatrix} T_{11}e^{-\Lambda\tau}T_{11}^{-1}m_{20}(0) \\ T_{21}e^{-\Lambda\tau}T_{11}^{-1}m_{20}(0) \end{bmatrix}. \tag{5.32}$$

By using the real Schur form, the similar discussions can be also developed. Thus, if the initial conditions $[m_{20}(0), \rho_{20}(0)]$ belong to n_2 dimensional eigenspace corresponding to negative eigenvalues of G, there exist the boundary-layer correction terms which decaying to zero as $\tau \to \infty$.

On the other hand, from (5.24.a) and (5.24.b), we have

$$m_{10}(\tau) = -\int_{\tau}^{\infty}[A_{12}m_{20}(s) - S_{12}\rho_{20}(s)]ds = c_1 m_{20}(\tau) \tag{5.33.a}$$

$$; \; c_1 = (A_{12}-S_{12}T_{21}T_{11}^{-1})(A_{22}-S_{22}T_{21}T_{11}^{-1})^{-1}$$

$$\rho_{10}(\tau) = -\int_{\tau}^{\infty}[-Q_{12}m_{20}(s) - A_{21}'\rho_{20}(s)]ds = c_2 m_{20}(\tau) \qquad (5.33.b)$$

$$; \; c_2 = -(Q_{12}+A_{21}'T_{21}T_{11}^{-1})(A_{22}-S_{22}T_{21}T_{11}^{-1})^{-1}.$$

[II] The correction terms near $t = t_f$

Setting $\varepsilon = 0$ in (5.11) gives

$$\begin{cases} dn_{10}/d\sigma = 0 & (5.34.a) \\ d\nu_{10}/d\sigma = 0 & (5.34.b) \\ dn_{20}/d\sigma = A_{21}n_{10} + A_{22}n_{20} - S_{12}'\nu_{10} - S_{22}\nu_{20} & (5.34.c) \\ d\nu_{20}/d\sigma = -Q_{12}'n_{10} - Q_{22}n_{20} - A_{12}'\nu_{10} - A_{22}'\nu_{20}. & (5.34.c) \end{cases}$$

Here, by taking $\lim_{\sigma\to-\infty} n_{10}(\sigma) = 0$ and $\lim_{\sigma\to-\infty}\nu_{10}(\sigma) = 0$ into account, we have

$$n_{10}(\sigma) = 0, \; \nu_{10}(\sigma) = 0 \quad ; \quad \sigma\in(-\infty,0].$$

Thus, we obtain

$$\frac{d}{d\sigma}\begin{bmatrix} n_{20} \\ \nu_{20} \end{bmatrix} = G\begin{bmatrix} n_{20} \\ \nu_{20} \end{bmatrix}. \qquad (5.35)$$

In the similar way, under the initial condition:

$$\nu_2(0,\varepsilon) = T_{22}T_{12}^{-1}n_2(0,\varepsilon) \qquad (5.36)$$

we have

$$\begin{bmatrix} n_{20}(\sigma) \\ \nu_{20}(\sigma) \end{bmatrix} = \begin{bmatrix} T_{12}e^{\Lambda\sigma}T_{12}^{-1}n_{20}(0) \\ T_{22}e^{\Lambda\sigma}T_{12}^{-1}n_{20}(0) \end{bmatrix}. \qquad (5.37)$$

By using (5.37) and (5.12.d), the initial condition $n_{20}(0)$ is given by

$$n_{20}(0) = -T_{12}\Lambda^{-1}T_{12}^{-1}dX_{20}/dt(t_f). \qquad (5.38)$$

By applying the asymptotic expansion method, the higher order terms in ε can be obtained, if necessary.

[Remarks]

Under the condition (H-2), these boundary-layer correction terms can be obtained via the dichotomy transformation (Wilde et al. [Wi.72]). It is shown in Appendix B that this transformation corresponds to a special case of the combined transformation (2.15).

5.2.3 Main results

By summarizing the above results, we obtain the following theorem.

Theorem 5-1 Let the following conditions be satisfied:

(A-1) A_{22} : Hurwitz matrix.

(A-2) (C_2, A_{22}, B_2) : stabilizable and detectable.

(A-3) Γ_{22} : nonsingular.

(A-4) Π : positive semidefinite.

Then, for ε sufficiently small, the optimal control and the corresponding responses satisfy

$$\begin{cases} x_1^o(t,\varepsilon) = [X_{10}(t)+\varepsilon X_{11}(t)] + \varepsilon m_{10}(\tau) + \varepsilon n_{10}(\sigma) + O(\varepsilon^2) & (5.39.a) \\ x_2^o(t,\varepsilon) = [X_{20}(t)+\varepsilon X_{21}(t)] + [m_{20}(\tau)+\varepsilon m_{21}(\tau)] + \varepsilon n_{20}(\sigma) + O(\varepsilon^2) & (5.39.b) \\ u^o(t,\varepsilon) = [U_0(t)+\varepsilon U_1(t)] + [v_0(\tau)+\varepsilon v_1(\tau)] + \varepsilon w_0(\sigma) + O(\varepsilon^2) & (5.39.c) \end{cases}$$

where
$$\begin{aligned} U_i(t) &= -R^{-1}[B_1'P_{1i}(t)+B_2'P_{2i}(t)], \quad ; \quad i = 0,1 \\ v_0(\tau) &= -R^{-1}B_2'\rho_{20}(\tau) \\ v_1(\tau) &= -R^{-1}[B_1'\rho_{10}(\tau)+B_2'\rho_{21}(\tau)] \\ w_0(\sigma) &= -R^{-1}[B_1'\nu_{10}(\sigma)+B_2'\nu_{20}(\sigma)] \end{aligned}$$

which are uniformly valid over the whole interval.

[Remarks]

1. When the control

$$u^s = [U_0(t)+\varepsilon U_1(t)] + [v_0(\tau)+\varepsilon v_1(\tau)] + \varepsilon w_0(\sigma) \tag{5.40}$$

is implemented, the suboptimally controlled system is given by

$$\begin{cases} dx_1/dt = A_{11}x_1 + A_{12}x_2 + B_1u^s \ ; & x_1(t_0,\varepsilon) = x_{10} \\ \varepsilon dx_2/dt = A_{21}x_1 + A_{22}x_2 + B_2u^s \ ; & x_2(t_0,\varepsilon) = x_{20}. \end{cases} \tag{5.41}$$

The above system can be regarded as the singularly perturbed initial value problems (O'Malley [OM.68];[OM.71.1],[OM.74]). Then, the condition (A-1) guarantees that the solutions of (5.41) converge to the reduced solution asymptotically.

2. The condition 「Q_0 : positive semidefinite」 is not necessary since it is self-evident under (A-1) and (A-2).

5.2.4 Numerical example

The above results have been applied to a control problem of changing a power level of a nuclear reactor system. We consider a point reactor model with one-delayed neutron group. Linearizing and normalizing the kinetic equations with respect to the desired values (n_d, c_d) yields (4.171) shown in Section 4.4.5.

The performance index to be minimized is given by

$$J = \frac{1}{2}\int_0^{t_f}[q_1x_1^2 + q_2x_2^2 + ru^2]dt. \tag{5.42}$$

The boundary conditions are given by

$$x_1(0,\varepsilon) = x_{10},\ x_2(0,\varepsilon) = x_{20} \tag{5.43.a}$$

$$x_2(t_f,\varepsilon) = 0,\quad dx_2(t_f,\varepsilon)/dt = 0. \tag{5.43.b}$$

In this problem, the conditions in Theorem 5-1 are all satisfied. By following the procedure discussed in the preceding section, the suboptimal responses were calculated, which are shown in Fig. 5.1.

The data used are as follows:

$$\begin{aligned} &\beta = 0.0064, && l = 5\times10^{-3}, && \lambda = 0.078, \\ &\tilde{a}_1 = 0.078, && \tilde{a}_2 = 1.28, && \tilde{b} = 200, \\ &q_1 = 10, && q_2 = 1, && r = 5\times10^5, \\ &x_1 = -0.1, && x_2 = -0.1, && t_f = 10. \end{aligned}$$

In Fig.5.1, in order to keep the power level at its desired level for $t > t_f$, the control need be implemented after $t > t_f$, which can be easily determined from (5.43.b). Through the numerical example, it can be recognized that the singular perturbation approach is especially helpful to solve approximately the T.P.B.V.P. with the special split boundary conditions such as this problem. That is, numerical difficulties are alleviated since the global problem can be decomposed into some local problems of low dimension.

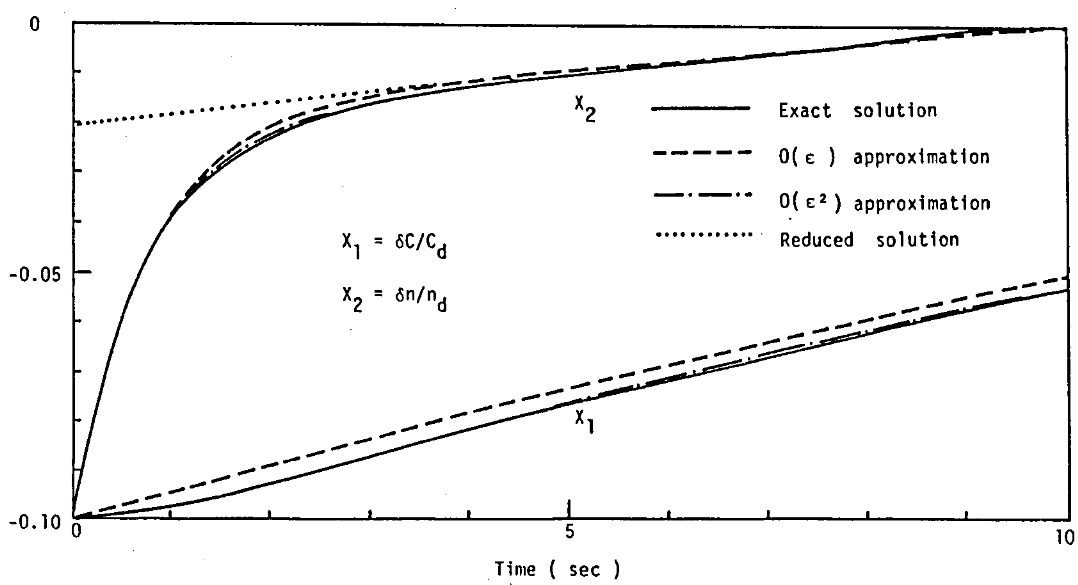

Fig. 5.1. Optimal and suboptimal responses.

5.3 Near Optimum Controller Design of Fixed-Endpoint Problems with Multi-Time-Scales

5.3.1 Near optimum controller design of two-time-scale systems

We consider the following singularly perturbed fixed-endpoint problem:

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u \tag{5.44}$$

where $\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\varepsilon & A_{22}/\varepsilon \end{bmatrix}$; $\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2/\varepsilon \end{bmatrix}$.

$$\varepsilon = \|\tilde{A}_0\| \|\tilde{A}_{22}^{-1}\| , \quad \tilde{A}_0 = \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}.$$

The performance index is

$$J = \frac{1}{2}\int_{t_0}^{t_f} [(x_1', x_2')Q\binom{x_1}{x_2} + u'Ru]dt. \tag{5.45}$$

where $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix}$: positive semidefinite matrix

R : positive definite matrix.

The boundary conditions are given by

$$\begin{cases} x_1(t_0) = x_{10}, & x_2(t_0) = x_{20} \\ x_1(t_f) = 0 , & x_2(t_f) = 0. \end{cases} \tag{5.46}$$

First, a global problem is decomposed into some local problems defined at respective time-scales. Second, the reduced-order local problems are solved independently and then their solutions are coordinated so as to approximate solutions of the global problem asymptotically. On the basis of such a decomposition-coordination principle, a mode-decoupled near optimum controller design will be developed.

(a) Decomposition into local problems

By using the boundary layer method (O'Malley et al. [OM.75.1]), a near optimum control is given by

$$
\begin{aligned}
u^s(t,\varepsilon) &= -R^{-1}[B_1'p_1(t,\varepsilon) + B_2'p_2(t,\varepsilon)] \\
&= U_0(t) + v_0(\tau) + w_0(\sigma) + O(\varepsilon)
\end{aligned}
\tag{5.47}
$$

where
$$
\begin{aligned}
U_0(t) &= -R^{-1}[B_1'P_{10}(t) + B_2'P_{20}(t)] \\
v_0(\tau) &= -R^{-1}B_2'\rho_0(\tau), \quad \tau = (t-t_0)/\varepsilon \\
w_0(\sigma) &= -R^{-1}B_2'\nu_0(\sigma), \quad \sigma = (t_f-t)/\varepsilon.
\end{aligned}
$$

The state variables of the canonical equations are represented by

$$
\begin{cases}
x_1 = X_{10}(t) + O(\varepsilon) \\
x_2 = X_{20}(t) + m_0(\tau) + n_0(\sigma) + O(\varepsilon) \\
p_1 = P_{10}(t) + O(\varepsilon) \\
p_2 = P_{20}(t) + \rho_0(\tau) + \nu_0(\sigma) + O(\varepsilon).
\end{cases}
\tag{5.48}
$$

Then, the local problems defined at respective time-scales are formulated as follows.

[I] t-time-scale problem

The t-time-scale terms satify the following equations:

$$
\begin{cases}
dX_{10}/dt = \mathcal{A}_0X_{10} - \mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'P_{10} \quad ; \quad X_{10}(t_0) = x_{10} & (5.49.a) \\
dP_{10}/dt = -\mathcal{Q}_0X_{10} - \mathcal{A}_0'P_{10} \quad ; \quad X_{10}(t_f) = 0 & (5.49.b) \\
X_{20}(t) = -A_{22}^{-1}[A_{21} + S_{22}\theta^{-1}(Q_{12}' - Q_{22}A_{22}^{-1}A_{21})]X_{10}(t) + \\
\qquad A_{22}^{-1}[S_{12}'-S_{22}\theta^{-1}(A_{12}' + Q_{22}A_{22}^{-1}S_{12}')]P_{10}(t) & (5.49.c) \\
P_{20}(t) = -\theta^{-1}(Q_{12}' - Q_{22}A_{22}^{-1}A_{21})X_{10}(t) - \theta^{-1}(A_{12}' + Q_{22}A_{22}^{-1}S_{12}')P_{10}(t) & (5.49.d)
\end{cases}
$$

where
$$
\begin{aligned}
&\mathcal{A}_0 = A_0 + \mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{C}_0', \quad A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} \\
&\mathcal{B}_0 = B_0, \qquad\qquad\quad B_0 = B_1 - A_{12}A_{22}^{-1}B_2 \\
&\mathcal{R}_0 = R + B_2'A_{22}'^{-1}Q_{22}A_{22}^{-1}B_2, \quad \mathcal{C}_0 = (Q_{12} - A_{21}'A_{22}'^{-1}Q_{22})A_{22}^{-1}B_2
\end{aligned}
\tag{5.50}
$$

$$Q_0 = [I, -A_{21}' A_{22}'^{-1}] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} \begin{bmatrix} I \\ -A_{22}^{-1} A_{21} \end{bmatrix} - C_0 R_0^{-1} C_0'$$

$$= C_r' H_r C_r$$

$$C_r = C_1 - C_2 A_{22}^{-1} A_{21}, \quad H_r = [I + (C_2 A_{22}^{-1} B_2) R^{-1} (C_2 A_{22}^{-1} B_2)']^{-1}$$

$$S_{12} = B_1 R^{-1} B_2', \quad S_{22} = B_2 R^{-1} B_2', \quad \theta = A_{22}' + Q_{22} A_{22}^{-1} S_{22}.$$

