

On The N-Fractional Calculus of Some Products of Some Power Functions

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Abstract

In this article, N-fractional calculus of products of power functions

$$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma, ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma, \text{ and } ((z-c)^{\alpha+\beta})_\gamma$$

are discussed again.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in \mathbb{C}$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii). (Γ ; Gamma function),

$$(iv) \quad (u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. N-fractional calculus of products of some power functions

In the following $\alpha, \beta, \gamma \in \mathbb{R}$, for our convenience.

Theorem 1. *Let*

$$P = P(\alpha, \beta, \gamma) := \frac{\sin \pi \alpha \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \alpha)} \quad (|P(\alpha, \beta, \gamma)| = M < \infty) \quad (1)$$

and

$$Q = Q(\alpha, \beta, \gamma) := P(\beta, \alpha, \gamma) \quad (|P(\beta, \alpha, \gamma)| = M < \infty) \quad (2)$$

When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$, we have ;

$$(i) \quad ((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = e^{-i\pi\gamma} P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha+\beta-\gamma}, \quad (3)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, (1 + \alpha - \gamma) \notin \mathbb{Z}_0^-),$$

$$(ii) \quad ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = e^{-i\pi\gamma} Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha+\beta-\gamma}, \quad (4)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, (1 + \beta - \gamma) \notin \mathbb{Z}_0^-)$$

$$(iii) \quad ((z-c)^{\alpha+\beta})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha+\beta-\gamma}, \quad (5)$$

where

$$z-c \neq 0, \quad \left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right| < \infty.$$

Proof of (i). We have

$$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k! \Gamma(\gamma+1-k)} ((z-c)^\alpha)_{\gamma-k} ((z-c)^\beta)_k \quad (6)$$

by Lemma (iv).

Next we have

$$((z-c)^\alpha)_{\gamma-k} = e^{-i\pi(\gamma-k)} \frac{\Gamma(\gamma-k-\alpha)}{\Gamma(-\alpha)} (z-c)^{\alpha-\gamma+k} \quad \left(\left| \frac{\Gamma(\gamma-k-\alpha)}{\Gamma(-\alpha)} \right| < \infty \right) \quad (7)$$

and

$$((z-c)^\beta)_k = e^{-i\pi k} \frac{\Gamma(k-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-k} \quad (8)$$

by Lemma (i), respectively.

Substitute (7) and (8) into (6), we have then

$$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = e^{-i\pi\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(\gamma-\alpha-k)\Gamma(k-\beta)}{k!\Gamma(\gamma+1-k)\Gamma(-\alpha)\Gamma(-\beta)} (z-c)^{\alpha+\beta-\gamma} \quad (9)$$

$$= e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha)}{\Gamma(-\alpha)} (z-c)^{\alpha+\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\beta]_k [-\gamma]_k}{k! [1+\alpha-\gamma]_k} \quad (10)$$

using the relationship

$$\Gamma(\lambda+1-k) = (-1)^{-k} \frac{\Gamma(\lambda+1)\Gamma(-\lambda)}{\Gamma(k-\lambda)}, \quad (11)$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \text{ with } [\lambda]_0 = 1.$$

(notation of Pochhammer).

Therefore, applying the following relationships

$$\sum_{k=0}^{\infty} \frac{[a]_k [b]_k}{k! [c]_k} = {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \left(\begin{array}{l} \operatorname{Re}(c-a-b) > 0, \\ c \notin \mathbb{Z}_0^- \end{array} \right) \quad (12)$$

and

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbb{Z}) \quad (13)$$

to (10), we obtain

$$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha)}{\Gamma(-\alpha)} (z-c)^{\alpha+\beta-\gamma} {}_2F_1(-\beta, -\gamma; 1+\alpha-\gamma; 1) \quad (14)$$

$$= e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha)\Gamma(1+\alpha+\beta)\Gamma(1+\alpha-\gamma)}{\Gamma(-\alpha)\Gamma(1+\alpha)\Gamma(1+\alpha+\beta-\gamma)} (z-c)^{\alpha+\beta-\gamma} \quad (15)$$

$$\left(\begin{array}{l} \operatorname{Re}(\alpha+\beta+1) > 0, \\ (1+\alpha-\gamma) \notin \mathbb{Z}_0^- \end{array} \right)$$

$$= e^{-i\pi\gamma} \frac{\sin \pi \alpha \cdot \sin \pi (\gamma - \alpha - \beta)}{\sin \pi (\alpha + \beta) \cdot \sin \pi (\gamma - \alpha)} \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha+\beta-\gamma}. \quad (16)$$

We have then (3) from (16) using the notation (1), under the conditions.

Proof of (ii). In the same way as the proof of (i) we obtain (4) using (2) instead of (1), under the conditions. (Simply, changing α and β in (3) we obtain (4), using (2).)

