

Deformations of $\mathbb{T}^*$ -Seifert fibrations

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We describe deformations of $\mathbb{T}^*$ -Seifert fibrations and make a remark on the relationship between deformations of an isolated singularity (X,p) with $\mathbb{T}^*$ action and deformations of the $\mathbb{T}^*$ -Seifert fiber space $X-p$. For the torus Seifert fibering case, we refer to [8]. Details of this note will appear elsewhere.

§1. $\mathbb{T}^*$ -Seifert fibrations.

Following Conner-Raymond [2], we construct $\mathbb{T}^*$ -Seifert fiberings as follows. Let W be a complex manifold and let N be a group acting analytically and properly discontinuously on W from the left. The quotient space $V=W/N$ has a natural structure of complex space and the projection $v:W \rightarrow V$ is holomorphic. We assume that V is compact hereafter. Consider the exponential sequence

$$(1.1) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{1} \mathbb{C} \xrightarrow{\exp} \mathbb{T}^* \rightarrow 1,$$

($\exp(z)=\exp 2\pi iz$). Contrary to the torus case, we let N act trivially on each of the groups in (1.1). By taking the sheaf of germs of holomorphic maps from W into each of the groups

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in (1.1), we get the exact sequence of sheaves over W

$$(1.2) \quad 0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_W^* \rightarrow 1.$$

If $\mathcal{J}$ is one of the sheaves in (1.2), we define, for each $\alpha \in N$ and an open set U in W ,

$$\alpha: \Gamma(U, \mathcal{J}) \rightarrow \Gamma(\alpha U, \mathcal{J})$$

by $(\alpha\sigma)(w) = \sigma(\alpha^{-1}w)$, $\sigma \in \Gamma(U, \mathcal{J})$, $w \in \alpha U$. Then we have a structure of N -sheaf on $\mathcal{J}$ ([2],[3],[8]). From (1.2), we get the cohomology exact sequence

$$(1.3) \quad \dots \xrightarrow{l^*} H^1(N, \mathcal{O}_W) \xrightarrow{\varepsilon^*} H^1(N, \mathcal{O}_W^*) \xrightarrow{c} H^2(N, \mathcal{J}) \rightarrow \dots$$

We call $\text{Ker } c$ the Picard group of the action (N, W) and denote it by $\text{Pic}(N, W)$. Each element m in the group $H^1(N, \mathcal{O}_W^*)$ defines a principal $\mathbb{C}^*$ -bundle $\varpi: B \rightarrow W$ and an action (N, B) covering (N, W) as follows. First, if we take a suitable open covering $\{W_\lambda\}_{\lambda \in \Lambda}$ of W , m is represented by a collection $\{m^{\lambda\mu}(w; \alpha)\}$, where for each $(\lambda, \mu) \in \Lambda^2$ and $\alpha \in N$, $m^{\lambda\mu}(; \alpha)$ is a non-vanishing holomorphic function on $W_\lambda \cap W_\mu$. The collection satisfies the cocycle condition

$$(1.4) \quad m^{\lambda\nu}(w; \alpha\beta) = m^{\lambda\mu}(w; \alpha) m^{\mu\nu}(\alpha^{-1}w; \beta),$$

$(\lambda, \mu, \nu) \in \Lambda^3$, $(\alpha, \beta) \in N^2$. In particular, if we set $\alpha = \beta = e$ (the identity of N), we get

$$m^{\lambda\nu}(w; e) = m^{\lambda\mu}(w; e) m^{\mu\nu}(w; e).$$

Thus the collection $\{m^{\lambda\mu}(w; e)\}$ defines a principal $\mathbb{C}^*$ -bundle $\varpi: B \rightarrow W$ with $\varpi^{-1}(W_\lambda) \simeq W_\lambda \times \mathbb{C}^*$. We let N act on $W_\lambda \times \mathbb{C}^*$ by

$$(1.5) \quad \alpha(w, t^\lambda) = (\alpha w, m^{\lambda\lambda}(\alpha w; \alpha) t^\lambda),$$