Here, we consider a fixed-endpoint problem corresponding to the canonical equations (5.49.a),(5.49.b). It is well known that such a fixed-endpoint problem can be transformed into the equivalent minimum energy problem (Brockett [Br.70]). Then, on the basis of this equivalence, the control of the fixed-endpoint problem is given as follows:

$$\begin{aligned} V_0^o(t) &= -R_0^{-1} B_0' P_{10}(t) \\ &= -R_0^{-1} B_0' \Pi(t;0,t_f) X_{10}(t) - R_0^{-1} B_0' \Phi'(t_0,t) W^{-1}(t_0,t_f) x_{10} \end{aligned} \tag{5.51}$$

where $\Pi(t;0,t_f)$ is a solution of the Riccati differential equation:

$$-dK_0(t)/dt = A_0' K_0(t) + K_0(t) A_0 - K_0(t) B_0 R_0^{-1} B_0' K_0(t) + Q_0 \;;\; K_0(t_f) = 0 \tag{5.52}$$

and $\Phi(t,t_0)$ is a transition matrix which satisfies

$$\begin{aligned} d\Phi(t,t_0)/dt &= [A_0 - B_0 R_0^{-1} B_0' \Pi(t;0,t_f)] \Phi(t,t_0) \quad ; \quad \Phi(t_0,t_0) = I \\ &= A_c(t) \Phi(t,t_0) \end{aligned} \tag{5.53}$$

and $W(t_0,t_f)$ is a controllability Grammian matrix:

$$dW(t,t_f)/dt = A_c(t) W(t,t_f) + W(t,t_f) A_c'(t) - B_0 R_0^{-1} B_0' \;;\; W(t_f,t_f) = 0 \tag{5.54.a}$$

or

$$W(t_0,t_f) = \int_{t_0}^{t_f} \Phi(t_0,\sigma) B_0 R_0^{-1} B_0' \Phi'(t_0,\sigma) d\sigma. \tag{5.54.b}$$

On the other hand, when we use the inverse Riccati transformation

(Mufti et al. [Muf.69], Kwon et al. [Kw.77], Breitenecker [Bre.83]):

$$X_{10}(t) = \mathcal{H}_0(t)P_{10}(t) + g_0(t) \tag{5.55}$$

we have

$$\begin{aligned} V_o^c(t) &= -\mathcal{R}_0^{-1}\mathcal{B}_0'P_{10}(t) \\ &= -\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{H}_0^{-1}(t)[X_{10}(t) - g_0(t)] \end{aligned} \tag{5.56}$$

$$\begin{aligned} \text{where} \quad d\mathcal{H}_0/dt &= \mathcal{A}_0\mathcal{H}_0(t)+\mathcal{H}_0(t)\mathcal{A}_0'+\mathcal{H}_0(t)\mathcal{Q}_0\mathcal{H}_0(t)-\mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0' \quad ; \quad \mathcal{H}_0(t_f) = 0 \\ dg_0/dt &= [\mathcal{A}_0 + \mathcal{H}_0(t)\mathcal{Q}_0]g_0(t) \;\; ; \;\; g_0(t_f) = x_1(t_f). \end{aligned} \tag{5.57.a,b}$$

Especially, in case of $x_1(t_f) = 0$, we have

$$g_0(t) = 0, \quad \forall t \in [t_0,t_f].$$

Further discussions on the inverse Riccati transformation are given in Appendix C.

[Remarks]

After algebraic manipulations, it can be shown (Appendix C) that the relation between the inverse Riccati transformation (5.55) and the controllability Grammian (5.54) is given by the following expression:

$$\begin{cases} \mathcal{H}_0^{-1}(t,t_f) = W^{-1}(t,t_f) + \Pi(t;0,t_f) & (5.58.a) \\ \qquad\qquad\quad = -\psi_{12}^{-1}(t_f,t)\psi_{11}(t_f,t) & \\ g(t) = [I - \mathcal{H}_0(t,t_f)\Pi(t;0,t_f)]\Phi(t,t_f)x_1(t_f) & (5.58.b) \\ \qquad\;\; = \psi_{11}^{-1}(t_f,t)x_1(t_f) & \end{cases}$$

where $[\psi_{ij}(t_f,t)]$ represents a transition matrix of the Hamiltonian system.

Here, we consider the t-time-scale control part of (5.47):

$$U_0(t) = -R^{-1}[B_1'P_{10}(t) + B_2'P_{20}(t)].$$

Then, by using the following formulas (Kando et al. [Ka.81.3]):

(i) $\theta^{-1} = A_{22}^{'-1} + \gamma S_{22} A_{22}^{'-1}$ (5.59.a)

where $\gamma = \beta^{'-1} H_{22}^{-1} + H_{22}^{-1}\beta^{-1} + H_{22}^{-1}\beta^{-1} S_{22}\beta^{'-1} H_{22}^{-1}$

$\beta = A_{22} - S_{22}H_{22}^{-1}$

$H_{22}A_{22} + H_{22}A_{22}' + H_{22}Q_{22}H_{22} - B_2 R^{-1} B_2' = 0$

and $H_{22} > 0$ exists uniquely and $\beta < 0$ under the condition: $\ulcorner(C_2, A_{22}, B_2)$: detectability and completely controllable$\lrcorner$ (Kwon et al. [Kw.77])

(ii) $\mathcal{R}_0 = M' R M$ (5.59.b)

where $M = (I - R^{-1}B_2' H_{22}^{-1} A_{22}^{-1} B_2)$

$= (I + R^{-1}B_2' H_{22}^{-1}\beta^{-1} B_2)^{-1}$

and after considerable algebraic manipulations, we have

$$\begin{aligned}
U_0(t) &= -R^{-1}[B_1'P_{10}(t) + B_2'P_{20}(t)] \\
&= -R^{-1}\{[B_1' - B_2'\theta^{-1}(A_{12}' + Q_{22}A_{22}^{-1}S_{12}')]\mathcal{H}_0^{-1}(t) - \\
&\qquad\qquad B_2'\theta^{-1}(Q_{12}' - Q_{22}A_{22}^{-1}A_{21})\}X_{10}(t) \\
&= -\mathcal{R}_0^{-1}(\mathcal{B}_0'\mathcal{H}_0^{-1}(t) - \mathcal{C}_0')X_{10}(t) \\
&= V_0(t) + \mathcal{R}^{-1}\mathcal{C}_0'X_{10}(t).
\end{aligned} \tag{5.60}$$

It should be noted that the above control corresponds to one of the following problem:

[t-time-scale problem]

$$\left\{\begin{aligned}
& dX_{10}/dt = \mathcal{A}_0 X_{10} + \mathcal{B}_0 V_0 \quad ; \quad X_{10}(t_0) = x_{10}, \quad X_{10}(t_f) = 0 \\
& J_t = \frac{1}{2}\int_{t_0}^{t_f}[X_{10}'\mathcal{Q}_0 X_{10} + V_0'\mathcal{R}_0 V_0]dt
\end{aligned}\right. \tag{5.61.a}$$

$$U_0(t) = V_0(t) + \mathcal{R}_0^{-1}\mathcal{C}_0'X_{10}(t). \tag{5.61.b}$$

Here, we assume the conditions:

(A-1) A_{22} : Hurwitz matrix

(A-2) (A_0, B_0) : completely controllable where the condition (A-1) is used in order to guarantee the asymptotic stability of the suboptimally controlled system.

[II] τ-time-scale problem

The τ-time-scale terms $[m_0(\tau),\ \rho_0(\tau)]$ must satisfy the following canonical equations:

$$\begin{cases} dm_0/d\tau = A_{22}m_0(\tau) - S_{22}\rho_0(\tau) \quad ; \quad m_0(0) = x_{20} - X_{20}(t_0) \\ d\rho_0/d\tau = -Q_{22}m_0(\tau) - A_{22}'\rho_0(\tau). \end{cases} \tag{5.62}$$

Here, by applying the following transformation:

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}' \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \; ; \; \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^{-1} = \begin{bmatrix} T_{22}' & -T_{12}' \\ -T_{21}' & T_{11}' \end{bmatrix}$$

and by choosing the initial condition as

$$\rho_0(0) = T_{11}^{'-1}T_{21}^{'-1}m_0(0) = T_{21}T_{11}^{-1}m_0(0) \tag{5.63}$$

there exist solutions $m_0(\tau)$ and $\rho_0(\tau)$ which decay exponentially to zero as $\tau \rightarrow \infty$:

$$\begin{cases} m_0(\tau) = T_{11}\exp(-\Lambda\tau)T_{11}^{-1}m_0(0) \\ \rho_0(\tau) = T_{21}\exp(-\Lambda\tau)T_{11}^{-1}m_0(0) \end{cases} \tag{5.64}$$

or $\quad \rho_0(\tau) = T_{21}T_{11}^{-1}m_0(\tau).$ $\qquad$ (5.65)

It is well known (Kučera [Kuc.72], O'Donnel [OD.66]) that the positive definite solution P_{22} of the Riccati algebraic equation:

$$A_{22}'K_{22} + K_{22}A_{22} - K_{22}S_{22}K_{22} + Q_{22} = 0 \tag{5.66}$$

has the form $P_{22} = T_{21}T_{11}^{-1}$. Accordingly, we have

$$\rho_0(\tau) = P_{22}m_0(\tau). \tag{5.67}$$

Alternatively, the Shur vector approach (Laub [Lau.79]) can be also used in the above procedure, which is superior to the generalized eigenvector approach used above.

[Remarks]

In the nonlinear singularly perturbed problem, it has been shown (Hoppensteadt [Ho.71]) that there is an n_2-dimensional manifold S_0 in E^{2n_2} with the property that the initial conditions $[m_0(0),\ \rho_0(0)] \in S_0$ imply that the boundary-layer solutions exist, which decay exponentially to zero as $\tau \to \infty$. Furthermore, it has been indicated (Freedman et al. [Fr.76.1],[Fr.76.2]) that there exists sufficiently small $\eta > 0$ such that if $|m_0(0)| < \eta$, these boundary-layer solutions exist. In the linear case (5.62), these results imply that if the initial conditions $[m_0(0),\ \rho_0(0)]$ of (5.63) belong to the eigenspace corresponding to the negative eigenvalues of the Hamiltonian matrix of (5.62), there exist the decaying terms $[m_0(\tau),\ \rho_0(\tau)]$ as $\tau \to \infty$, and there are no restrictions with respect to the size of the region of attraction.

The τ-time-scale control part of (5.47) is given by

$$\begin{aligned} v_0(\tau) &= -R^{-1}B_2'\rho_0(\tau) \\ &= -R^{-1}B_2'P_{22}m_0(\tau). \end{aligned} \tag{5.68.a}$$

It is noted that the above control corresponds to one of the following τ-time-scale problem:

[τ-time-scale problem]

$$\begin{cases} dm_0/d\tau = A_{22}m_0(\tau) + B_2v_0(\tau) \quad ; \quad m_0(0) = x_{20} - X_{20}(t_0) \\ J_\tau = \frac{1}{2}\int_0^\infty [m_0'(\tau)Q_{22}m_0(\tau) + v_0'(\tau)Rv_0(\tau)]d\tau. \end{cases} \tag{5.68.b}$$

Here, the following condition is assumed:

(A-3) $(C_2,\ A_{22},\ B_2)$: completely controllable and observable.

[III] σ-time-scale problem

The σ-time-scale terms $[n_0(\sigma), \nu_0(\sigma)]$ must satisfy the following canonical equations:

$$\begin{cases} dn_0/d\sigma = -A_{22}n_0(\sigma) + S_{22}\nu_0(\sigma) \quad ; \quad n_0(0) = -X_{20}(t_f) \\ d\nu_0/d\sigma = Q_{22}n_0(\sigma) + A_{22}'\nu_0(\sigma). \end{cases} \tag{5.69}$$

Similarly as the τ-time-scale case, if we choose the initial conditions as

$$\nu_0(0) = T_{22}T_{12}^{-1}n_0(0) \tag{5.70}$$

there exist solutions $n_0(\sigma)$ and $\nu_0(\sigma)$ which decay exponentially to zero as $\sigma \to \infty$. Then, we have the σ-time-scale control as follows:

$$\begin{aligned} w_0(\sigma) &= -R^{-1}B_2'\nu_0(\sigma) \\ &= -R^{-1}B_2'N_{22}n_0(\sigma) \end{aligned} \tag{5.71}$$

where $\nu_0(\sigma) = N_{22}n_0(\sigma) = T_{22}T_{12}^{-1}n_0(\sigma)$. (5.72)

It is known (Wilde [Wi.72], O'Donnel [OD.66]) that the negative definite solution N_{22} of the Riccati algebraic equation (5.66) has the form $N_{22} = T_{22}T_{12}^{-1}$, which exist under condition (A-3).

It is noted that the above control corresponds to one of the following problem:

[σ-time-scale problem]

$$\begin{cases} dn_0/d\sigma = -A_{22}n_0(\sigma) - B_2w_0(\sigma) \quad ; \quad n_0(0) = -X_{20}(t_f) \\ J_\sigma = \frac{1}{2}\int_0^\infty [n_0'(\sigma)Q_{22}n_0(\sigma) + w_0'(\sigma)Rw_0(\sigma)]d\sigma. \end{cases} \tag{5.73}$$

In this way, the global problem (5.44)-(5.46) have been decomposed into the reduced-order local problems (5.61), (5.68) and (5.73), which are defined at respective time-scales.

(b) Near optimum controller design

By coordinating controls of local problems defined above, a near optimum controller is given by

$$u^s(t) = U_0(t) + v_0(\tau) + w_0(\sigma). \tag{5.74}$$

Then, from (5.51), (5.61.b), (5.68.a) and (5.71), the open-loop control is given by:

$$u_o^s(t) = -\tilde{\mathcal{R}}_0^{-1}(\tilde{\mathcal{B}}_0'\tilde{\Pi}(t;0,t_f)-\tilde{\mathcal{C}}_0')\tilde{X}_{10}(t) - \tilde{\mathcal{R}}_0^{-1}\tilde{\mathcal{B}}_0'\tilde{\Phi}'(t_0,t)\tilde{W}^{-1}(t_0,t_f)x_{10}$$
$$- R^{-1}\tilde{B}_2'\tilde{P}_{22}\tilde{m}_0(t) - R^{-1}\tilde{B}_2'\tilde{N}_{22}\tilde{n}_0(t) \tag{5.75.a}$$

where $\tilde{X}_{10}(t)$, $\tilde{m}_0(t)$ and $\tilde{n}_0(t)$ satisfy

$$\begin{cases} d\tilde{X}_{10}/dt = [\tilde{\mathcal{A}}_0-\tilde{\mathcal{B}}_0\tilde{\mathcal{R}}_0^{-1}\tilde{\mathcal{B}}_0'\tilde{\Pi}(t;0,t_f)]\tilde{X}_{10}(t) - \tilde{\mathcal{B}}_0\tilde{\mathcal{R}}_0^{-1}\tilde{\mathcal{B}}_0'\tilde{\Phi}'(t_0,t)\tilde{W}^{-1}(t_0,t_f)x_{10}, \\ \qquad\qquad ; \quad \tilde{X}_{10}(t_0) = x_{10} \\ d\tilde{m}_0/dt = [\tilde{A}_{22} - \tilde{S}_{22}\tilde{P}_{22}]\tilde{m}_0(t) \quad ; \quad \tilde{m}_0(t_0) = x_{20} - \tilde{X}_{20}(t_0), \\ d\tilde{n}_0/dt = -[\tilde{A}_{22} - \tilde{S}_{22}\tilde{N}_{22}]\tilde{n}_0(t) \quad ; \quad \tilde{n}_0(t_f) = -\tilde{X}_{20}(t_f). \end{cases} \tag{5.75.b,c,d}$$

On the other hand, by using (5.60), we have

$$u_c^s(t) = -\mathcal{R}_0^{-1}(\mathcal{B}_0'\mathcal{H}_0^{-1}(t)-\mathcal{C}_0')X_{10}(t) - R^{-1}B_2'P_{22}m_0(\tau) - R^{-1}B_2'N_{22}n_0(\sigma). \tag{5.76}$$

In order to transform the above into the state feedback form, by using the expressions (5.48), we have

$$u_c^s(t) = -[(I-R^{-1}\tilde{B}_2'\tilde{P}_{22}\tilde{A}_{22}^{-1}\tilde{B}_2)\tilde{\mathcal{R}}_0^{-1}(\tilde{\mathcal{B}}_0'\tilde{\mathcal{H}}_0^{-1}(t)-\tilde{\mathcal{C}}_0')+R^{-1}\tilde{B}_2'\tilde{P}_{22}\tilde{A}_{22}^{-1}\tilde{A}_{21}]x_1(t)$$
$$- R^{-1}\tilde{B}_2'\tilde{P}_{22}x_2(t) - R^{-1}\tilde{B}_2'(\tilde{N}_{22}-\tilde{P}_{22})\tilde{n}_0(t) \tag{5.77}$$

where the open-loop part is defined by (5.75.d).

