

Symmetry-breaking bifurcation of positive solutions to a one-dimensional Liouville type equation

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We consider the two-point boundary value problem for the one-dimensional Liouville type equation

$$(1) \quad \begin{cases} u'' + \lambda|x|^l e^u = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ and $l > 0$.

Jacobsen and Schmitt [2] studied the exact multiplicity of radial solutions of the problem for the multi-dimensional Liouville type equation

$$(2) \quad \begin{cases} \Delta u + \lambda|x|^l e^u = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $l \geq 0$ and $B := \{x \in \mathbf{R}^n : |x| < 1\}$. They proved the following (i)–(iii):

- (i) if $1 \leq N \leq 2$, then there exists $\lambda_* > 0$ such that (2) has exactly two radial solutions for $0 < \lambda < \lambda_*$, a unique radial solution for $\lambda = \lambda_*$ and no radial solution for $\lambda > \lambda_*$;
- (ii) if $3 \leq N < 10 + 4l$, then (2) has infinitely many radial solutions for $\lambda = (l + 2)(N - 2)$ and a finite but large number of radial solutions when $|\lambda - (l + 2)(N - 2)|$ is sufficiently small;
- (iii) if $N \geq 10 + 4l$, then (2) has a unique radial solution for $0 < \lambda < (l + 2)(N - 2)$ and no radial solution for $\lambda \geq (l + 2)(N - 2)$.

We note here that every solution of (2) is positive in B , by the strong maximum principle. Result (i)–(iii) were established by Joseph and Lundgren [3] for the case $l = 0$, that is, for the Liouville equation

$$(3) \quad \begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\Omega = B$. Gidas, Ni and Nirenberg's theorem ([1]) shows that every positive solution of (3) is radially symmetric when $\Omega = B$. However, when Ω is an annulus $A := \{x \in \mathbf{R}^N : a < |x| < b\}$, $a > 0$, problem (3) may have non-radial solutions. Indeed, Lin [4] proved that (3) has infinitely many symmetry-breaking bifurcation points when $N = 2$ and $\Omega = A$. Nagasaki and Suzuki [6] found that large non-radial solutions of (3) when $N = 2$ and $\Omega = A$. More precisely, for each sufficiently large $\mu > 0$, there exist (λ, u) such that $\lambda > 0$, u is a non-radial solution of (3) and $\int_A e^u dx = \mu$ when $N = 2$ and $\Omega = A$.

Recently, Miyamoto [5] considered the problem for the Liouville type equation (2) and proved the following result.

Theorem A ([5]). *Let n_0 be the largest integer that is smaller than $1 + \frac{l}{2}$ and let $\alpha_n := 2 \log \frac{2l+4}{l+2-2n}$. All the radial solutions of (2) with $N = 2$ can be written explicitly as*

$$\lambda(\alpha) = 2(l+2)^2(e^{-\alpha/2} - e^{-\alpha}), \quad U(r; \alpha) = \log \frac{e^\alpha}{(1 + (e^{\alpha/2} - 1)r^{l+2})^2}.$$

The radial solutions can be parameterized by the L^∞ -norm, it has one turning point at $\lambda = \lambda(\alpha_0) = (l+2)/2$, and it blows up as $\lambda \downarrow 0$. For each $n \in \{1, 2, \dots, n_0\}$, $(\lambda(\alpha_n), U(r; \alpha_n))$ is a symmetry breaking bifurcation point from which an unbounded branch consisting of non-radial solutions of (2) with $N = 2$ emanates, and $U(r; \alpha)$ is nondegenerate if $\alpha \neq \alpha_n$, $n = 0, 1, \dots, n_0$. Each non-radial branch is in $(0, \lambda(\alpha_0)) \times \{u > 0\} \subset \mathbf{R} \times H_0^2(B)$.

When $N = 2$, radial solutions of problems (2) and (3) can be written explicitly, and hence, Lin [4] and Miyamoto [5] succeeded to show the existence of bifurcation points. That is difficult even if we know exact solutions, much more difficult if we do not know them usually. When $N \neq 2$, we do not find exact radial solutions of (2). However, the structure of eigenvalues and eigenfunctions of the linearized problem in the dimension 1 is well-known, and then, by the comparison function introduced in [7], we can find the Morse indices of even solutions of (1). Then we obtain the existence of a symmetry-breaking bifurcation point of (1).