$\alpha \in N$, $(w, t^\lambda) \in W_\lambda \times \mathbb{T}^*$. Then we get an action of N on $\tilde{\omega}^{-1}(W_\lambda)$. The cocycle condition (1.4) shows that the actions $(N, \tilde{\omega}^{-1}(W_\lambda))$ and $(N, \tilde{\omega}^{-1}(W_\mu))$ coincide on $\tilde{\omega}^{-1}(W_\lambda \cap W_\mu)$ and we get a global action (N, B) covering (N, W) . Clearly the action is properly discontinuous. It is fixed point free if and only if the isotropy subgroup N_w has no fixed points on the fiber $\tilde{\omega}^{-1}(w) (\simeq \mathbb{T}^*)$, i.e. if $c(m)$ is a Bieberbach class ([2]). Thus if $c(m)$ is a Bieberbach class, the quotient $M = N \backslash B$ is a complex manifold. Since the action (N, B) is compatible with the canonical right action of $\mathbb{T}^*$ on B , M admits a $\mathbb{T}^*$ -action and we have the diagram

$$(1.6) \quad \begin{array}{ccc} (N, B, \mathbb{T}^*) & \xrightarrow{\mu} & (M, \mathbb{T}^*) = (N \backslash B, \mathbb{T}^*) \\ \tilde{\omega} \downarrow & & \downarrow \pi \\ (N, W) & \xrightarrow{\nu} & V = N \backslash W = M / \mathbb{T}^*. \end{array}$$

We call $M \xrightarrow{\pi} V$ the $\mathbb{T}^*$ -Seifert fibration determined by m . The fiber $\pi^{-1}(\nu(w))$ over a point $\nu(w) \in V$ is given by $\pi^{-1}(\nu(w)) = N_w \backslash \tilde{\omega}^{-1}(w) \simeq N_w \backslash \mathbb{T}^*$. When $N_w \neq \{e\}$, we call the fiber a (multiple) singular fiber of $M \rightarrow V$. It is not difficult to show

Lemma 1. 1. Given a principal $\mathbb{T}^*$ -bundle $\tilde{\omega}: B \rightarrow W$ and properly discontinuous actions of N on B and W so that $\tilde{\omega}$ is equivariant. Assume that the action (N, B) is compatible with the canonical $\mathbb{T}^*$ action. Then there is an element m in $H^1(N, \mathcal{Q}_W^*)$ such that (N, B) is equivalent to the one constructed from m as above.

§2. Deformations of $\mathbb{T}^*$ -Seifert fibrations.

Definition 2. 1. Let

$$\begin{array}{ccc} B & \longrightarrow & M \\ \downarrow & & \downarrow \\ W & \longrightarrow & V \end{array}$$

be a $\mathbb{C}^*$ -Seifert fibration as constructed in §1. A deformation of it consists of

- (I) A deformation $\mathcal{B} \xrightarrow{\Pi} \mathcal{W} \xrightarrow{\omega} S$ of the principal $\mathbb{C}^*$ -bundle $\tilde{\omega}: B \rightarrow W$ ([4] Definition 1.8) (we let $o \in S$ be the specific point so that $\omega^{-1}(o) = W_0 \simeq W$, $\Pi^{-1}(W_0) = B_0 \simeq B$ and $\Pi|_{B_0} \simeq \tilde{\omega}$),
- (II) Properly discontinuous actions $(N, \mathcal{B})$ and $(N, \mathcal{W})$ such that
- (a) Π and ω are equivariant (we let N act trivially on S),
 - (b) $(N, B_0) \simeq (N, B)$ and $(N, W_0) \simeq (N, W)$,
 - (c) $(N, \mathcal{B})$ is compatible with the canonical $\mathbb{C}^*$ action on $\mathcal{B}$.

If we set $W_s = \omega^{-1}(s)$ and $B_s = \Pi^{-1}(W_s)$ for each $s \in S$, Lemma 1.1 shows that (N, B_s) is equivalent to the one constructed from a cohomology class $m(s) \in H^1(N, \mathcal{O}_{W_s}^*)$. Since $m(s)$ depends "holomorphically" on s , if $c(m)$ is a Bieberbach class so is $c(m(s))$ for every sufficiently small s . Therefore if S is "small", the quotient $\mathcal{M} = N \backslash \mathcal{B}$ is a complex manifold. We have the diagram

