

Higher codimensional Ueda theory for a compact submanifold with unitary flat normal bundle (arXiv:1606.01837)

Takayuki Koike (Kyoto University)

1. Configurations and our main interest

X : complex manifold,
 $Y \subset X$: compact complex submanifold of codimension $r \geq 1$
 with *unitary flat* normal bundle.
 i.e. $N_{Y/X} \in \text{Image}(H^1(Y, U(r)) \rightarrow H^1(Y, GL_r(\mathcal{O}_Y))) =: \mathcal{P}_r(Y)$.

Main interest —
 Compare a neighborhood of Y in X and a neighborhood of the zero section in $N_{Y/X}$. $\square$

Main interest in local description:
 Let $\{U_j\}$ be an open covering of Y and $\{V_j\}$ be a neighborhood of U_j in X . Is there a *defining functions system* $w_j = (w_j^1, w_j^2, \dots, w_j^r)$ of U_j in V_j with $w_j = T_{jk} w_k$?
 (i.e. $w_j^\lambda = (T_{jk} w_k)^\lambda := \sum_{\mu=1}^r (T_{jk})_{\mu}^{\lambda} \cdot w_k^\mu$ holds on each V_{jk} , where $T_{jk} \in U(r)$ is the transition matrix of $N_{Y/X}$)

2. Obstruction classes and the type of the pair

As $T_{jk} \in U(r)$ is the transition matrix of $N_{Y/X}$, one can take w_j with $dw_j = T_{jk} dw_k$ on each U_{jk} : i.e.

$$(T_{jk} w_k)^\lambda = w_j^\lambda + O(|w_j|^2) = w_j^\lambda + \sum_{|\alpha| \geq 2} f_{kj,\alpha}^\lambda(z_j) \cdot w_j^\alpha$$

for some $f_{kj,\alpha}^\lambda$ ($\alpha \in \mathbb{Z}_{\geq 0}^r$, $|\alpha| := \sum_{\lambda=1}^r \alpha_\lambda$, $w_j^\alpha := \prod_{\lambda=1}^r (w_j^\lambda)^{\alpha_\lambda}$).

$$u_1(Y, X) := \{ \{ (U_{jk}, \sum_{|\alpha|=2} \sum_{\lambda=1}^r f_{kj,\alpha}^\lambda \cdot (\partial/\partial w_j^\alpha) \otimes (dw_j)^\alpha) \} \} \\ \in H^1(Y, N_{Y/X} \otimes S^2 N_{Y/X}^*)$$

If $u_1(Y, X) = 0$, then one can take $\{w_j\}$ such that

$$(T_{jk} w_k)^\lambda = w_j^\lambda + O(|w_j|^3) = w_j^\lambda + \sum_{|\alpha| \geq 3} f_{kj,\alpha}^\lambda(z_j) \cdot w_j^\alpha.$$

$$u_2(Y, X) := \{ \{ (U_{jk}, \sum_{|\alpha|=3} \sum_{\lambda=1}^r f_{kj,\alpha}^\lambda \cdot (\partial/\partial w_j^\alpha) \otimes (dw_j)^\alpha) \} \} \\ \in H^1(Y, N_{Y/X} \otimes S^3 N_{Y/X}^*)$$

If $u_2(Y, X) = 0, \dots$

Properties of $u_n(Y, X)$ and Definition of the type

- $u_n(Y, X) \in H^1(Y, N_{Y/X} \otimes S^{n+1} N_{Y/X}^*)$.
- " $u_n(Y, X) = 0$ " does not depend on the choice of $\{w_j\}$.
- $\text{type}(Y, X) := \max\{n \mid u_n(Y, X) = 0 \forall n < n\} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$

3. Main result

$\mathcal{E}_0^{(r)}(Y) := \{E \in \mathcal{P}_r(Y) \mid \#(\text{Image } \rho_E) < \infty\}$, where $\rho_E: \pi_1(Y, *) \rightarrow U(r)$ is the monodromy of E .