Let $m(U)$ be the Morse index of a solution U to (1), that is, the number of negative eigenvalues μ of

$$(4) \quad \begin{cases} \phi'' + \lambda|x|^l e^{U(x)} \phi + \mu \phi = 0, & x \in (-1, 1), \\ \phi(-1) = \phi(1) = 0, \end{cases}$$

A solution U of (1) is said to be degenerate if $\mu = 0$ is an eigenvalue of (4). Otherwise, it is said to be nondegenerate.

The main result is as follows.

Theorem 1. *For each $\alpha > 0$, there exists a unique $(\lambda(\alpha), U(x; \alpha))$ such that (1) with $\lambda = \lambda(\alpha)$ has a unique positive even solution $U = U(x; \alpha)$ such that $\|U\|_\infty = \alpha$. Moreover, there exist α_* , α_1 , α_2 and α_3 such that $\alpha_* < \alpha_1 \leq \alpha_2 \leq \alpha_3$ and the following (i)–(vii) hold:*

- (i) *if $0 < \alpha < \alpha_*$, then $m(U) = 0$ and $U(x; \alpha)$ is nondegenerate;*
- (ii) *if $\alpha = \alpha_*$, then $m(U) = 0$ and $U(x; \alpha)$ is degenerate;*
- (iii) *if $\alpha_* < \alpha < \alpha_1$, then $m(U) = 1$ and $U(x; \alpha)$ is nondegenerate;*
- (iv) *if $\alpha = \alpha_1$, then $m(U) = 1$ and $U(x; \alpha)$ is degenerate;*
- (v) *if $\alpha = \alpha_2$, then $m(U) = 1$, $U(x; \alpha)$ is degenerate and (U, λ) is a non-even bifurcation point, that is, for each $\varepsilon > 0$, there exists (λ, u) such that u is a non-even positive solution of (1) and $|\lambda - \lambda(\alpha_2)| + \|u - U(\cdot, \alpha_2)\|_\infty < \varepsilon$;*
- (vi) *if $\alpha = \alpha_3$, then $m(U) = 2$ and $U(x; \alpha)$ is degenerate;*
- (vii) *if $\alpha > \alpha_3$, then $m(U) = 2$ and $U(x; \alpha)$ is nondegenerate.*

Here and Hereafter, we use the notation $\|U\|_\infty = \sup_{x \in [-1,1]} U(x)$.

For the proof of Theorem 1, see [8]. Here, we give a sufficient condition for the second eigenvalue of the linearized problem to be negative for the following problem

$$(5) \quad \begin{cases} u'' + \lambda h(x)f(u) = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0. \end{cases}$$

where $\lambda > 0$ and $h \in C^1([-1, 0] \cup (0, 1]) \cap C[-1, 1]$, $h(-x) = h(x)$, $h(x) > 0$ and $h'(x) \geq 0$ for $x > 0$, $f \in C^1[0, \infty)$, $f(s) > 0$ and $f'(s) \geq 0$ for $s > 0$. Namely we will show the following result, which plays a crucial role in the proof of Theorem 1.

Proposition 1. *Assume that, for each sufficiently large $\alpha > 0$, there exist $\lambda(\alpha) > 0$ and $U(x; \alpha)$ such that $U(x; \alpha)$ is a positive even solution of (5) at $\lambda = \lambda(\alpha)$ and $\|U(\cdot; \alpha)\|_\infty = \alpha$. Assume moreover that there exist $s_0 > 0$ and $\delta > 0$ such that*

$$(6) \quad \frac{l(x)(g(s) - 1) - 4}{g(s) + l(x) + 3} \geq \delta, \quad x \in (0, 1], \quad s \geq s_0,$$

where $l(x) = xh'(x)/h(x)$ and $g(s) = sf'(s)/f(s)$. Let $\mu_2(\alpha)$ be the second eigenvalue of

$$(7) \quad \begin{cases} \phi'' + \lambda(\alpha)h(x)f'(U(x; \alpha))\phi + \mu\phi = 0, & x \in (-1, 1), \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

Then $\mu_2(\alpha) < 0$ for all sufficiently large $\alpha > 0$.