The above controls (5.75.a) and (5.77) are connected by the relation:

$$\tilde{\mathcal{H}}_0^{-1}(t) = \tilde{W}^{-1}(t,t_f) + \tilde{\Pi}(t;0,t_f). \tag{5.78}$$

We summarize the above results as follows:

<u>Theorem 5-2</u> Let the following conditions be satisfied:

(A-1) $\tilde{A}_{22}$: Hurwitz matrix.

(A-2) $(\tilde{A}_0, \tilde{B}_0)$: completely controllable.

(A-3) $(\tilde{C}_2, \tilde{A}_{22}, \tilde{B}_2)$: completely observable and controllable.

Then, the suboptimal open-loop and partial feedback controls are given by (5.75.a) and (5.77), respectively.

[Remarks]

1. If the condition (A-2) is weakened to the stabilizability (Kwon [Kw.77], Sandell [San.74]), the t-time-scale control (5.60) is given by:

$$U_0(t) = -\mathcal{R}_0^{-1}(\mathcal{B}_0' \mathcal{H}_0^{\#}(t) - \mathcal{C}_0')X_{10}(t) \tag{5.79}$$

where $\mathcal{H}_0^{\#}(t)$ denotes the generalized inverse.

2. When the open-loop control (5.75.a) is implemented, the suboptimally controlled system is given by

$$\begin{cases} dx_1 = A_{11}x_1 + A_{12}x_2 + B_1u_o^s \ ; \quad x_1(t_0) = x_{10} \\ \epsilon dx_2 = A_{21}x_1 + A_{22}x_2 + B_2u_o^s \ ; \quad x_2(t_0) = x_{20}. \end{cases} \tag{5.80}$$

It has been shown (Wilde [Wi.73]) that if unstable modes are presented in A_{22}, the solutions of (5.80) diverge near the boundary-layer. Then, the condition (A-1) guarantees the asymptotic stability of the above system. Now, we suppose that the matrix A_{22} has k eigenvalues with negative real parts, i.e., $Re[\lambda_i(A_{22})] \leq -\mu<0$ while the remaining (n_2-k) eigenvalues have positive real parts satisfying $Re[\lambda_i(A_{22})] \geq \mu>0$. In the nonlinear singularly perturbed initial value problems, this case is regarded as the conditionally stable case treated by Levin [Lev.56], Chang [Cha.69] and Hoppensteadt [Ho.71]. They have shown that there is a k-dimensional stable manifold S_0 so that the boundary-layer corrections which decay exponentially to zero as $\tau \to \infty$ exist if $x_{20} - X_{20}(t_0) = m_0(0) \in S_0$. In the linear case (5.80), these results imply that if $m_0(0)$ belong to the k-dimensional eigenspace corresponding to $Re[\lambda(A_{22})] \leq -\mu < 0$, the decaying solutions exist. Actually, in many physical systems, A_{22} is a stable matrix and then we always obtain the stable solutions for any initial condition $m_0(0)$. On the other hand, when the partial feedback control (5.77) is implemented, the stability of the system (5.80) is always guaranteed, because $\mathrm{Re}[\lambda(A_{22}-S_{22}P_{22})] \le -\mu<0$.

5.3.2 Extension to multi-time-scale systems

The problem of singular perturbation modelling for two-time-scale systems has been already studied in Section 2.3, where the modelling concepts and algorithms have been given.

In this section, firstly, an extension of these results to multi-time-scale systems is discussed.

Let the following conditions be satisfied in (5.44).

(H-1) $\max[2\tilde{a}_f^j(\tilde{a}_s^j+\tilde{a}_{12}^j\tilde{l}_0^j),\tilde{a}_f^j(\tilde{a}_s^j+\tilde{a}_{12}^j\tilde{l}_0^j+8\tilde{a}_s^j\tilde{a}_{12}^j\tilde{l}_0^j/(\tilde{a}_s^j+\tilde{a}_{12}^j\tilde{l}_0^j))] < 1$

where $\tilde{a}_s^j = \|\tilde{\alpha}_j\|$, $\tilde{a}_f^j = \|\tilde{\alpha}_{jj}^{-1}\|$, $\tilde{a}_{12}^j = \|\tilde{\alpha}_{1j}\|$, $\tilde{l}_0^j = \|\tilde{\alpha}_{jj}^{-1}\tilde{\alpha}_{j1}\|$

(H-2) $\varepsilon_j = \tilde{a}_s^j\tilde{a}_f^j \ll 1$

(H-3) $\tilde{a}_f^j\tilde{a}_{12}^j \ll 1$

where
$$\left[\begin{array}{c|c} \tilde{A}_{jj} & \tilde{\alpha}_{1j} \\ \hline \tilde{\alpha}_{j1} & \tilde{\alpha}_{jj} \end{array}\right] = \left[\begin{array}{c|ccc} \tilde{A}_{jj} & \tilde{A}_{j,j+1} & \cdots & \tilde{A}_{j,N+1} \\ \hline \tilde{A}_{j+1,j} & \tilde{A}_{j+1,j+1} & \cdots & \tilde{A}_{j+1,N+1} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{N+1,j} & \tilde{A}_{N+1,j+1} & \cdots & \tilde{A}_{N+1,N+1} \end{array}\right]$$
$$\tilde{\alpha}_j = \tilde{A}_{jj} - \tilde{\alpha}_{1j}\tilde{\alpha}_{jj}^{-1}\tilde{\alpha}_{j1}, \quad \tilde{\alpha}_{N+1} = \tilde{A}_{N+1,N+1} \ ; \ j = 1,\ldots,N.$$

Then, similarly as the case of two-time-scale systems, it can be shown that the system (5.44) satisfies the multi-time-scale property and the eigenvalues are approximated by

$$\lambda(\tilde{A}) = \lambda(\tilde{\alpha}_1)\cup\lambda(\tilde{\alpha}_2)\cup\ldots\cup\lambda(\tilde{\alpha}_{N+1})[1+O(\varepsilon_M)] \ ; \ \varepsilon_M = \max[\varepsilon_1,\ldots,\varepsilon_N]$$

where $\lambda(\tilde{\alpha}_{jj}) \approx \lambda(\tilde{\alpha}_{j+1})\cup\lambda(\tilde{\alpha}_{j+1,j+1})$. (5.81)

For example, a nuclear reactor model with temperature feedback (Kando et al. [Ka.71]) and a single-machine power system (Hassan [Has.81]) can be regarded as three-time-scale systems.

By applying the modelling algorithm shown in Section 2.3 repeatedly, the system (5.44) can be identified as the following singularly perturbed system with small multi-parameters $\varepsilon_1, \ldots, \varepsilon_N$:

$$dx_i/dt = \tilde{A}_{ii}x_i + \sum_{\substack{j=1 \\ (j\neq i)}}^{N+1} \tilde{A}_{ij}x_j + \tilde{B}_i u \quad ; \quad x_i(t_0)=x_{i0}, \quad x_i(t_f)=0$$

$$; \quad i = 1,\ldots,N+1 \qquad (5.82)$$

where $x_i \in R^{n_i}$, $u \in R^r$, $\tilde{A}_{ij} = A_{ij}/(\varepsilon_1\ldots\varepsilon_{i-1})$, $\tilde{B}_i = B_i/(\varepsilon_1\ldots\varepsilon_{i-1})$, $\varepsilon_i = \tilde{a}_s^i\tilde{a}_f^i$.

The performance index is given by

$$J = \frac{1}{2}\int_0^{t_f}\{ [x_1',\ldots,x_{N+1}'] Q\begin{bmatrix} x_1 \\ \vdots \\ x_{N+1}\end{bmatrix} + u'Ru\}dt. \qquad (5.83)$$

Here, we consider a near optimum controller design of the fixed-endpoint problem with multi-time-scale property given above.

Then, by applying the design procedure for two-time-scale systems shown in the preceding section to multi-time-scale systems repeatedly, and after considerable algebraic manipulations, we obtain finally the following results:

$$u^s(t) = U_0(t) + \sum_{j=1}^{N}\hat{v}_j(\tau_j) + \sum_{j=1}^{N}\hat{w}_j(\sigma_j) = u_j|_{j=N} \qquad (5.84)$$

$$; \quad u_j = u_{j-1} + \tilde{L}_j\tilde{m}_j^j + \tilde{F}_j\tilde{n}_j^j$$

$$= u_{j-1} + \tilde{L}_j\{ x_{j+1} - (\tilde{M}_{j0}+\tilde{N}_{j0}\tilde{L}_0)x_1 - \sum_{k=1}^{j-1}[(\tilde{G}_j^k+\tilde{H}_j^k\tilde{L}_k)\tilde{m}_k^k +$$

$$+(\tilde{G}_j^k+\tilde{H}_j^k\tilde{F}_k)\tilde{n}_k^k]\} + (\tilde{F}_j-\tilde{L}_j)\tilde{n}_j^j(t) \qquad (5.85)$$

where
$$\begin{cases} \tilde{m}_1^1 + \tilde{n}_1^1 = x_2 - (\tilde{M}_{10}+\tilde{N}_{10})x_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \tilde{m}_k^k + \tilde{n}_k^k = x_{k+1} - (\tilde{M}_{k0}+\tilde{N}_{k0}\tilde{L}_0)x_1 - \sum_{h=1}^{k-1}[(\tilde{G}_k^h+\tilde{H}_k^h\tilde{L}_h)\tilde{m}_h^h+(\tilde{G}_k^h+\tilde{H}_k^h\tilde{F}_h)\tilde{n}_h^h] \end{cases} \qquad (5.86.a)$$

$$\begin{bmatrix} \tilde{M}_{10} + \tilde{N}_{10}\tilde{L}_0 \\ \cdots\cdots\cdots\cdots \\ \tilde{M}_{N0} + \tilde{N}_{N0}\tilde{L}_0 \end{bmatrix} = -\tilde{\alpha}_{jj}^{-1}\Big|_{j=1}[\tilde{\alpha}_{j1}\Big|_{j=1} + \tilde{\beta}_1\tilde{L}_0] \qquad (5.86.b)$$

$$\begin{bmatrix} \tilde{G}^{j}_{j+1} + \tilde{H}^{j}_{j+1}\tilde{L}_j \\ \cdots\cdots\cdots\cdots \\ \tilde{G}^{j}_{N} + \tilde{H}^{j}_{N}\tilde{L}_j \end{bmatrix} = -\tilde{\alpha}^{-1}_{j+1,j+1}[\tilde{\alpha}_{j+1,j} + \tilde{\beta}_{j+1}\tilde{L}_j] \tag{5.86.c}$$

and $\tilde{L}_j$ and $\tilde{F}_j$ represent feedback gains of local problems.

Here, local problems defined at respective time-scales are formulated as follows:

[t-time-scale problem]

$$\begin{cases} dX_{10}/dt = \mathcal{A}_0 X_{10} + \mathcal{B}_0 V_0 \quad ; \quad X_{10}(t_0) = x_{10}, \quad X_{10}(t_f) = 0 \\ J_t = \frac{1}{2}\int_{t_0}^{t_f} [X_{10}' \mathcal{Q}_0 X_{10} + V_0' \mathcal{R}_0 V_0]dt \end{cases} \tag{5.87}$$

where

$$\mathcal{A}_0 = A_0 + \mathcal{B}_0 \mathcal{R}_0^{-1} \mathcal{C}_0', \quad A_0 = \alpha_1$$

$$\mathcal{B}_0 = B_0, \quad B_0 = B_1 - \alpha_{1j}|_{j=1} \alpha^{-1}_{jj}|_{j=1} \beta_1$$

$$\mathcal{R}_0 = R + \beta_1' \alpha'^{-1}_{jj}|_{j=1} \hat{Q}_{22} \alpha^{-1}_{jj}|_{j=1} \beta_1$$

$$\mathcal{C}_0 = (\hat{Q}_{12} - \alpha'_{j1}|_{j=1} \alpha'^{-1}_{jj}|_{j=1} \hat{Q}_{22}) \alpha^{-1}_{jj}|_{j=1} \beta_1$$

$$\mathcal{Q}_0 = [I, -\alpha'_{j1}|_{j=1} \alpha'^{-1}_{jj}|_{j=1}] \begin{bmatrix} Q_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}' & \hat{Q}_{22} \end{bmatrix} \begin{bmatrix} I \\ -\alpha^{-1}_{jj}|_{j=1} \alpha_{j1}|_{j=1} \end{bmatrix} - \mathcal{C}_0 \mathcal{R}_0^{-1} \mathcal{C}_0'$$

$$\alpha_{1j} = \begin{bmatrix} A_{j,j+1} & \cdots & A_{j,N+1} \end{bmatrix}, \quad \alpha_{jj} = \begin{bmatrix} A_{j+1,j+1} & \cdots & A_{j+1,N+1} \\ \vdots & & \vdots \\ A_{N+1,j+1} & \cdots & A_{N+1,N+1} \end{bmatrix}$$

$$\alpha_{j1} = \begin{bmatrix} A_{j+1,j} \\ \vdots \\ A_{N+1,j} \end{bmatrix}, \quad \alpha_{jj} = A_{jj} - \alpha_{1j} \alpha^{-1}_{jj} \alpha_{j1}$$

$$\beta_j = \begin{bmatrix} B'_{j+1}, \cdots, B'_{N+1} \end{bmatrix}', \quad \gamma_j = B_j - \alpha_{1j} \alpha^{-1}_{jj} \beta_j$$

$$\left[\begin{array}{c|c} Q_{jj} & \hat{Q}_{j,j+1} \\ \hline \hat{Q}'_{jj+1} & \hat{Q}_{j+1,j+1} \end{array}\right] = \left[\begin{array}{c|ccc} Q_{jj} & Q_{j,j+1} & \cdots & Q_{j,N+1} \\ \hline Q'_{j,j+1} & Q_{j+1,j+1} & \cdots & Q_{j+1,N+1} \\ \vdots & \vdots & & \vdots \\ Q'_{j,N+1} & Q'_{j+1,N+1} & \cdots & Q_{N+1,N+1} \end{array}\right].$$

Then, the inverse Riccati approach yields

$$U_0(t) = V_0(t) + \mathcal{R}_0^{-1}\mathcal{C}_0' X_{10}(t)$$
$$= -\mathcal{R}^{-1}(\mathcal{B}_0'\mathcal{H}_0^{-1}(t) - \mathcal{C}_0')X_{10}(t)$$
$$= L_0 X_{10}(t) \tag{5.88.a}$$

where $d\mathcal{H}_0/dt = \mathcal{H}_0\mathcal{A}_0' + \mathcal{A}_0\mathcal{H}_0 + \mathcal{H}_0\mathcal{Q}_0\mathcal{H}_0 - \mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0' \;\; ; \;\; \mathcal{H}_0(t_f) = 0.$

Alternatively, the open-loop approach gives

$$U_0(t) = -\mathcal{R}_0^{-1}(\mathcal{B}_0'\Pi(t;0,t_f)-\mathcal{C}_0')X_{10}(t) - \mathcal{R}_0^{-1}\mathcal{B}_0'\phi'(t_0,t)W^{-1}(t_0,t_f)x_{10}. \tag{5.88.b}$$

[τ_j-time-scale problem]

$$\begin{cases} dm_j^j/d\tau_j = \mathcal{A}_j m_j^j(\tau_j) + \mathcal{B}_j v_j(\tau_j) \quad ; \quad j = 1,\ldots,N \\ J_{\tau_j} = \frac{1}{2}\int_0^\infty [\, m_j^{j\prime}\mathcal{Q}_j m_j^j + v_j'\mathcal{R}_j v_j]\, d\tau_j. \end{cases} \tag{5.89}$$