$\mathcal{E}_1^{(r)}(Y) := \bigcup \left\{ E \in \mathcal{P}_r(Y) \mid \pi^* E \in \mathcal{S}_A^{(r)}(\tilde{Y}) \text{ for some } A > 0 \right\}$, where $\pi: \tilde{Y} \rightarrow Y$: finite normal covering

$$\mathcal{S}_A^{(r)}(\tilde{Y}) := \left\{ \bigoplus_{\lambda=1}^r L_\lambda \mid L_\lambda \in \mathcal{P}_1(\tilde{Y}), d\left(\mathbb{I}_{\tilde{Y}}^{(1)}, \bigotimes_{\lambda=1}^r L_\lambda^{\alpha_\lambda}\right) \geq \frac{1}{(2|a|)^A} \right\} \\ \text{for } a = (a_\lambda)_{\lambda=1}^r \in \mathbb{Z}^r \text{ with } |a| \geq 1$$

Main Theorem

Assume $\text{type}(Y, X) = \infty$ and $N_{Y/X} \in \mathcal{E}_0^{(r)}(Y) \cup \mathcal{E}_1^{(r)}(Y)$. Then the following holds:

- (i) There exists a non-singular holomorphic foliation $\mathcal{F}$ of codimension r on some neighborhood V of Y which includes Y as a leaf with $\text{Hol}_{\mathcal{F}} Y = \rho_{N_{Y/X}}$.
- (ii) For each hypersurface S such that $Y \subset S$ and $N_{Y/S}$ is unitary flat, there exists a non-singular holomorphic foliation $\mathcal{G}_S$ of codimension 1 on V with the following properties by shrinking V if necessary: $\mathcal{G}_S$ includes $S \cap V$ as a leaf with $U(1)$ -linear holonomy, and each leaf of $\mathcal{F}$ is holomorphically immersed into a leaf of $\mathcal{G}_S$. $\square$

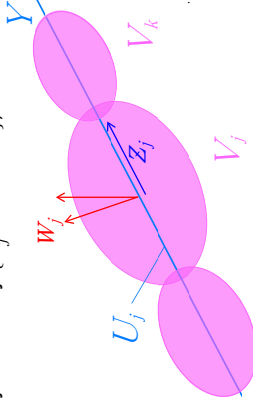
The assertions (i) and (ii) in local description:

(i) means that one can take $\{w_j\}$ with $w_j = T_{jk} w_k$ holds on each V_{jk} (The leaves of $\mathcal{F}$ is locally defined by $\{w_j = \text{constant}\}$).

(ii) means that there exists a convergent power series $F^\lambda \in \mathbb{C}\{X^1, X^2, \dots, X^r\}$ such that, by setting

$$\hat{w}^\lambda := F^\lambda(w_j^1, w_j^2, \dots, w_j^r),$$

$\{\hat{w}_j\}$ is a new defining functions system with $w_j = \hat{T}_{jk} w_k$ holds on each V_{jk} for some $\hat{T}_{jk} \in U(r)$ and $\{\hat{w}_j^\lambda = 0\} = S \cap V_j$ (The leaves of $\mathcal{G}_S$ is locally defined by $\{w_j^\lambda = \text{constant}\}$).



4. History

Arnol'd: The case where Y is an elliptic curve [A].

Ueda: The case where $r = 1$ [U].

K-, Ogawa: The case where $r = 2$ [K], [KO].

5. Application

Theorem (ii) can be applied to the "semi-positivity problem" on a nef line bundle, since the assertion (ii) implies the unitary flatness of the line bundle $[S]$ on V . For example:

Corollary (Application to the semi-positivity problem)

Let X be a complex manifold of dimension n and L be a holomorphic line bundle on X . Take $D_1, D_2, \dots, D_{n-1} \in |L|$. Assume that $C := \bigcap_{\lambda=1}^{n-1} D_\lambda$ is a smooth elliptic curve, $L|_C \in \mathcal{E}_1^{(1)}(C)$, and $\{D_\lambda\}_{\lambda=1}^{n-1}$ intersects transversally along C . Then L is semi-positive (i.e. L admits a C^∞ Hermitian metric with semi-positive curvature). $\square$

Note that L as in Corollary has C as a stable base locus: $C = SB(L) := \bigcap_{m \geq 1} \text{Bs } L^m$.

Example

Let (V, L) be a del Pezzo manifold of degree 1 (i.e. V is a projective manifold of dimension n and L is an ample line bundle on V with $K_V^{-1} \cong L^{n-1}$ and the self-intersection number (L^n) is equal to 1), and $C \subset V$ be an intersection of general $n-1$ elements of $|L|$. For each point $q \in C$ with $L|_C \otimes [-q] \in \mathcal{E}_1^{(0)}(C) \cup \mathcal{E}_1^{(1)}(C)$, the anti-canonical bundle of the blow-up of V at q is semi-positive. $\square$

This example can be regarded as a generalization of the known semi-positivity criterion for the anti-canonical bundle of the blow-up of $\mathbb{P}^2$ at 9 points [A], [U], [B].

References

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