In the case where $h(x) = |x|^l$, $l > 0$ and $f(s) = e^s$, it follows that $l(x) = xh'(x)/h(x) = l$ for $x \in (0, 1]$ and $g(s) = sf'(s)/f(s) = s$, and hence (6) is satisfied.

We conclude that if U is a positive even solution of (1) and $\|U\|_\infty \leq 1$, then $m(U) = 0$. Indeed, let μ_1 be the first eigenvalue of (4) and let ϕ_1 be an eigenfunction corresponding to μ_1 . We may assume that $\phi_1(x) > 0$ on $(-1, 1)$. Integrating the equality

$$(\phi_1(x)U'(x) - \phi_1'(x)U(x))' = \mu_1\phi_1(x)U(x) + \lambda|x|^l e^{U(x)}\phi_1(x)(U(x) - 1)$$

on $[-1, 1]$, we have

$$\mu_1 \int_{-1}^1 \phi_1(x)U(x)dx = \lambda \int_{-1}^1 |x|^l e^{U(x)}\phi_1(x)(1 - U(x))dx > 0.$$

Consequently, we have $\mu_1 > 0$, which means $m(U) = 0$. By applying Proposition 1, we can conclude that $m(U(\cdot; \alpha)) = 0$ for $0 < \alpha \leq 1$ and $m(U(\cdot; \alpha)) \geq 2$ for all sufficiently large $\alpha > 1$. Then, using the Leray-Schauder degree, we can find a bifurcation point.

To prove Proposition 1, we need the following two lemmas.

Lemma 1. *Let ϕ_2 be an eigenfunction corresponding to the second eigenvalue $\mu_2(\alpha)$ of (7). Then ϕ_2 is odd, $\phi_2(0) = \phi_2(1) = 0$ and $\phi_2(x) \neq 0$ for $x \in (0, 1)$.*

Proof. Let M_1 be the first eigenvalue of

$$\begin{cases} \Phi'' + \lambda(\alpha)h(x)f'(U(x; \alpha))\Phi + M\Phi = 0, & x \in (0, 1), \\ \Phi(0) = \Phi(1) = 0, \end{cases}$$

and let Φ_1 be an eigenfunction corresponding to M_1 . Then $\Phi_1(0) = \Phi_1(1) = 0$ and $\Phi_1(x) \neq 0$ on $(0, 1)$. Set

$$\Phi(x) = \begin{cases} \Phi_1(x), & x \in [0, 1], \\ -\Phi_1(-x), & x \in [-1, 0]. \end{cases}$$

Noting the fact that $\lim_{x \rightarrow -0} \Phi''(x) = \lim_{x \rightarrow -0} (-\Phi_1''(-x)) = -\Phi_1''(0) = 0$, we easily check that Φ is a solution of

$$\begin{cases} \Phi'' + \lambda(\alpha)h(x)f'(U(x; \alpha))\Phi + M_1\Phi = 0, & x \in (-1, 1), \\ \Phi(-1) = \Phi(1) = 0, \end{cases}$$

and Φ is odd, $\Phi(x) \neq 0$ on $(0, 1)$ and $\Phi(0) = 0$. Therefore, M_1 is an eigenvalue of (7) and Φ is an eigenfunction corresponding to M_1 . Since Φ has exactly one zero in $(-1, 1)$, M_1 must be μ_2 and hence $\phi_2(x)$ must be $c\Phi(x)$ for some $c \neq 0$. $\square$

Lemma 2. Assume that $w \in C[a, b]$ is positive and concave on (a, b) . Let $\rho \in (0, 1/2)$. Then $w(x) \geq \rho \max_{\xi \in [a, b]} w(\xi)$ for $x \in [(1 - \rho)a + \rho b, \rho a + (1 - \rho)b]$.