In case of (i) $j = 1,\ldots,N-1$:

$$\mathcal{A}_j = \alpha_{j+1} + \mathcal{B}_j\mathcal{R}_j^{-1}\mathcal{C}_j', \quad \mathcal{B}_j = \gamma_{j+1}$$
$$\mathcal{R}_j = R + \beta_{j+1}'\alpha_{j+1,j+1}'^{-1}\hat{Q}_{j+2,j+2}\alpha_{j+1,j+1}^{-1}\beta_{j+1}$$
$$\mathcal{C}_j = (\hat{Q}_{j+1,j+2} - \alpha_{j+1,1}'\alpha_{j+1,j+1}'^{-1}\hat{Q}_{j+2,j+2})\alpha_{j+1,j+1}^{-1}\beta_{j+1}$$
$$\mathcal{Q}_j = [I, -\alpha_{j+1,1}'\alpha_{j+1,j+1}'^{-1}]\begin{bmatrix} Q_{j+1,j+1} & \hat{Q}_{j+1,j+2} \\ \hat{Q}_{j+1,j+2}' & \hat{Q}_{j+2,j+2}\end{bmatrix}\begin{bmatrix} I \\ -\alpha_{j+1,j+1}^{-1}\alpha_{j+1,1}\end{bmatrix} - \mathcal{C}_j\mathcal{R}_j^{-1}\mathcal{C}_j'.$$

In case of (ii) $j = N$:

$$\mathcal{A}_N = A_{N+1,N+1}, \quad \mathcal{B}_N = B_{N+1}, \quad \mathcal{R}_N = R, \quad \mathcal{Q}_N = Q_{N+1,N+1}.$$

Then, we have

$$\hat{v}_j(\tau_j) = v_j(\tau_j) + \mathcal{R}_j^{-1}\mathcal{C}_j' m_j^j(\tau_j)$$
$$= -\mathcal{R}_j^{-1}(\mathcal{B}_j'\mathcal{P}_j - \mathcal{C}_j')m_j^j(\tau_j)$$
$$= L_j m_j^j(\tau_j) \quad ; \quad j = 1,\ldots,N-1 \tag{5.90.a}$$

where $\mathcal{P}_j$ is a positive definite solution of the following Riccati equation:

$$\mathcal{A}_j'\mathcal{K}_j + \mathcal{K}_j\mathcal{A}_j - \mathcal{K}_j\mathcal{B}_j\mathcal{R}_j^{-1}\mathcal{B}_j'\mathcal{K}_j + \mathcal{Q}_j = 0. \tag{5.90.b}$$

In case of $j = N$:

$$\hat{v}_N(\tau_N) = -R^{-1}B'_{N+1}P_{N+1,N+1}m_N^N(\tau_N)$$

$$= L_N m_N^N(\tau_N). \tag{5.90.c}$$

[σ_j-time-scale problem]

$$\begin{cases} dn_j^j/d\sigma_j = -\mathcal{A}_j n_j^j(\sigma_j) - \mathcal{B}_j w_j(\sigma_j) \\ J_{\sigma_j} = \frac{1}{2}\int_0^\infty [\, n_j^{j\prime}\mathcal{Q}_j n_j^j + w_j'\mathcal{R}_j w_j] d\sigma_j . \end{cases} \tag{5.91}$$

Then, we have

$$\hat{w}_j(\sigma_j) = w_j(\sigma_j) + \mathcal{R}_j^{-1}\mathcal{C}_j' n_j^j(\sigma_j)$$

$$= -\mathcal{R}_j^{-1}(\mathcal{B}_j'\mathcal{N}_j - \mathcal{C}_j')n_j^j(\sigma_j)$$

$$= F_j n_j^j(\sigma_j) \tag{5.92}$$

where $\mathcal{N}_j$ is a negative definite solution of the Riccati algebraic equation (5.90.b).

The mode-decoupled controller (5.84) is a partial-feedback type and the open-loop part is defined by

$$d\tilde{n}_j^j/dt = (\tilde{\mathcal{A}}_j + \tilde{\mathcal{B}}_j\tilde{F}_j)\tilde{n}_j^j(t) \;;\; \tilde{n}_j^j(t_f) = -\tilde{X}_{j+1,0}(t_f) - \sum_{k=1}^{j-1}(\tilde{G}_j^k + \tilde{H}_j^k\tilde{F}_k)\tilde{n}_k^k(t_f),$$

$$\tilde{n}_1^1(t_f) = -\tilde{X}_{20}(t_f). \tag{5.93}$$

It is noted that these control terms can be calculated straightforwards in terms of the original system matrices $\tilde{A}_{ij}$, $\tilde{B}_i$ etc..

We summarize the above results as follows:

<u>Theorem 5-3</u> Let the following conditions be satisfied:

(A-1) $(\tilde{A}_0, \tilde{B}_0)$: completely controllable

(A-2) $(\tilde{\alpha}_j, \tilde{\gamma}_j)$: completely controllable, $j = 2,\ldots,N+1$

(A-3) $(\tilde{C}_j, \tilde{\alpha}_j)$: completely observable, $j = 2,\ldots,N+1$

(A-4) $\tilde{\alpha}_{j+1}$: Hurwitz matrix, $j = 1,\ldots,N$

where

$$\tilde{C}_j'\tilde{H}_j\tilde{C}_j = \tilde{\mathcal{Q}}_{j-1}, \quad \tilde{C}_j = C_j - \hat{C}_{j+1}\tilde{\alpha}_{jj}^{-1}\tilde{\alpha}_{j1}$$

$$\tilde{H}_j = [\,I+(\hat{C}_{j+1}\tilde{\alpha}_{jj}^{-1}\tilde{\beta}_j)R^{-1}(\hat{C}_{j+1}\tilde{\alpha}_{jj}^{-1}\tilde{\beta}_j)'\,]^{-1}$$

$$\begin{bmatrix} \tilde{C}_j'\tilde{C}_j & \tilde{C}_j'\hat{C}_{j+1} \\ \hat{C}_{j+1}'\tilde{C}_j & \hat{C}_{j+1}'\hat{C}_{j+1} \end{bmatrix} = \begin{bmatrix} Q_{jj} & \hat{Q}_{j,j+1} \\ \hat{Q}_{j,j+1}' & \hat{Q}_{j+1,j+1} \end{bmatrix}.$$

Then, the near optimum control is given by (5.84).

For example, in case of two-time-scale systems, the near optimum control has been given already by (5.77).

In case of three-time-scale systems, we obtain

$$\begin{aligned} u^s(t) &= u_j\big|_{j=2} \\ &= \{\tilde{L}_0-\tilde{L}_1(\tilde{M}_{10}+\tilde{N}_{10}\tilde{L}_0)-\tilde{L}_2[(\tilde{M}_{20}+\tilde{N}_{20}\tilde{L}_0)-(\tilde{G}_2^1+\tilde{H}_2^1\tilde{L}_1)(\tilde{M}_{10}+\tilde{N}_{10}\tilde{L}_0)]\}x_1 \\ &\quad + [\tilde{L}_1-\tilde{L}_2(\tilde{G}_2^1+\tilde{H}_2^1\tilde{L}_1)]x_2 + \tilde{L}_2x_3 + [(\tilde{F}_1-\tilde{L}_1)-\tilde{L}_2\tilde{H}_2^1(\tilde{F}_1-\tilde{L}_1)]\tilde{n}_1^1(t) \\ &\quad + (\tilde{F}_2-\tilde{L}_2)\tilde{n}_2^2(t) \end{aligned} \tag{5.94}$$

where
$$\tilde{L}_0 = -\tilde{\mathcal{R}}_0^{-1}(\tilde{\mathcal{B}}_0'\tilde{\mathcal{H}}_0^{-1}(t)-\tilde{\mathcal{C}}_0')$$

$$\tilde{L}_1 = -\tilde{\mathcal{R}}_1^{-1}(\tilde{\mathcal{B}}_1'\tilde{\mathcal{P}}_1-\tilde{\mathcal{C}}_1'), \quad \tilde{F}_1 = -\tilde{\mathcal{R}}_1^{-1}(\tilde{\mathcal{B}}_1'\tilde{\mathcal{N}}_1-\tilde{\mathcal{C}}_1')$$

$$\tilde{L}_2 = -R^{-1}\tilde{B}_3'\tilde{P}_{33}, \quad \tilde{F}_2 = -R^{-1}\tilde{B}_3'\tilde{N}_{33}$$

$$\tilde{G}_2^1 + \tilde{H}_2^1\tilde{L}_1 = -\tilde{A}_{33}^{-1}(\tilde{A}_{32}+\tilde{B}_3\tilde{L}_1)$$

$$\begin{bmatrix} \tilde{M}_{10} + \tilde{N}_{10}\tilde{L}_0 \\ \tilde{M}_{20} + \tilde{N}_{20}\tilde{L}_0 \end{bmatrix} = -\begin{bmatrix} \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{bmatrix} + \begin{bmatrix} \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix} \tilde{L}_0 \right\}$$

and the open-loop parts are defined by

$$\begin{cases} d\tilde{n}_1^1/dt = (\tilde{\mathcal{A}}_1-\tilde{\mathcal{B}}_1\tilde{\mathcal{R}}_1^{-1}\tilde{\mathcal{B}}_1'\tilde{\mathcal{N}}_1)\tilde{n}_1^1(t) \;; & \tilde{n}_1^1(t_f) = -\tilde{X}_{20}(t_f) \\ d\tilde{n}_2^2/dt = (\tilde{A}_{33}-\tilde{B}_3R^{-1}\tilde{B}_3'\tilde{N}_{33})\tilde{n}_2^2(t) \;; & \tilde{n}_2^2(t_f) = -\tilde{X}_{30}(t_f)-(\tilde{G}_2^1+\tilde{H}_2^1\tilde{F}_1)\tilde{n}_1^1(t_f). \end{cases}$$

Thus, once a time-scale hierarchy is identified by applying the modelling algorithms based on norm conditions, the near optimum controllers which can be expressed in terms of original system matrices without involving small parameter ε_i, can be obtained systematically via computer-oriented recursive design procedures.

5.4 Near Optimum Controller Design of Regulator Problem With Multi-Time Scales

5.4.1 General terminal penalty problem

We consider the regulator problem with a general terminal penalty:

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u \quad ; \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{5.95}$$

where $\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\varepsilon & A_{22}/\varepsilon \end{bmatrix} \; ; \; \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2/\varepsilon \end{bmatrix}.$

$$J = \frac{1}{2}[x_1'(t_f), x_2'(t_f)]\tilde{\Pi}\begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} + \frac{1}{2}\int_{t_0}^{t_f} [[x_1', x_2'] Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u' R u] dt \tag{5.96}$$

where $\tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} \\ \tilde{\Pi}_{12}' & \tilde{\Pi}_{22} \end{bmatrix}, \; Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}$: positive semidefinite matrices

R : positive definite matrix.

The special terminal penalty problems have been extensively developed so far by numerous researchers (Kokotovic et al. [Ko.76]). However, in the general terminal problem given above, the situation is more complicated since the Riccati solutions have unbounded terminal conditions (Vasil'eva et al. [Va.78]). Glizer & Dmitriev [Gl.75.1], [Gl.78],[Gl.75.2],[Gl.76] have investigated this problem via the Riccati feedback approach, where the boundary-layer jump phenomenon appears near the terminal-time in the solution $\tilde{K}_{11}$, too. Generally, in the singularly perturbed regulator problems, some degenerated problems whose solutions exhibit slow behaviors of the system can be formulated, i.e., the limit problem, the t-time-scale problem and the natural reduced problem. The relationships among them are briefly discussed in Appendix D. In the general terminal penalty problem, as $\varepsilon \to 0$, an optimal control $u^o(t,\varepsilon)$ converges to the slow control $\hat{U}_0(t)$ composed of the reduced Riccati solutions $H^0_{ij}(t)$ (Glizer et al. [Gl.78]), which is equivalent to the control $U_0(t)$ of the t-time-scale problem. However, it is noted that the control $\hat{U}_0(t)$ (i.e., $U_0(t)$) is not equivalent to the control $\bar{u}(t)$ of the natural reduced problem, which is obtained by setting $\varepsilon = 0$ directly in the original system, except the special case, e.g., $B_2 = 0$ (Glizer et al. [Gl.75.2]). Another special cases have been studied in references [Gl.76],[Dm.78].

In consideration of this situation, on the basis of the decomposition-coordination principle from the temporal viewpoint, the general terminal penalty problem will be investigated in this section.

Now, we consider the following problems with multi-time-scale property:

$$dx_i/dt = \tilde{A}_{ii}x_i + \sum_{\substack{j=1 \\ (j \neq i)}}^{N} \tilde{A}_{ij}x_j + \tilde{B}_i u \quad ; \quad x_i(t_0) = x_{i0}\ , \ i=1,\dots,N+1 \tag{5.97}$$

where $\tilde{A}_{ij} = A_{ij}/(\varepsilon_1 \cdots \varepsilon_{j-1})$, $\tilde{B}_i = B_i/(\varepsilon_1 \cdots \varepsilon_{j-1})$.

$$J = \frac{1}{2}[x_1'(t_f),\dots,x_{N+1}'(t_f)]\begin{bmatrix} \tilde{\Pi}_{11} & \cdots & \tilde{\Pi}_{1,N+1} \\ \vdots & & \vdots \\ \tilde{\Pi}'_{1,N+1} & \cdots & \tilde{\Pi}_{N+1,N+1} \end{bmatrix}\begin{bmatrix} x_1(t_f) \\ \vdots \\ x_{N+1}(t_f) \end{bmatrix}$$

$$+ \frac{1}{2}\int_{t_0}^{t_f}\{[x_1',\dots,x_{N+1}']\begin{bmatrix} Q_{11} & \cdots & Q_{1,N+1} \\ \vdots & & \vdots \\ Q'_{1,N+1} & \cdots & Q_{N+1,N+1} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_{N+1} \end{bmatrix} + u'Ru\}dt. \tag{5.98}$$

On the basis of the design procedure shown in the preceding section, we obtain the following results:

<u>Theorem 5-4</u> Let the following conditions be satisfied:

(A-1) $(\tilde{\alpha}_j, \tilde{\gamma}_j)$: completely controllable, $j = 2,\dots,N+1$

(A-2) $(\tilde{C}_j, \tilde{\alpha}_j)$: completely observable, $j = 2,\dots,N+1$

(A-3) $\tilde{\alpha}_{j+1}$: nonsingular, $j = 1,\ldots,N$

(A-4) $\hat{\Pi}_{j+1}$: nonsingular, $j = 1,\ldots,N$

$$\text{where} \quad \hat{\Pi}_j = \tilde{\Pi}_{jj} - \tilde{\Pi}_{j,row}\tilde{\Pi}_{j+1,mat}^{-1}\tilde{\Pi}'_{j,row}, \quad \hat{\Pi}_{N+1} = \tilde{\Pi}_{N+1,N+1},$$

$$\tilde{\Pi}_{j,row} = (\tilde{\Pi}_{j,j+1}\cdots\tilde{\Pi}_{j,N+1}), \quad \tilde{\Pi}_{j,mat} = \begin{bmatrix} \tilde{\Pi}_{jj} & \cdots & \tilde{\Pi}_{j,N+1} \\ \vdots & & \vdots \\ \tilde{\Pi}'_{j,N+1} & \cdots & \tilde{\Pi}_{N+1,N+1} \end{bmatrix}.$$

Then, the mode-decoupled near optimum control is given by

$$u^s(t) = u_j\big|_{j=N} \quad ; \quad u_j \text{ is defined by (5.85)} \tag{5.100}$$

where $\tilde{L}_0 = -\tilde{\mathcal{R}}_0^{-1}(\tilde{\mathcal{B}}_0'\tilde{\mathcal{K}}_0(t) - \tilde{\mathcal{C}}_0')$ and $\tilde{\mathcal{K}}_0(t)$ is a solution of the Riccati differential equation:

$$d\tilde{\mathcal{K}}_0/dt = -\tilde{\mathcal{K}}_0\tilde{\mathcal{A}}_0 - \tilde{\mathcal{A}}_0'\tilde{\mathcal{K}}_0 + \tilde{\mathcal{K}}_0\tilde{\mathcal{B}}_0\tilde{\mathcal{R}}_0^{-1}\tilde{\mathcal{B}}_0'\tilde{\mathcal{K}}_0 - \tilde{\mathcal{Q}}_0 \; ; \; \tilde{\mathcal{K}}_0(t_f) = \hat{\Pi}_1. \tag{5.101}$$