Proof of (iii). We have

$$((z-c)^{\alpha+\beta})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} (z-c)^{\alpha+\beta-\gamma} \quad (17)$$

$$\left(\left| \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right| < \infty \right)$$

directly by Lemma (i).

Corollary 1.

When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$, we have ;

$$(i) \quad (z^\alpha \cdot z^\beta)_\gamma = e^{-i\pi\gamma} P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha+\beta-\gamma}, \quad (18)$$

$$(\operatorname{Re}(\alpha+\beta+1) > 0, (1+\alpha-\gamma) \notin \mathbb{Z}_0^-),$$

$$(ii) \quad (z^\beta \cdot z^\alpha)_\gamma = e^{-i\pi\gamma} Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha+\beta-\gamma}, \quad (19)$$

$$(\operatorname{Re}(\alpha+\beta+1) > 0, (1+\beta-\gamma) \notin \mathbb{Z}_0^-)$$

$$(iii) \quad (z^{\alpha+\beta})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha+\beta-\gamma}, \quad (20)$$

where

$$\left| \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right| < \infty,$$

and P and Q are the ones shown by (1) and (2) respectively.

Proof. Set $c=0$ in Theorem 1.

Theorem 2. When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$, we have ;

$$(i) \quad ((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = P(\alpha, \beta, \gamma) ((z-c)^{\alpha+\beta})_\gamma, \quad (21)$$

$$(\operatorname{Re}(\alpha+\beta+1) > 0, (1+\alpha-\gamma) \notin \mathbb{Z}_0^-),$$

$$(ii) \quad ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = Q(\alpha, \beta, \gamma) ((z-c)^{\alpha+\beta})_\gamma, \quad (22)$$

$$(\operatorname{Re}(\alpha+\beta+1) > 0, (1+\beta-\gamma) \notin \mathbb{Z}_0^-)$$

$$(iii) \quad \frac{1}{P} ((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = \frac{1}{Q} ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = ((z-c)^{\alpha+\beta})_\gamma, \quad (23)$$

$$(\operatorname{Re}(\alpha+\beta+1) > 0, (1+\alpha-\gamma) \notin \mathbb{Z}_0^-, (1+\beta-\gamma) \notin \mathbb{Z}_0^-, PQ \neq 0)$$

where

$$z - c \neq 0, \quad \left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right| < \infty,$$

and P and Q are the ones shown by (1) and (2) respectively.

Proof. It is clear by Theorem 1.

Corollary 2. When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$, we have ;

$$(i) \quad (z^\alpha \cdot z^\beta)_\gamma = P(\alpha, \beta, \gamma) (z^{\alpha+\beta})_\gamma, \quad (24)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, (1 + \alpha - \gamma) \notin \mathbb{Z}_0^-),$$

$$(ii) \quad (z^\beta \cdot z^\alpha)_\gamma = Q(\alpha, \beta, \gamma) (z^{\alpha+\beta})_\gamma, \quad (25)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, (1 + \beta - \gamma) \notin \mathbb{Z}_0^-)$$

$$(iii) \quad \frac{1}{P}(z^\alpha \cdot z^\beta)_\gamma = \frac{1}{Q}(z^\beta \cdot z^\alpha)_\gamma = (z^{\alpha+\beta})_\gamma, \quad (26)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, (1 + \alpha - \gamma) \notin \mathbb{Z}_0^-, (1 + \beta - \gamma) \notin \mathbb{Z}_0^-, PQ \neq 0)$$

where

$$\left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right| < \infty,$$

and P and Q are the ones shown by (1) and (2) respectively.

Proof. Set $c = 0$ in Theorem 2.

Corollary 3.

(i) When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$ and

$$P(\alpha, \beta, \gamma) = Q(\alpha, \beta, \gamma) = 1, \quad (27)$$

we have

$$((z - c)^\alpha \cdot (z - c)^\beta)_\gamma = ((z - c)^\beta \cdot (z - c)^\alpha)_\gamma = ((z - c)^{\alpha+\beta})_\gamma. \quad (28)$$

$$(\operatorname{Re}(\alpha + \beta + 1) > 0, (1 + \alpha - \gamma) \notin \mathbb{Z}_0^-, (1 + \beta - \gamma) \notin \mathbb{Z}_0^-)$$

(ii) When $\gamma = m \in \mathbb{Z}_0^+$ we have

$$((z - c)^\alpha \cdot (z - c)^\beta)_m = ((z - c)^\beta \cdot (z - c)^\alpha)_m = ((z - c)^{\alpha+\beta})_m. \quad (29)$$

Proof of (i). We have (28) from Theorem 2 (23), clearly.

Proof of (ii). When $\gamma = m \in \mathbb{Z}_0^+$ we have (29) from Theorem 2 (23), since

$$P(\alpha, \beta, m) = Q(\alpha, \beta, m) = 1. \quad (30)$$

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