Proof. We take $c \in [a, b]$ for which $w(c) = \max_{\xi \in [a, b]} w(\xi)$. Then $w(c) > 0$. Since w is positive and concave on (a, b) , we have

$$w(x) \geq \frac{w(c)(x - a)}{c - a} \geq \frac{w(c)(x - a)}{b - a} =: l_1(x), \quad x \in [a, c],$$

and

$$w(x) \geq \frac{w(c)(b - x)}{b - c} \geq \frac{w(c)(b - x)}{b - a} =: l_2(x), \quad x \in [c, b].$$

Hence $w(x) \geq \min\{l_1(x), l_2(x)\}$ on $[a, b]$. We conclude that if $x \in [(1 - \rho)a + \rho b, (a + b)/2]$, then

$$\min\{l_1(x), l_2(x)\} = l_1(x) \geq l_1((1 - \rho)a + \rho b) = \rho w(c),$$

and if $x \in [(a + b)/2, \rho a + (1 - \rho)b]$, then

$$\min\{l_1(x), l_2(x)\} = l_2(x) \geq l_2(\rho a + (1 - \rho)b) = \rho w(c).$$

The proof is complete. $\square$

Now we are ready to show Proposition 1.

Proof of Proposition 1. Let $\alpha > 0$ be sufficiently large. We use the following comparison function $y(x)$ introduced in [7]:

$$y(x) = xU(x; \alpha) - (x - 1)^2U'(x; \alpha).$$

This function $y(x)$ satisfies $y(0) = y(1) = 0$, $y(x) > 0$ on $(0, 1)$, and

$$y'' + \lambda(\alpha)h(x)f'(U(x; \alpha))y = \lambda(\alpha)x^{-1}h(x)H(x; \alpha)f(U(x; \alpha))$$

for $x \in (0, 1]$, where

$$H(x; \alpha) = (1 - x)^2l(x) + x(3x - 4) + x^2g(U(x; \alpha)).$$

Let $\phi_2(x; \alpha)$ be an eigenfunction corresponding to $\mu_2(\alpha)$. From Lemma 1, it follows that $\phi_2(0; \alpha) = \phi_2(1; \alpha) = 0$ and $\phi_2(x; \alpha) \neq 0$ for $x \in (0, 1)$. Without loss of generality, we may assume that $\phi_2(x; \alpha) > 0$ for $x \in (0, 1)$ and $\max_{\xi \in [0,1]} \phi_2(\xi; \alpha) = 1$. We observe that

$$(y'\phi_2 - y\phi_2')' = \mu_2(\alpha)\phi_2 y + \lambda(\alpha)x^{-1}h(x)H(x; \alpha)f(U(x; \alpha))\phi_2, \quad x \in (0, 1].$$

Integrating this equality on $(0, 1)$, we obtain

$$(8) \quad \mu_2(\alpha) \int_0^1 \phi_2(x; \alpha)y(x)dx + \lambda(\alpha) \int_0^1 x^{-1}h(x)H(x; \alpha)f(U(x; \alpha))\phi_2(x; \alpha)dx = 0.$$

Since

$$\begin{aligned} H(x) &= [g(U(x; \alpha)) + l(x) + 3] \left(x - \frac{l(x) + 2}{g(U(x; \alpha)) + l(x) + 3} \right)^2 \\ &\quad + \frac{l(x)[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3} \\ &\geq \frac{l(x)[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3}, \end{aligned}$$

we have

$$\begin{aligned} (9) \quad &\int_0^1 x^{-1}h(x)H(x; \alpha)f(U(x; \alpha))\phi_2(x; \alpha)dx \\ &\geq \int_0^1 x^{-1}h(x) \frac{l(x)[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3} f(U(x; \alpha))\phi_2(x; \alpha)dx. \end{aligned}$$

Since $U''(x; \alpha) = -\lambda(\alpha)h(x)f(U(x; \alpha)) < 0$ on $(0, 1]$, we find that $U'(x; \alpha)$ is decreasing in $x \in (0, 1]$. From $U'(0; \alpha) = 0$ it follows that $U'(x; \alpha) < 0$ for $x \in (0, 1]$, which implies that $U(x; \alpha)$ is also decreasing in $x \in (0, 1]$. Then there exists $x(\alpha) \in (0, 1)$ such that $U(x; \alpha) \geq s_0$ for $x \in [0, x(\alpha)]$ and $U(x; \alpha) < s_0$ for $x \in (x(\alpha), 1]$. Since $U(x; \alpha)$ is concave on $(0, 1)$, we conclude that