The open-loop control part of (5.100) is defined by (5.93) and the terminal conditions are modified as follows:

$$\tilde{n}_j^j(t_f) = \tilde{\Gamma}_{j+1}\tilde{X}_{10}(t_f) - \tilde{X}_{j+1,0}(t_f) - \sum_{k=1}^{j-1} \tilde{n}_j^k(t_f) \tag{5.102}$$

where

$$\tilde{n}_j^k(t_f) = (\tilde{G}_j^k + \tilde{H}_j^k\tilde{F}_k)\tilde{n}_k^k(t_f), \quad k = 1,\ldots,j-1$$

$$\begin{bmatrix} \tilde{\Gamma}_2 \\ \vdots \\ \tilde{\Gamma}_{N+1} \end{bmatrix} \tilde{X}_{10}(t_f) = -\tilde{\Pi}_{2,mat}^{-1}\tilde{\Pi}'_{1,row}\tilde{X}_{10}(t_f).$$

For instance, in the case of three-time-scale systems, we have the following results from (5.101) and (5.102):

$$\tilde{\mathcal{K}}_0(t_f) = \tilde{\Pi}_{11} - [\tilde{\Pi}_{12},\tilde{\Pi}_{13}] \begin{bmatrix} \hat{\Pi}_2^{-1}, & -\hat{\Pi}_2^{-1}\tilde{\Pi}_{23}\tilde{\Pi}_{33}^{-1} \\ -\tilde{\Pi}_{33}^{-1}\tilde{\Pi}'_{23}\hat{\Pi}_2^{-1}, & \tilde{\Pi}_{33}^{-1} + \tilde{\Pi}_{33}^{-1}\tilde{\Pi}'_{23}\hat{\Pi}_2^{-1}\tilde{\Pi}_{23}\tilde{\Pi}_{33}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\Pi}'_{12} \\ \tilde{\Pi}'_{13} \end{bmatrix}$$

$$\tilde{n}_1^1(t_f) = (-\hat{\Pi}_2^{-1}\tilde{\Pi}'_{12} + \hat{\Pi}_2^{-1}\tilde{\Pi}_{23}\tilde{\Pi}_{33}^{-1}\tilde{\Pi}'_{13})\tilde{X}_{10}(t_f) - \tilde{X}_{20}(t_f)$$

$$\tilde{n}_2^2(t_f) = [\tilde{\Pi}_{33}^{-1}\tilde{\Pi}'_{23}\hat{\Pi}_2^{-1}\tilde{\Pi}'_{12} + (\tilde{\Pi}_{33}^{-1} + \tilde{\Pi}_{33}^{-1}\tilde{\Pi}'_{23}\hat{\Pi}_2^{-1}\tilde{\Pi}_{23}\tilde{\Pi}_{33}^{-1})\tilde{\Pi}'_{13}]\tilde{X}_{10}(t_f) - \tilde{X}_{30}(t_f) - \tilde{n}_2^1(t_f)$$

where $\tilde{n}_2^1(t_f) = (\tilde{G}_2^1 + \tilde{H}_2^1\tilde{F}_1)\tilde{n}_1^1(t_f) = -\tilde{A}_{33}^{-1}(\tilde{A}_{32} + \tilde{B}_3\tilde{F}_1)\tilde{n}_1(t_f)$

$$\hat{\Pi}_2 = \tilde{\Pi}_{22} - \tilde{\Pi}_{23}\tilde{\Pi}_{33}^{-1}\tilde{\Pi}_{23}'.$$

[Remarks]

1. In the general terminal problems, it is difficult to obtain the Riccati asymptotic expressions of the form: $\tilde{H}_{ij}(t,\varepsilon) + \tilde{P}_{ij}(\tau,\varepsilon)$ due to nonlinearities and singularities at terminal conditions (Valil'eva [Va.78], Glizer et al. [Gl.78]). In this section, instead of obtaining such a pure feedback type, a mode-decoupled partial feedback control has been derived on the basis of the decomposition-coordination principle. The open-loop part functions to correct the final jump phenomena near the terminal time.

2. The optimal control $u^o(t,\varepsilon)$ converges to the control $U_0(t)$ of the t-time-scale problem as $\varepsilon \to 0$. In the general terminal penalty case, the t-time-scale problem is not necessarily equivalent to the natural reduced problem (Glizer et al. [Gl.78]) though both are equivalent in the special terminal penalty case. It seems that this result implies in which process of the controller design the time-scale decomposition should be performed.

5.4.2 Infinite-time control problem

(a) Near optimum control

In this problem, the partial feedback control shown in the preceding section leads to a pure feedback one.

We assume the following conditions (Kando et al. [Ka.80]):

(A-1) $(\tilde{A}_0, \tilde{B}_0)$: completely controllable.

(A-2) $(\tilde{C}_0, \tilde{A}_0)$: completely observable

where $\tilde{C}_0'\tilde{H}_0\tilde{C}_0 = \tilde{Q}_0$, $\tilde{C}_0 = C_1 - \hat{C}_2\tilde{\alpha}_{jj}^{-1}\big|_{j=1}\tilde{\alpha}_{j1}\big|_{j=1}$

$\tilde{H}_0 = [I + (\hat{C}_2\tilde{\alpha}_{jj}^{-1}\big|_{j=1}\tilde{\beta}_j\big|_{j=1})R^{-1}(\hat{C}_2\tilde{\alpha}_{jj}^{-1}\big|_{j=1}\tilde{\beta}_j\big|_{j=1})']^{-1}.$

(A-3) $(\tilde{\alpha}_j, \tilde{\gamma}_j)$: completely controllable, $j = 2,\ldots,N+1$.

(A-4) $(\tilde{C}_j, \tilde{\alpha}_j)$: completely observable, $j = 2,\ldots,N+1$

where $\tilde{C}_j'\tilde{H}_j\tilde{C}_j = \tilde{Q}_{j-1}$, $\tilde{C}_j = C_j - \hat{C}_{j+1}\tilde{\alpha}_{jj}^{-1}\tilde{\alpha}_{j1}$

$\tilde{H}_j = [I + (\hat{C}_{j+1}\tilde{\alpha}_{jj}^{-1}\tilde{\beta}_j)R^{-1}(\hat{C}_{j+1}\tilde{\alpha}_{jj}^{-1}\tilde{\beta}_j)']^{-1}.$

(A-5) $\tilde{\alpha}_{j+1}$: nonsingular, $j = 1,\ldots,N$.

Then, the near optimum control is given by

$$u^s(t) = u_j\big|_{j=N} \tag{5.103}$$

where $\quad u_j = u_{j-i} + \tilde{L}_j[x_{j+1} - (\tilde{M}_{j0}+\tilde{N}_{j0}\tilde{L}_0)x_1 - \sum_{k=1}^{j-1}(\tilde{G}_j^k+\tilde{H}_j^k\tilde{L}_k)\tilde{m}_k^k]$

$\quad u_0 = -\tilde{\mathcal{R}}_0^{-1}(\tilde{\mathcal{B}}_0'\tilde{\mathcal{P}}_0 - \tilde{\mathcal{C}}_0')x_1 = \tilde{L}_0x_1$

and $\tilde{\mathcal{P}}_0$ is a positive definite solution of the Riccati algebraic equation of (5.101)

and $\quad \begin{cases} \tilde{m}_1^1 = x_2 - (\tilde{M}_{10}+\tilde{N}_{10}\tilde{L}_0)x_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \tilde{m}_k^k = x_{k+1} - (\tilde{M}_{k0}+\tilde{N}_{k0}\tilde{L}_0)x_1 - \sum_{h=1}^{k-1}(\tilde{G}_k^h+\tilde{H}_k^h\tilde{L}_h)\tilde{m}_h^h, \quad k = 2,\ldots,N. \end{cases}$

A two-time-scale case has been investigated on the basis of the quasi-steady-state concepts (Chow et al. [Cho.76.2]) also.

(b) Asymptotic stability of a suboptimally controlled system

For simplicity, we consider the case of two-time-scale systems. The suboptimally controlled system with two-time-scale property is given by

$$\begin{cases} dx_1/dt = \hat{A}_{11}x_1 + \hat{A}_{12}x_2 \quad ; \quad x_1(t_0) = x_{10} \\ \varepsilon dx_2/dt = \hat{A}_{21}x_1 + \hat{A}_{22}x_2 \quad ; \quad x_2(t_0) = x_{20} \end{cases} \tag{5.104}$$

where $\quad \hat{A}_{11} = A_{11} - B_1G_1, \quad \hat{A}_{12} = A_{12} - S_{12}P_{22}$

$\quad \hat{A}_{21} = A_{21} - B_2G_1, \quad \hat{A}_{22} = A_{22} - S_{22}P_{22}$

$\quad G_1 = (I-R^{-1}B_2'P_{22}A_{22}^{-1}B_2)\mathcal{R}_0^{-1}(\mathcal{B}_0'\mathcal{P}_0-\mathcal{C}_0')+R^{-1}B_2'P_{22}A_{22}^{-1}A_{21}.$

Noting $\hat{A}_{22}$ is a Hurwitz matrix under the conditions (A-3)-(A-5), and applying the transformation:

$$x_f = x_2 + \hat{L}_0x_1 \quad ; \quad \hat{L}_0 = \hat{A}_{22}^{-1}\hat{A}_{21} \tag{5.105}$$

to (5.104) yields

$$\begin{cases} dx_1/dt = \hat{A}_s x_1 + \hat{A}_{12} x_2 \\ \varepsilon dx_2/dt = \varepsilon \hat{L}_0 \hat{A}_s x_1 + \hat{A}_f x_f \end{cases} \tag{5.106}$$

where $\hat{A}_s = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} = \mathcal{A}_0 - \mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{P}_0$

$\hat{A}_f = \hat{A}_{22} + \varepsilon\hat{L}_0\hat{A}_{12}$.

Here, by following the vector Lyapunov approach (Bernussou et al. [Be.82], Šiljak [Sil.78]), we will investigate the asymptotic stability of (5.106). As Lyapunov functions of decoupled systems:

$$\begin{cases} dx_1 dt = \hat{A}_s x_1 & (5.107.a) \\ \varepsilon dx_f/dt = \hat{A}_{22} x_f & (5.107.b) \end{cases}$$

we choose the following:

$$\begin{cases} v_s(x_1) = (\ x_1' P_s x_1\ )^{1/2} & (5.108.a) \\ v_f(x_f) = (\ x_f' P_f x_f\ )^{1/2} & (5.108.b) \end{cases}$$

where $(\mathcal{A}_0 - \mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{P}_0)' P_s + P_s(\mathcal{A}_0 - \mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{P}_0) + D_s' D_s = 0$

$(A_{22} - S_{22}P_{22})' P_f + P_f(A_{22} - S_{22}P_{22}) + D_f' D_f = 0.$

Applying the estimates:

$$\begin{cases} \lambda_m(P_s)^{1/2}\|x_1\| \le v_s \le \lambda_M(P_s)^{1/2}\|x_1\| & (5.109.a) \\ \lambda_m(P_f)^{1/2}\|x_f\| \le v_f \le \lambda_M(P_f)^{1/2}\|x_f\| & (5.109.b) \end{cases}$$

$$\begin{cases} \text{grad } v_s \le \lambda_M(P_s)/\lambda_m(P_s)^{1/2} & (5.109.c) \\ \text{grad } v_f \le \lambda_M(P_f)/\lambda_m(P_f)^{1/2} & (5.109.d) \end{cases}$$

where $\lambda_M(A) = \max \lambda(A)$, $\lambda_m(A) = \min \lambda(A)$

yields the following differential inequality:

$$\begin{bmatrix} dv_s/dt \\ \varepsilon dv_f/dt \end{bmatrix} \le \begin{bmatrix} -\alpha_{11} & \alpha_{12} \\ \varepsilon\alpha_{21} & -\alpha_{22} + \varepsilon\beta_{22} \end{bmatrix} \begin{bmatrix} v_s \\ v_f \end{bmatrix} \tag{5.110}$$

where $\alpha_{11} = \frac{1}{2}\lambda_m(D_s D_s P_s^{-1})$, $\quad \alpha_{12} = \dfrac{\lambda_M(P_s)}{\lambda_m(P_s)^{1/2}\lambda_m(P_f)^{1/2}}\|\hat{A}_{12}\|$

$$\alpha_{21} = \frac{\lambda_M(P_f)}{\lambda_m(P_s)^{1/2}\lambda_m(P_f)^{1/2}}\|\hat{L}_0\|\|\hat{A}_s\|, \quad \alpha_{22} = \frac{1}{2}\lambda_m(D_f' D_f P_f^{-1})$$

$$\beta_{22} = \frac{\lambda_M(P_f)}{\lambda_m(P_f)}\|\hat{L}_0\|\|\hat{A}_{12}\|.$$

By using a stability condition for a Metzler matrix, i.e., a Hicks condition (Šiljak [Sil.78]), we obtain

$$0 \le \varepsilon < \frac{1}{\left(\dfrac{\beta_{22}}{\alpha_{22}} + \dfrac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)} = \varepsilon^* \tag{5.111}$$

where
$$\varepsilon^* = \frac{1}{\dfrac{2\lambda_M(P_f)}{\lambda_m(P_f)\lambda_m(D_f' D_f P_f^{-1})}\left[\|\hat{L}_0\|\|\hat{A}_{12}\| + 2\dfrac{\lambda_M(P_s)\ \|\hat{A}_{12}\|\|\hat{A}_s\|\|\hat{L}_0\|}{\lambda_m(P_s)\lambda_m(D_s' D_s P_s^{-1})}\right]}$$

$$P_s = \mathcal{P}_0, \quad D_s' D_s = \mathcal{Q}_0 + \mathcal{P}_0\mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{P}_0$$

$$P_f = P_{22}, \quad D_f' D_f = Q_{22} + P_{22}S_{22}P_{22}$$

which gives an upper bound of ε such that the suboptimally controlled system (5.104) is asymptotically stable. If necessary, this estimate can be improved via parametric optimization techniques (Geromel et al. [Geo.82], Soliman et al. [So.82]).

5.4.3 Performance degradation

For simplicity, we consider the two-time-scale case (5.95) and (5.96).

Under the conditions given in Theorem 5-4, this general terminal problem has a unique asymptotic solution for ε sufficiently small throughout $t_0 \le t \le t_f$:

$$\begin{cases} u^o(t,\varepsilon) = U_0(t) + v_0(\tau) + w_0(\sigma) + O(\varepsilon) \\ x_1^o(t,\varepsilon) = X_{10}(t) + O(\varepsilon) \\ x_2^o(t,\varepsilon) = X_{20}(t) + m_0(\tau) + n_0(\sigma) + O(\varepsilon). \end{cases} \tag{5.112}$$

Here, the terms $[U_0(t),\ X_{10}(t),\ X_{20}(t)]$, $[v_0(\tau),\ m_0(\tau)]$, $[w_0(\sigma),\ n_0(\sigma)]$ satisfy the t-time-scale problem, the τ-time-scale problem and the σ-time-scale problem, respectively:

$$\begin{cases} dX_{10}/dt = (\mathcal{A}_0 - \mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{K}_0)X_{10}(t) \quad ; \quad X_{10}(t_0) = x_{10}, & (5.113.a) \\ X_{20}(t) = -A_{22}^{-1}[A_{21} + S_{22}\theta^{-1}(Q_{12}' - Q_{22}A_{22}^{-1}A_{21})]X_{10}(t) + & \\ \qquad A_{22}^{-1}[S_{12}' - S_{22}\theta^{-1}(A_{12}' + Q_{22}A_{22}^{-1}S_{12}')]\mathcal{K}_0(t)X_{10}(t) & (5.113.b) \end{cases}$$

$$dm_0/d\tau = (A_{22} - S_{22}P_{22})m_0(\tau) \quad ; \quad m_0(0) = x_{20} - X_{20}(t_0) \tag{5.114}$$

$$dn_0/d\sigma = -(A_{22} - S_{22}N_{22})n_0(\sigma) \quad ; \quad n_0(0) = -\Pi_{22}^{-1}\Pi_{12}'X_{10}(t_f) - X_{20}(t_f). \tag{5.115}$$

When the partial feedback control

$$u^s = -(ML_0 + R^{-1}B_2'P_{22}A_{22}^{-1}A_{21})x_1 - R^{-1}B_2'P_{22}x_2 - R^{-1}B_2'(N_{22}-P_{22})n_0(\sigma) \tag{5.116}$$

where $M = I - R^{-1}B_2'P_{22}A_{22}^{-1}B_2$, $L_0 = \mathcal{R}_0^{-1}(\mathcal{B}_0'\mathcal{K}_0(t) - \mathcal{C}_0')$

is implemented, the suboptimally controlled system is given by

$$\begin{cases} dx_1^s/dt = [A_{11}-B_1(ML_0+R^{-1}B_2'P_{22}A_{22}^{-1}A_{21})]x_1^s+[A_{12}-S_{12}P_{22}]x_2^s-S_{12}(N_{22}-P_{22})n_0(\sigma) \\ \varepsilon dx_2^s/dt = [A_{21}-B_2(ML_0+R^{-1}B_2'P_{22}A_{22}^{-1}A_{21})]x_1^s+[A_{22}-S_{22}P_{22}]x_2^s-S_{22}(N_{22}-P_{22})n_0(\sigma) \end{cases}$$

$$; \quad x_1^s(t_0)=x_{10},\ x_2^s(t_0)=x_{20}. \tag{5.117}$$

Here, we shall set asymptotic series solutions of (5.117) formally:

$$\begin{cases} x_1^s(t,\varepsilon) = \hat{X}_{10}(t) + O(\varepsilon) & ;\tau = (t-t_0)/\varepsilon \\ x_2^s(t,\varepsilon) = \hat{X}_{20}(t) + \hat{m}_0(\tau) + \hat{n}_0(\sigma) + O(\varepsilon) & ;\sigma = (t_f-t)/\varepsilon. \end{cases} \tag{5.118}$$

Then, after algebraic manipulations, the following relations between (5.112) and (5.118) can be proved (Kando et al. [Ka.83.1]):

$$\hat{X}_{10}(t) = X_{10}(t),\quad \hat{X}_{20}(t) = X_{20}(t),\quad \hat{m}_0(\tau) = m_0(\tau),\quad \hat{n}_0(\sigma) = n_0(\sigma). \tag{5.119}$$

Thus, we obtain the differences:

$$x_1^o(t,\varepsilon) - x_1^s(t,\varepsilon) = O(\varepsilon) \tag{5.120.a}$$

$$x_2^o(t,\varepsilon) - x_2^s(t,\varepsilon) = O(\varepsilon) \qquad (5.120.b)$$

$$u^o(t,\varepsilon) - u^s(t,\varepsilon) = O(\varepsilon). \qquad (5.120.c)$$

Then, by applying Hadlock's results [Hal.77] under the additional hypothesis $\ulcorner A_{22}$: Hurwitz matrix$\lrcorner$, the performance degradation is estimated by:

$$J^o - J^s = O(\varepsilon^2). \qquad (5.121)$$

In the case of the infinite-time control case, the above estimation can be derived directly by using the Lyapunov equation, too.

Similarly as above, in the case of the fixed-endpoint problem, we can prove the estimation (5.121) as the performance degradation.

In the multi-time-scale systems, by using the boundary layer method with multi-parameters (O'Malley [OM.74],[OM.71.2]), we can obtain

$$J^o - J^s = O(\varepsilon_M^2) \ ; \quad \varepsilon_M = \max[\varepsilon_1, \ldots, \varepsilon_N] \qquad (5.122)$$

where the time-scale separation ratio ε_i is defined in Section 5.3.2.

[Remarks]

On the basis of modelling procedures and computer-oriented decomposition techniques for model reduction controller design, mode-decoupled near optimum controllers of multi-time-scale systems can be designed systematically, which can be expressed in terms of the original matrices $\tilde{A}_{ij}$, $\tilde{B}_i$ etc. without including ε explicitly and without depending on the scaling factor. This design method needs computational labors of time-scale modelling but does not demand computational iterations for convergence such as the hierarchical control methods (Singh et al. [Sin.76], Mahmoud [Mam.79], Lehtomaki et al. [Leh.77], Gruver et al. [Gr.80]).

5.4.4 Numerical example

We consider a reactor model with temperature feedback. The one-group neutron kinetic equations are given by

$$\begin{cases} dn/dt = (\delta k - \beta)/l \cdot n + \lambda c \\ dc/dt = \beta/l \cdot n - \lambda c \end{cases} \tag{5.123}$$

where n : neutron density

c : concentration of delayed neutron precursors.

The heat transfer equations are given by

$$\begin{cases} M_f dT_f/dt = Q_0 n - 2h(T_f - T_m) \\ M_m dT_m/dt = 2h(T_f - T_m) - 2M_m(1/\tau_0)T_m \end{cases} \tag{5.124}$$

where T_f ; mean temperature of fuel, T_m : mean temperature of water. The total reactivity is the sum of the contribution provided by the tempertature feedback of fuel and of moderator, and by the control rod action:

$$\delta k = \delta k_r - \alpha_f T_f - \alpha_m T_m. \tag{5.125}$$

The other notations follow the conventional practice (Schultz [Sc.61]). Linearizing and normalizing the above equations with respect to the desired values gives

$$\begin{cases} dx_1/dt = a_{11}x_1 + a_{14}x_4 \\ dx_2/dt = a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ dx_3/dt = a_{32}x_2 + a_{33}x_3 \\ dx_4/dt = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + bu \end{cases} \tag{5.126}$$

where $a_{11} = -a_{14} = -\lambda$

$$a_{22} = -1/\tau_1, \quad a_{23} = \frac{\tau_0}{(\tau_0+2\tau_2)\tau_1}, \quad a_{24} = \frac{M_m}{(\tau_0+2\tau_2)M_f}$$

$$a_{32} = -a_{33} = 1/\tau_2', \qquad a_{41} = -a_{44} = \beta/l$$

$$a_{42} = -\frac{\alpha_f}{l}(\tau_0/2+\tau_2)\frac{1}{M_m} Q_0 n^d, \qquad a_{43} = -\frac{\alpha_m}{l}(\tau_0/2)\frac{1}{M_m} Q_0 n^d$$

$$b = 1/l$$

$$x_1 = \delta c/c^d,\ x_2 = \delta T_f/T_f^d,\ x_3 = \delta T_m/T_m^d,\ x_4 = \delta n/n^d,\ u = \delta k_r$$

$$\tau_0 = L/v,\ \tau_1 = M_f/2h,\ \tau_2 = M_m/2h,\ \tau_2' = \tau_0\tau_2/(\tau_0+2\tau_2).$$

The performance index is given by

$$J = \frac{1}{2}X'(t_f)\Pi X(t_f) + \frac{1}{2}\int_0^{t_f}[X'QX + u'Ru]dt. \tag{5.127}$$

The following data are used (Kando et al. [Ka.71]):

$$\begin{array}{lll} \beta = 0.0065, & M_f = 0.01, \ \alpha_f = 2\times10^{-5}, \ Q_0 = 0.54, & \tau_0 = 2 \\ l = 10^{-3}, & M_m = 0.15, \ \alpha_m = 1\times10^{-4}, & \tau_1 = 1 \\ \lambda = 0.077, & h = 0.005, & \tau_2 = 15. \end{array}$$

[I] Two-time-scale case

Firstly, by following the modelling procedure studied in Section 2.3, it must be verified whether the norm conditions are satisfied or not.

Grouping the state variables into $n_1 = 1$ and $n_2 = 3$ yields

(H-1) $\max[0.3589,\ 0.3293]<1$, or $0\le\varepsilon_1 < \min[0.0591,\ 0.0644]=\varepsilon_1^*$

(H-2) $\varepsilon_1 = \|\tilde{A}_{22}^{-1}\|\,\|\tilde{A}_0\| = 0.0212<<1$

(H-3) $\|\tilde{A}_{22}^{-1}\|\,\|\tilde{A}_{12}\| = 0.1126<<1$

where $\|\tilde{A}_{22}^{-1}\| = 1.4621$, $\|\tilde{A}_0\| = 0.01453$, $\|\tilde{A}_{12}\| = 0.077$, $\|\tilde{L}_0\| = 1.4052$.
The exact eigenvalues are

$$\lambda(\tilde{A}) = \{-0.01465,\ -1.0237,\ -1.2143,\ -6.3910\} \tag{5.128.a}$$

and the approximate eigenvalues are

$$\begin{cases} \lambda(\tilde{A}_0) = -0.01453 \\ \lambda(\tilde{A}_{22}) = \{-1.0633,\ -1.1943,\ -6.309\}. \end{cases} \tag{5.128.b}$$

Thus, a nuclear model possesses a strong singularly perturbed structure.

By following the results shown in Section 5.4.2, the near optimum control and cost of the infinite-time control problem were calculated as follows:

$$\begin{cases} u^s(t) = -0.60874\times10^{-2}x_1 - 0.24239\times10^{-2}x_2 - 0.860889\times10^{-3}x_3 \\ \qquad\qquad - 0.33384\times10^{-2}x_4 & (5.129) \\ J^s = 0.6778384\times10^{-1} & (5.130) \end{cases}$$

where the weight matrices $Q = I$ and $R = 2\times10^4$ were used.

On the other hand, the optimal ones were given by

$$\begin{cases} u^o(t) = -0.6106\times10^{-2}x_1 - 0.23753\times10^{-2}x_2 - 0.84733\times10^{-3}x_3 \\ \qquad\qquad - 0.33384\times10^{-2}x_4 & (5.131) \\ J^o = 0.6778377\times10^{-1}. & (5.132) \end{cases}$$

Furthermore, an upper bound ε^* for the asymptotic stability was given by

$$0\leq\varepsilon=0.0212<\varepsilon^*=0.1604. \qquad (5.133)$$

Figure 5.2 shows the optimal and suboptimal responses.

[II] Three-time-scale case

Secondly, the validity of norm conditions for multi-time-scale systems shown in Section 5.3.2 was tested.

Grouping the states into $n_1 = 1$, $n_2 = 2$ and $n_3 = 1$ yields

(H-1)$'$ $\max[0.5998,\ 0.4950]<1$, or $0<\varepsilon_2<\min[0.4553,\ 0.5517]=\varepsilon_2^*$

(H-2)$'$ $\varepsilon_2 = \|\tilde{\alpha}_{jj}^{-1}\|\|\tilde{\alpha}_j\| = 0.2731$; $j = 2$

(H-3)$'$ $\|\tilde{\alpha}_{jj}^{-1}\|\|\tilde{\alpha}_{1j}\| = 0.1442$; $j = 2$

where $\|\tilde{\alpha}_{jj}^{-1}\| = 0.1538$, $\|\tilde{\alpha}_j\| = 1.7759$, $\|\tilde{\alpha}_{1j}\| = 0.9375$, $\|\tilde{\alpha}_{jj}^{-1}\tilde{\alpha}_{j1}\| = 0.1857$; $j = 2$.

The exact eigenvalues has been given in (5.128.a) and the approximate

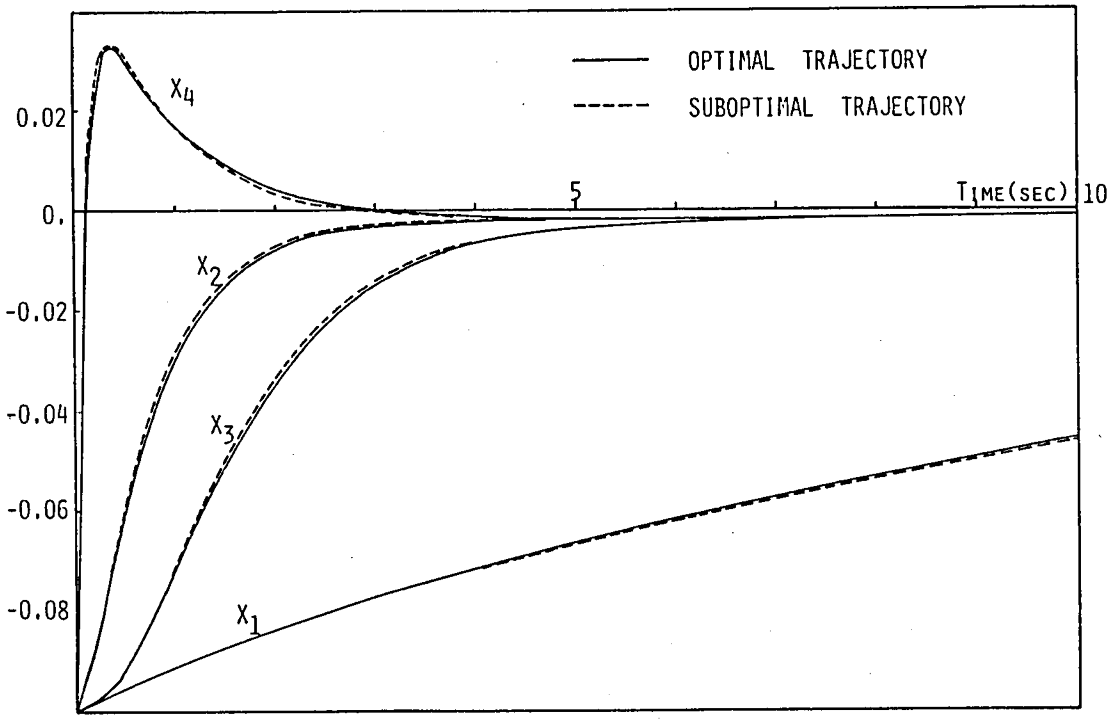

Fig. 5.2. Optimal and suboptimal responses.

(Two-time-scale case)

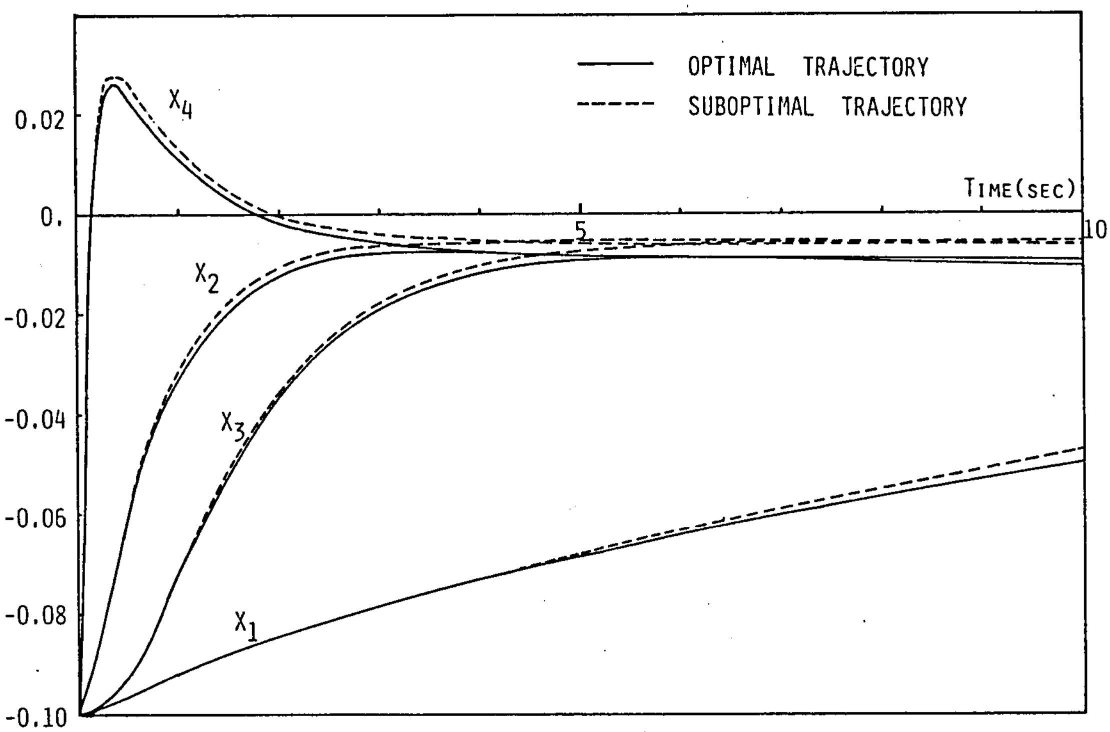

Fig. 5.3. Optimal and suboptimal responses.