$$U(x; \alpha) \geq \alpha(1 - x), \quad x \in [0, 1],$$

which shows that if $x \in [0, (\alpha - s_0)/\alpha]$, then $U(x; \alpha) \geq s_0$. Therefore, $x(\alpha) \geq (\alpha - s_0)/\alpha$, which implies

$$(10) \quad \lim_{\alpha \rightarrow \infty} x(\alpha) = 1.$$

We take $s_1 \geq s_0$ for which $x(\alpha) \geq 3/4$ for $\alpha \geq s_1$. If $\alpha \geq s_1$, then (6) implies

$$\begin{aligned} (11) \quad &\int_0^{x(\alpha)} x^{-1}h(x) \frac{l(x)[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3} f(U(x; \alpha))\phi_2(x; \alpha)dx \\ &\geq \delta f(s_0) \int_0^{x(\alpha)} x^{-1}h(x)\phi_2(x; \alpha)dx \\ &\geq \delta f(s_0) \int_{1/4}^{3/4} x^{-1}h(x)\phi_2(x; \alpha)dx. \end{aligned}$$

Recalling $\max_{\xi \in [0,1]} \phi_2(\xi) = 1$, we have

$$\begin{aligned}
 (12) \quad & \int_{x(\alpha)}^1 x^{-1} h(x) \frac{l(x)[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3} f(U(x; \alpha)) \phi_2(x; \alpha) dx \\
 & \geq - \int_{x(\alpha)}^1 x^{-1} h(x) \frac{(l(x) + 4)f(U(x; \alpha)) \phi_2(x; \alpha)}{g(U(x; \alpha)) + l(x) + 3} dx \\
 & \geq -f(s_0) \int_{x(\alpha)}^1 x^{-1} h(x) \frac{l(x) + 4}{l(x) + 3} dx.
 \end{aligned}$$

Now we will show that there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. Assume to the contrary that there exists $\{\alpha_n\}_{n=1}^\infty$ such that $\mu_2(\alpha_n) \geq 0$ and $\alpha_n \geq s_1$ for $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \alpha_n = \infty$.

Since $\phi_2(x; \alpha_n) > 0$ and

$$\phi_2''(x; \alpha_n) = -h(x)f'(U(x; \alpha_n))\phi_2(x; \alpha_n) - \mu_2(\alpha_n)\phi_2(x; \alpha_n) \leq 0, \quad x \in (0, 1),$$

we find that $\phi_2(x; \alpha_n)$ is concave on $(0, 1)$. From Lemma 2 with $\rho = 1/4$, $a = 0$ and $b = 1$, it follows that

$$\phi_2(x; \alpha_n) \geq \frac{1}{4} \max_{\xi \in [0,1]} \phi_2(\xi; \alpha_n) = \frac{1}{4}, \quad x \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

By (11), we have

$$\begin{aligned}
 (13) \quad & \int_0^{x(\alpha)} x^{-1} h(x) \frac{l(x)[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l(x) + 3} f(U(x; \alpha)) \phi_2(x; \alpha) dx \\
 & \geq \frac{\delta f(s_0)}{4} \int_{1/4}^{3/4} x^{-1} h(x) dx.
 \end{aligned}$$

Combining (8) with (9), (12) and (13), we have

$$\begin{aligned}
 0 & \geq -\mu_2(\alpha_n) \int_0^1 \phi_2(x; \alpha_n) y(x) dx \\
 & \geq \lambda(\alpha_n) f(s_0) \left[\frac{\delta}{4} \int_{1/4}^{3/4} x^{-1} h(x) dx - \int_{x(\alpha_n)}^1 x^{-1} h(x) \frac{l(x) + 4}{l(x) + 3} dx \right],
 \end{aligned}$$

which implies

$$\int_{x(\alpha_n)}^1 x^{-1} h(x) \frac{l(x) + 4}{l(x) + 3} dx \geq \frac{\delta}{4} \int_{1/4}^{3/4} x^{-1} h(x) dx > 0, \quad n \in \mathbb{N}.$$

This contradicts the fact (10). Consequently, there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. This completes the proof. $\square$

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