(Three-time-scale case)

eigenvalues are

$$\begin{cases} \lambda(\tilde{\alpha}_1) = -0.01453 \\ \lambda(\tilde{\alpha}_2) = -0.9991,\ -1.2334 \\ \lambda(\tilde{\alpha}_3) = -6.5. \end{cases} \tag{5.134}$$

Thus, the nuclear reactor model exhibits a three-time-scale behavior too. By following the results for the general terminal penalty case, the near optimum controller of partial-feedback type was calculated. The corresponding responses are shown in Fig. 5.3. Here, the weight matrices: $\tilde{\Pi} = \text{diag.}[1,1,1,0.4]$, $Q = I$, $R = 2\times10^4$ were used.

Furthermore, the suboptimal and optimal costs were calculated as follows:

$$J^s = 0.5620547\times10^{-1} \tag{5.135}$$

$$J^o = 0.5528956\times10^{-1}. \tag{5.136}$$

These results indicate the effectiveness of near optimum controller design studied in this chapter.

5.5 Concluding Remarks

As indicated by the general terminal penalty case of the regulator problem, it is an important problem to investigate in which design process we should implement the time-scale decomposition. If we use the temporal decomposition-coordination approach while paying attention to this problem, it provides us a useful tool for the design problems of the multi-time-scale systems formulated on the basis of the modelling concepts and algorithms shown in Chapter 2 and Section 5.3.2.

In this chapter, various linear quadratic problems for the continuous-time versions, i.e., the control problem of the system having the special terminal conditions such as a power level change of a nuclear reactor, the fixed-endpoint problem, and the regulator problem with the general terminal penalty have been considered by applying this decomposition-coordination approach which has the similar theoretical framework to the spatial decomposition-coordination approach used in the hierarchical control methods. The design procedures of the mode-decoupled near optimum controls have been developed, and some sufficient conditions for their validity have been studied. Firstly, the two-time-scale cases have been dealt with, and then these results have been extended to the multi-time-scale cases. As these results, the computer-oriented and the systematic decomposition procedures for the reduced order controller design have been given.

Thus, an application of the temporal decomposition-coordination approach to the high dimensional and the ill-conditioned two-point-boundary-value problems brings about the simplification of the problem structure as well as the reduced order controller design, which enable us to reduce the computational labors and to alleviate the ill-conditioning.

CHAPTER 6. DISCUSSION AND CONCLUSIONS

By following multi-time-scale modelling concepts and computer-aided algorithms which are based on structural properties of singularly perturbed systems, the time-scale hierarchical structure involved in large-scale systems is identified. Then, after the global problem with multi-time-scale property is decomposed into some local problems which are defined at respective time-scales, they are solved independently, and thier solutions are combined together in a proper way so as to approximate global solutions asymptotically. Within such a theoretical framework of a decomposition-coordination principle which has a common ground with other theoretical methods for large-scale system design and analysis, e.g., the model reduction methods and the hierachical control methods, various problems associated with modelling and control of continuous-time and discrete-time systems have been tackled not from a spacial but from a temporal viewpoint.

In this chapter, the results obtained are briefly summarized and additional remarks with respect to some subjects which deserve further research are also given.

In Chapter 2, the problem of singular perturbation modelling has been considered along with discussions for formulations of singularly perturbed discrete-time versions. Modelling concepts and computer-oriented algorithms based on norm criterions to test a validity of modelling have been investigated. It is noted that the time-scale decomposition methods such as the subspace iteration method and the q.s.s. method are concepts based on coordinates transformations, but the singular perturbation method is not necessarily coordinate-free concept. As the result, all of two-time-scale systems can not be always identified with the singularly perturbed forms, e.g., the large-scale systems with high frequency oscillations. The modelling concepts and algorithms shown in this chapter give us the useful criterion to verify the validity of this singular perturbation modelling, and provide us the beneficial data to predict an accuracy of solutions derived in the following system design and analysis phase. However, since the modelling concepts developed here are based on only eigenvalue gaps, which cause the multi-time-scale behaviors, it cannot be applied directly to nonlinear systems or time-varying systems. Modelling concepts and systematic algorithms for these systems as well as high oscillation systems must be further investigated.

In Chapter 3, on the basis of these modelling results, the boundary layer method for discrete-time versions has been formulated, which gives an equivalent asymptotic series solution to one via the iterative time-scale decomposition methods. This method provides us a useful tool for controller design procedures of the slow-time-scale version treated in the following chapter.

The studies for discrete-time systems with two-time-scale property can be categorized into two groups, i.e., the sampled control systems and the discrete-time systems represented as difference formulas essentially. In Chapter 4, various design problems for the sampled control systems have been considered. These studies are classified into two cases, i.e., the single rate sampling case and the multirate sampling case. Furthermore, the single rate case is divided into the slow-time-scale version and the fast-time-scale version via a selection of sampling periods.

In Section 4.2 - 4.5, the former version has been treated. This version coincides with the state space model corresponding to the singularly perturbed difference equation, and thus the studies discussed in these sections can be applied to design and analysis of the essential discrete-time systems such as an economic and an ecology models as well as the sampled control systems. In this version, the controller design has been performed from a viewpoint of slow-time-scale, where the two-time-scale property of discrete-time system itself is utilized. This design policy provides us a composite solution of both discrete-time slow and fast subsystems. However, the fast mode is treated as dead beat, and this fact is not preferable from a viewpoint of the digital redesign problem (Kuo [Kuo.81]) because the performance degradation between the original continuous-time problem and the corresponding slow-time-scale version becomes a serious problem in some cases.

On the other hand, in Section 4.6, the latter version has been studied, where the system behavior has been analyzed from a viewpoint of fast-time-scale and as the result, the two-time-scale separation of eigenvalues is not preserved in this version. The solutions of the global system can be expressed as a hybrid combination of the continuous-time slow subsystem which dominates the system behavior over the whole interval, and the discrete-time fast subsystem which functions only near the initial time. Thus, a main part of the controller design is essentially performed in the continuous-time domain. In this version, the time delay due to the computation time of the control law cannot be neglected because of the short sampling period, and the control law counting it is required anew (Mita [Mt.85]).

In the last section, 4.7, the relationships between the original continuous-time problem and the corresponding two single rate sampling problems have been clarified, and on the basis of these studies, the multirate controller design procedure has been developed within a framework of the decomposition-coordination principle, and the usefulness of these results has been discussed from a viewpoint of the digital redesign problem. These multirate controller design procedures enable us not only to reduce the computational labors required in the derivation of the feedback gains and the determination of the sampling rates, but also to alleviate the various difficulties resulting in the single rate controller design. Several subjects which deserve further reseach are the extension of the two sample rate case to the general multirate case, and the multirate Kalman filter design problem.

In Chapter 5, a variety of linear quadratic problems of continuous-time systems with multi-time-scale property have been considered by applying the temporal decomposition-coordination approach, which is in contrast to the spatial decomposition-coordination approach used in the hierarchical control methods. As indicated by the general terminal penalty case of the regulator problem, it is an important subject to investigate in which design process we should implement the time-scale decomposition. While taking notice of this subject, the near optimum controller design procedures have been developed, and some sufficient conditions for their validity have been studied. Once the time-scale structure involved in large-scale systems is identified via the multi-time-scale modelling concepts and algorithms shown in Chapter 2 and Section 5.3.2, the mode-decoupled controllers can be obtained systematically on the basis of the computer-oriented recursive design algorithms. For the convenience of an application of these results, an equipment of software for the multi-time-scale modelling algorithms and for these recursive controller design algorithms will be required.

Thus, it has been recognized anew that the singular perturbations and the multi-time-scale methods bring about the simplification of the problem structure as well as the reduced order controller design, which enable us to reduce the computational labors and to alleviate the ill-conditioning, and these methods have proved useful for studies on the broad categories of modelling, design and analysis of large-scale systems in both continuous-time and discrete-time domains.

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APPENDICES

Appendix A. Asymptotic Series Expansions and Riccati Iterative Solutions of NARE

We consider the following NARE:

$$\varepsilon A_{22}L - LA_{11} + \varepsilon LA_{12}L - A_{21} = 0. \tag{A.1}$$

Under the assumption that A_{11} is nonsingular, by applying the implicit function theorem, we see that the solution $L(\varepsilon)$ of (A.1) is analytic in ε at $\varepsilon = 0$, i.e.,

$$L(\varepsilon) = \sum_{j=0}^{k} \hat{l}_j \varepsilon^j + O(\varepsilon^{k+1}) \quad ; \quad \hat{l}_0 = -A_{21}A_{11}^{-1}. \tag{A.2}$$

By equating coefficients of ε^k for $k > 0$, we have

$$\hat{l}_{k+1} = [A_{22}\hat{l}_k + \hat{l}_0 A_{12}\hat{l}_k + \hat{l}_1 A_{12}\hat{l}_{k-1} + \dots + \hat{l}_k A_{12}\hat{l}_0]A_{11}^{-1}. \tag{A.3}$$

On the other hand, we will consider the Riccati iteration:

$$L_{k+1} = (\varepsilon A_{22}L_k + \varepsilon L_k A_{12}L_k - A_{21})A_{11}^{-1} \quad ; \quad L_0 = -A_{21}A_{11}^{-1}. \tag{A.4}$$

Then, the k-th iteration L_k can be written as

$$L_k = \sum_{j=0}^{k} l_j \varepsilon^j + (\varepsilon^{k+1}) \quad ; \quad l_0 = -A_{21}A_{11}^{-1}. \tag{A.5}$$

Here, by using the mathematical induction, we will prove the following relation:

$$\hat{l}_k = l_k \quad ; \quad k = 0,1,\dots \tag{A.6}$$

Firstly, in case of $k = 0$, we have $\hat{l}_0 = l_0 = -A_{21}A_{11}^{-1}$.
For $k \le i$, we assume

$$\begin{cases} L_k = \sum_{j=0}^{k} l_j \varepsilon^j + O(\varepsilon^{k+1}) \\ l_k = [A_{22}l_{k-1} + l_0 A_{12}l_{k-1} + l_1 A_{12}l_{k-2} + \dots + l_{k-1}A_{12}l_0]A_{11}^{-1}. \end{cases} \tag{A.7}$$

Then, in order to obtain L_{i+1}, by substituting (A.7) into (A.4) and by arranging, we have

$$
\begin{aligned}
L_{i+1} &= (\varepsilon A_{22} L_i + \varepsilon L_i A_{12} L_i - A_{21}) A_{11}^{-1} \\
&= \sum_{j=0}^{i} l_j \varepsilon^j + \varepsilon^{i+1} (A_{22} l_i + l_0 A_{12} l_i + l_1 A_{12} l_{i-1} + \cdots \\
&\qquad + l_i A_{12} l_0) A_{11}^{-1} + O(\varepsilon^{i+2}). \qquad (A.8)
\end{aligned}
$$

Here, by setting as

$$l_{i+1} = [A_{22} l_i + l_0 A_{12} l_i + l_1 A_{12} l_{i-1} + \cdots + l_i A_{12} l_0] A_{11}^{-1} \qquad (A.9)$$

we obtain

$$L_{i+1} = \sum_{j=0}^{i} l_j \varepsilon^j + \varepsilon^{i+1} l_{i+1} + O(\varepsilon^{i+2}). \qquad (A.10)$$

Accordingly, in case of $k = i+1$, (A.7) is satisfied too. Thus, we have proved the relation (A.6).

Similarly as above, we can prove the following relation:

$$\hat{h}_k = h_k \quad ; \quad k = 0,1,\ldots \qquad (A.11)$$

where the solution $H(\varepsilon)$ of the Sylvester equation (3.4) is represented as

$$H_k = \sum_{j=0}^{k} \hat{h}_j \varepsilon^j + O(\varepsilon^{k+1}). \qquad (A.12)$$

The k-th iterative solution H_k of the iterative scheme:

$$H_{k+1} = A_{11}^{-1} [\varepsilon H_k (A_{22} + LA_{12}) + \varepsilon A_{12} L H_k + A_{12}] \qquad (A.13)$$

can be written as

$$H_{k+1} = \sum_{j=0}^{k} h_j \varepsilon^j + O(\varepsilon^{k+1}) \qquad (A.14)$$

where

$$h_k = A_{11}^{-1} h_{k-1} (A_{22} + LA_{12}) + h_0 L h_{k-1}.$$

Thus, the iterative schemes (A.4) and (A.13) seem to be excellent algorithms for the purpose of computing the $O(\varepsilon^{k+1})$ approximate solutions of the mode-decoupling transformation for discrete-time versions.

Appendix B. The Boundary Layer Method and the Dichotomy Transformation

We consider the following equation:

$$\begin{cases} \varepsilon dz_2/dt = A_{22}z_2 + B_2u & \text{(B.1)} \\ J = \frac{1}{2}\int_{t_0}^{t_f}(z_2'Q_{22}z_2 + u'Ru)dt. & \text{(B.2)} \end{cases}$$

The boundary conditions are

$$\begin{cases} z_2(t_0,\varepsilon) = c_1(\varepsilon) = x_2(\varepsilon) - X_2(t_0,\varepsilon) & \text{(B.3.a)} \\ dz_2/dt(t_f,\varepsilon) = c_2(\varepsilon) = -dX_2/dt(t_f,\varepsilon). & \text{(B.3.b)} \end{cases}$$

Then, we have the canonical equations:

$$\begin{cases} \varepsilon dz_2/dt = A_{22}z_2 - S_{22}p_2 \\ \varepsilon dp_2/dt = -Q_{22}z_2 - A_{22}'p_2. \end{cases} \qquad \text{(B.4)}$$

By using the boundary layer method, we will seek solutions of the following form:

$$\begin{cases} z_2(t,\varepsilon) = Z_2(t,\varepsilon) + m_2(\tau,\varepsilon) + \varepsilon n_2(\sigma,\varepsilon) & ; \ \tau = (t-t_0)/\varepsilon \\ p_2(t,\varepsilon) = P_2(t,\varepsilon) + \rho_2(\tau,\varepsilon) + \varepsilon \nu_2(\sigma,\varepsilon) & ; \ \sigma = (t-t_f)/\varepsilon \end{cases} \qquad \text{(B.5)}$$

where the respective terms have asymptotic series expansions in ε.

Under the condition (H-2) given in Section 5.2, we obtain

$$Z_2(t,\varepsilon) = 0, \quad P_2(t,\varepsilon) = 0. \qquad \text{(B.6)}$$

Accordingly, we have

$$\begin{cases} z_2(t,\varepsilon) = \sum_{j=0}^{\infty} m_{2j}(\tau)\varepsilon^j + \varepsilon\sum_{j=0}^{\infty} n_{2j}(\sigma)\varepsilon^j \\ p_2(t,\varepsilon) = \sum_{j=0}^{\infty} \rho_{2j}(\tau)\varepsilon^j + \varepsilon\sum_{j=0}^{\infty} \nu_{2j}(\sigma)\varepsilon^j \end{cases} \qquad \text{(B.7)}$$

where $\begin{bmatrix} m_{20}(\tau) \\ \rho_{20}(\tau) \end{bmatrix} = (5.32)$, $\begin{bmatrix} m_{2j}(\tau) \\ \rho_{2j}(\tau) \end{bmatrix} = \begin{bmatrix} T_{11}e^{\Lambda\tau}T_{11}^{-1} \\ T_{21}e^{\Lambda\tau}T_{11}^{-1} \end{bmatrix}$; $m_{2j}(0) = d^jc_1/d\varepsilon^j\big|_{\varepsilon=0}$ (B.8.a)

$\begin{bmatrix} n_{20}(\sigma) \\ \nu_{20}(\sigma) \end{bmatrix} = (5.37)$, $\begin{bmatrix} n_{2j}(\sigma) \\ \nu_{2j}(\sigma) \end{bmatrix} = \begin{bmatrix} T_{12}e^{\Lambda\sigma}T_{12}^{-1} \\ T_{22}e^{\Lambda\sigma}T_{12}^{-1} \end{bmatrix}$; $dn_{2j}/d\sigma(0) = d^jc_2/d\varepsilon^j\big|_{\varepsilon=0}$. (B.8.b)

On the other hand, by applying the dichotomy transformation (Wilde [Wi.72]):

$$\begin{bmatrix} z_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} I & I \\ P & N \end{bmatrix} \begin{bmatrix} y_2 \\ q_2 \end{bmatrix} \tag{B.9}$$

where P and N are positive and negative definite solutions of the Riccati algebraic equation:

$$KA_{22} + A_{22}'K - KS_{22}K + Q_{22} = 0 \tag{B.10}$$

to (B.4), we have

$$\begin{bmatrix} \varepsilon\dot{y}_2 \\ \varepsilon\dot{q}_2 \end{bmatrix} = \begin{bmatrix} A_{22} - S_{22}P & 0 \\ 0 & A_{22} - S_{22}N \end{bmatrix} \begin{bmatrix} y_2 \\ q_2 \end{bmatrix} \quad \begin{matrix} ; & y_2(t_0,\varepsilon) = c_1(\varepsilon) \\ ; & dq_2/dt(t_0,\varepsilon) = c_2(\varepsilon). \end{matrix} \tag{B.11}$$

From (B.11), we obtain

$$\begin{cases} y_2(t,\varepsilon) = l_2(\tau,\varepsilon) & ; \ \tau = (t-t_0)/\varepsilon \\ q_2(t,\varepsilon) = \varepsilon r_2(\sigma,\varepsilon) & ; \ \sigma = (t-t_f)/\varepsilon \end{cases} \tag{B.12}$$

where $dl_2/d\tau = (A_{22} - S_{22}P)l_2(\tau,\varepsilon) \quad ; \quad l_2(0,\varepsilon) = c_1(\varepsilon)$ (B.13.a)

$dr_2/d\sigma = (A_{22} - S_{22}N)r_2(\sigma,\varepsilon) \quad ; \quad dr_2/d\sigma(0,\varepsilon) = c_2(\varepsilon).$ (B.13.b)

From (B.9), (B.12) and (B.13), we have

$$\begin{cases} z_2(t,\varepsilon) = \sum_{j=0}^{\infty} l_{2j}(\tau)\varepsilon^j + \varepsilon\sum_{j=0}^{\infty} r_{2j}(\sigma)\varepsilon^j & \text{(B.14.a)} \\ p_2(t,\varepsilon) = P\sum_{j=0}^{\infty} l_{2j}(\tau)\varepsilon^j + \varepsilon N\sum_{j=0}^{\infty} r_{2j}(\sigma)\varepsilon^j & \text{(B.14.b)} \end{cases}$$

where $l_{2j}(\tau) = \exp(A_{22}-S_{22}P)\tau l_{2j}(0) \ ; \ l_{2j}(0) = d^jc_1/d\varepsilon^j$ (B.15.a)

$r_{2j}(\sigma) = \exp(A_{22}-S_{22}N)\sigma r_{2j}(0) \ ; \ dr_{2j}/d\sigma(0) = d^jc_2/d\varepsilon^j\big|_{\varepsilon=0}.$ (B.15.b)

Furthermore, under the condition (H-2), by using the relation:

$$\begin{cases} P = T_{21}T_{11}^{-1}, & A_{22} - S_{22}P = -T_{11}\Lambda T_{11}^{-1} \\ N = T_{22}T_{12}^{-1}, & A_{22} - S_{22}N = T_{12}\Lambda T_{12}^{-1} \end{cases} \tag{B.16}$$

it is easily shown that (B.7) is equivalent to (B.14).

[Remarks]

Setting $\tilde{L} = -P$, $\tilde{L}^s = N^{-1}$ and $x_f = N\eta$ in the combined transformation (2.15), we obtain the dichotomy transformation:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & I \\ P & N \end{bmatrix} \begin{bmatrix} x_s \\ \eta \end{bmatrix}. \tag{B.17}$$

Thus, the dichotomy transformation corresponds to a special case of the combined transformation.

Appendix C. The Inverse Riccati Approach

It has been shown by Brockett [Bro.70] that by setting as

$$v(t) = u(t) + R^{-1}B'\Pi(t;0,t_f)x(t) \tag{C.1}$$

the fixed endpoint problem can be transformed into the equivalent minimum energy problem. As the result, the optimal feedback control is given by

$$u^o(t) = -R^{-1}B'[W^{-1}(t,t_f) + \Pi(t;0,t_f)]x + R^{-1}B'W^{-1}(t,t_f)\Phi(t,t_f)x_f \tag{C.2}$$

where

$$dW(t,t_f)/dt = A_c(t)W(t,t_f) + W(t,t_f)A_c'(t) - BR^{-1}B' \quad ; \quad W(t_f,t_f) = 0 \tag{C.3}$$

or

$$W(t,t_f) = \int_t^{t_f} \Phi(t,\sigma)BR^{-1}B'\Phi'(t,\sigma)d\sigma \tag{C.4}$$

and $d\Phi(t,t_0)/dt = A_c(t)\Phi(t,t_0) \quad ; \quad A_c(t) = A - BR^{-1}B'\Pi(t;0,t_f)$ (C.5)

and $\Pi(t;0,t_f)$ is a solution of the following Riccati equation:

$$-dK/dt = KA + A'K - KBR^{-1}B'K + Q \quad ; \quad K(t_f) = 0. \tag{C.6}$$

Here, by letting

$$H^{-1}(t,t_f) = W^{-1}(t,t_f) + \Pi(t;0,t_f) \tag{C.7}$$

and by using the relation

$$dW^{-1}/dt = -W^{-1}(dW/dt)W^{-1} \tag{C.8}$$

and the expressions (C.3) and (C.6), after algebraic manipulations, we obtain

$$dH/dt = AH + HA' + HQH - BR^{-1}B' \;;\quad H(t_f,t_f) = 0. \tag{C.9}$$

Furthermore, by letting

$$g(t) = [I - H(t,t_f)\Pi(t;0,t_f)]\Phi(t,t_f)x_f \tag{C.10}$$

and by using the expression (C.5), we have the following equation:

$$dg/dt = [A + H(t,t_f)Q]g(t) \quad ; \quad g(t_f) = x_f. \tag{C.11}$$

Then, by using the gains $H(t,t_f)$ and $g(t)$, the optimal control (C.2) can be expressed by

$$u^o(t) = -R^{-1}B'H^{-1}(t,t_f)[x(t) - g(t)]. \tag{C.12}$$

Here, we consider the two point boundary value problems (T.P.B.V.P.):

$$\begin{bmatrix} dx/dt \\ dp/dt \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix}\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \;;\quad \begin{bmatrix} x(t_0) \\ x(t_f) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_f \end{bmatrix}. \tag{C.13}$$

By denoting the state transition matrix by

$$\Psi(t,t_0) = \begin{bmatrix} \psi_{11}(t,t_0), & \psi_{12}(t,t_0) \\ \psi_{21}(t,t_0), & \psi_{22}(t,t_0) \end{bmatrix} \tag{C.14}$$

the feedback control is given (Kwon [Kw.77]) by

$$u^o(t) = -R^{-1}B'[-\psi_{12}^{-1}(t_f,t)\psi_{11}(t_f,t)x(t) + \psi_{12}^{-1}(t_f,t)x_f]. \tag{C.15}$$

In the above expression, setting as

$$\begin{cases} H(t,t_f) = -\psi_{11}^{-1}(t_f,t)\psi_{12}(t_f,t) \\ g(t) = \psi_{11}^{-1}(t_f,t)x_f \end{cases} \tag{C.16}$$

leads to the control (C.12) too. Furthermore, by applying the above transformation (C.16) to the T.P.B.V.P. (C.13) and after algebraic manipulations, we can obtain the equations (C.9) and (C.11) which $H(t,t_f)$ and $g(t)$ satify, and the following relation:

$$x(t) = H(t,t_f)p(t) + g(t). \tag{C.17}$$

When we consider the T.P.B.V.P. generally, we frequently use the method of invariant imbedding (Mufti et al. [Muf.69], Breitenecker [Bre.83]) and set as

$$x(t) = H(t,t_f)p(t) + g(t).$$

The above expression is called also to be the inverse Riccati transformation in contrast with the conventional Riccati transformation (Kokotovic et al. [Ko.76]).

In particular, in the case of $x_f = 0$, we have

$$g(t) = 0, \text{ for } {}^{\forall}t\epsilon[t_0,t_f] \tag{C.18}$$

and the optimal cost of the fixed-endpoint problem is given by

$$J^o = \frac{1}{2}x_0'H^{-1}(t_0,t_f)x_0. \tag{C.19}$$

Appendix D. Reduced Problems in General Terminal Penalty Problems

By setting $\varepsilon = 0$ in the singularly perturbed Riccati differential equations of the general terminal problem, we have

$$\begin{cases} dH_{11}^0/dt = -\hat{A}'H_{11}^0 - H_{11}^0\hat{A} + H_{11}^0\hat{B}R^{-1}\hat{B}'H_{11}^0 - \hat{Q}\ ;\ H_{11}^0(t_f) = \tilde{\Pi}_{11}-\tilde{\Pi}_{12}\tilde{\Pi}_{22}^{-1}\tilde{\Pi}_{12}' & \text{(D.1.a)} \\ H_{12}^0 = H_{11}^0E_1 - E_2 & \text{(D.1.b)} \\ H_{22}^0A_{22} + A_{22}'H_{22}^0 - H_{22}^0S_{22}H_{22}^0 + Q_{22} = 0 & \text{(D.1.c)} \end{cases}$$

where $\hat{A} = A_{11} + E_1A_{21} + S_{12}E_2' + E_1S_{22}E_2'$

$$\hat{B} = B_1 + E_1B_2$$

$$\hat{Q} = (I, E_2H_{22}^{0-1}) \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix} \begin{pmatrix} I \\ H_{22}^{0-1}E_2' \end{pmatrix}$$

$$E_1 = (S_{12}H_{22}^0 - A_{12})\alpha^{-1} \quad , \quad \alpha = A_{22} - S_{22}H_{22}^0$$

$$E_2 = (A_{21}'H_{22}^0 + Q_{12})\alpha^{-1} \quad , \quad S_{12} = B_1R^{-1}B_2', \quad S_{22} = B_2R^{-1}B_2'$$

and $(\tilde{\Pi}_{11} - \tilde{\Pi}_{12}\tilde{\Pi}_{22}^{-1}\tilde{\Pi}_{12}')$ is a positive semidefinite matrix when $\tilde{\Pi}$ is a positive semidefinite matrix.

From the above expressions, the limit problem is defined as follows:

$$\begin{cases} d\hat{X}_{10}/dt = \hat{A}\hat{X}_{10} + \hat{B}\hat{V}_0 \quad ; \quad \hat{X}_{10}(t_0) = x_{10} \\ J = \frac{1}{2}\hat{X}_{10}'(t_f)(\tilde{\Pi}_{11}-\tilde{\Pi}_{12}\tilde{\Pi}_{22}^{-1}\tilde{\Pi}_{12}')\hat{X}_{10}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[\hat{X}_{10}'\hat{Q}\hat{X}_{10} + \hat{V}_0'\hat{R}\hat{V}_0]dt. \end{cases} \tag{D.2}$$

That is, the Riccati differential equation of the above problem leads to (D.1.a).

Then, by using the Riccati feedback approach (Kokotovic et al. [Ko.72.3]), the slow part $\hat{U}_0(t)$ of the optimal control $u^o(t,\varepsilon)$ is given as follows:

$$\hat{U}_0(t) = -R^{-1}[(B_1'H_{11}^0 + B_2'H_{12}^{0'})\hat{X}_{10} + B_2'H_{22}^0\hat{X}_{20}] \tag{D.3}$$

where $d\hat{X}_{10}/dt = (\hat{A} - \hat{B}R^{-1}\hat{B}'H_{11}^0)\hat{X}_{10}(t) \quad ; \quad \hat{X}_{10}(t_0) = x_{10}$ (D.4.a)

$$\hat{X}_{20}(t) = -\alpha^{-1}(A_{21} - S_{12}'H_{11}^0 - S_{22}H_{12}^{0'})\hat{X}_{10}(t) \tag{D.4.b}$$

which is composed of the reduced-order Riccati solutions shown above.

On the other hand, similarly as Section 5.3, on the basis of the decomposition-coordination approach, the mode decoupled slow control part $U_0(t)$ is given as follows:

$$\begin{aligned} U_0(t) &= -R^{-1}[B_1'P_{10}(t) + B_2'P_{20}(t)] \\ &= V_0(t) + \mathcal{R}_0^{-1}\mathcal{C}_0'X_{10}(t) \\ &= -\mathcal{R}_0^{-1}(\mathcal{B}_0'\mathcal{K}_0(t)-\mathcal{C}_0')X_{10}(t) \end{aligned} \tag{D.5}$$

where the t-time-scale problem is defined as follows:

$$\begin{cases} dX_{10}/dt = \mathcal{A}_0 X_{10}(t) + \mathcal{B}_0 V_0(t) \quad ; \quad X_{10}(t_0) = x_{10} \\ J = \frac{1}{2}X_{10}'(t_f)(\tilde{\Pi}_{11}-\tilde{\Pi}_{12}\tilde{\Pi}_{22}^{-1}\tilde{\Pi}_{12}')X_{10}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[X_{10}'\mathcal{Q}_0 X_{10} + V_0'\mathcal{R}_0 V_0]dt \end{cases} \tag{D.6}$$

and $V_0(t) = -\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{K}_0(t)X_{10}(t)$ (D.7.a)

$$d\mathcal{K}_0/dt = -\mathcal{A}_0'\mathcal{K}_0-\mathcal{K}_0\mathcal{A}_0+\mathcal{K}_0\mathcal{B}_0\mathcal{R}_0^{-1}\mathcal{B}_0'\mathcal{K}_0-\mathcal{Q}_0; \quad \mathcal{K}_0(t_f)=\tilde{\Pi}_{11}-\tilde{\Pi}_{12}\tilde{\Pi}_{22}^{-1}\tilde{\Pi}_{12}' \tag{D.7.b}$$

and furthermore $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{Q}_0$ and $\mathcal{R}_0$ etc. are defined in (5.50), respectively.

Then, we obtain the following relationships among the controls defined above:

$$\begin{cases} \lim_{\varepsilon\to 0} u^o(t,\varepsilon) = U_0(t) = \hat{U}_0(t) \neq \hat{V}_0(t) \quad ; \quad t\in(t_0,t_f) & \text{(D.8.a)} \\ \mathcal{K}_0(t) = H_{11}^0(t) & \text{(D.8.b)} \\ V_0(t) = M^{-1}\hat{V}_0(t) & \text{(D.8.c)} \end{cases}$$

where $M = [I - R^{-1}B_2'H_{22}^0A_{22}^{-1}B_2]$

$\quad = [I + R^{-1}B_2'H_{22}^0\alpha^{-1}B_2]^{-1}$.

The above results can be proved similarly as proofs given in [Ka.81.3], [Had.71].

On the other hand, setting $\varepsilon = 0$ directly in the original problem (5.95) and (5.96) yields the natural reduced problem as follows:

$$\begin{cases} d\bar{x}_1/dt = A_0\bar{x}_1 + B_0\bar{u} \quad ; \quad \bar{x}_1(t_0) = x_{10} \\ \bar{x}_2 = -A_{22}^{-1}A_{21}\bar{x}_1 - A_{22}^{-1}B_2\bar{u} \end{cases} \tag{D.9.a}$$

$$J = \frac{1}{2}[\bar{x}_1'(t_f),\bar{x}_2'(t_f)]\begin{bmatrix}\tilde{\Pi}_{11} & \tilde{\Pi}_{12} \\ \tilde{\Pi}_{12}' & \tilde{\Pi}_{22}\end{bmatrix}\begin{bmatrix}\bar{x}_1(t_f) \\ \bar{x}_2(t_f)\end{bmatrix} + \frac{1}{2}\int_{t_0}^{t_f}\{\bar{x}_1'\bar{Q}\bar{x}_1 - 2\bar{x}_1'\mathcal{C}_0\bar{u} + \bar{u}'\mathcal{R}_0\bar{u}\}dt \tag{D.9.b}$$

where $\bar{Q} = [I,-A_{21}'A_{22}'^{-1}]\begin{bmatrix}Q_{11} & Q_{12} \\ Q_{12}' & Q_{22}\end{bmatrix}\begin{bmatrix}I \\ -A_{22}^{-1}A_{21}\end{bmatrix}$.

In the general terminal problem, it has been shown by Glizer et al.

[Gl.78] that the control $U_0(t)$ ((D.5)) of the t-time-scale problem which is equivalent to $\hat{U}_0(t)$ ((D.3)) is not equivalent to the control $\bar{u}(t)$ ((D.9.a)) except the special case (Glizer et al [Gl.75.2]).