$$\begin{array}{ccc} N \backslash \mathcal{B} = \mathcal{M} & & \\ \downarrow & \searrow & \\ S & \swarrow & N \backslash \mathcal{W} = \mathcal{U} \end{array}$$

where $\mathcal{M} \rightarrow \mathcal{U}$ is a deformation of $M \rightarrow V$ over S . Given a deformation as in Definition 2.1, then we have, from (I), the Kodaira-Spencer fundamental sheaf diagram ([4] (4.2)_P)

$$(2.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathcal{O}_W & \longrightarrow & \Pi_W & \longrightarrow & T_W \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Sigma_W & \longrightarrow & \Gamma_W & \longrightarrow & T_W \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Xi_W & \longrightarrow & \Xi_W & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

If we denote by Π_X the holomorphic tangent bundle of a complex manifold, we have $\mathcal{O}_W = \mathcal{O}_W(\Pi_W)$, $\Sigma_W = \mathcal{O}_W(\Pi_B/\mathbb{C}^*)$ and $T_W = W \times \Pi_{S,o}$

($\Pi_{S,o}$ = the holomorphic tangent space of S at o). Moreover, if we denote by $\Pi_{B/W}$, the bundle of tangent vectors of B which are tangential to the fibers of ϖ , we have $\Xi_W = \mathcal{O}_W(\Pi_{B/W}/\mathbb{C}^*)$. Note that each sheaf in (2.1) has a natural structure of N -sheaf.

From the second row we get the connecting homomorphism $\delta: H^0(N, T_W) \rightarrow H^1(N, \Sigma_W)$. Since N acts trivially on S , we have $H^0(N, T_W) = \Pi_{S,o}^N = \Pi_{S,o}$.

Thus we get the infinitesimal deformation map

$$\eta: \Pi_{S,o} \longrightarrow H^1(N, \Sigma_W).$$

It is not difficult show that $H^1(N, \Sigma_W)$ is the set of isomorphism classes of first order infinitesimal deformations of the $\mathbb{C}^*$ -Seifert fibration $M \rightarrow V$. Noting that $\Xi_W \simeq \mathcal{O}_W$, we have, from the first column of (2.1), the cohomology exact sequence

$$\begin{aligned} \dots &\longrightarrow H^0(N, \Sigma_W) \xrightarrow{\psi^0} H^0(N, \mathcal{O}_W) \xrightarrow{\delta^0} H^1(N, \mathcal{O}_W) \\ &\xrightarrow{\phi^1} H^1(N, \Sigma_W) \xrightarrow{\psi^1} H^1(N, \mathcal{O}_W) \xrightarrow{\delta^1} H^2(N, \mathcal{O}_W) \longrightarrow \dots \end{aligned}$$

If we set $C = \text{Ker } \psi^1 = \text{Im } \phi^1$, $F = \text{Ker } \delta^1 = \text{Im } \psi^1$, we get a decomposition

of $H^1(N, \Sigma_W)$ into vector groups

$$(2.2) \quad 0 \longrightarrow C \longrightarrow H^1(N, \Sigma_W) \longrightarrow F \longrightarrow 0.$$

As in [8] §4, we can show that C represents the Picard deformations of $M \rightarrow V$ and that none of elements in C is obstructed. The group $H^1(N, \theta_W)$ is the set of isomorphism classes of first order infinitesimal deformations of the action (N, W) and the map δ^1 gives the first obstruction to constructing a deformation of $M \rightarrow V$ from the given deformation of (N, W) . Thus we may say that F represents the "base deformations" of $M \rightarrow V$.

§3. $\dim_{\mathbb{C}} W = 1$.

When W is one dimensional, the groups C and F are computed as follows. We may assume that W is simply connected without loss of generality. Let g denote the genus of the compact Riemann surface $V = N \setminus W$. The image of the set $\{w \in W \mid N_w \neq \{e\}\}$ by the map v consists of a finite number of points $p_1, \dots, p_r$ on V . Let d denote the divisor $\sum_{i=1}^r p_i$ on V and let $\theta_{V|d}$ denote the sheaf of germs of holomorphic vector fields on V which vanish on d . By [8] Lemma 2.1 and Proposition 3.4, we have

$$H^p(N, \theta_W) = H^p(V, \theta_{V|d}) \text{ for } p \geq 0.$$

Theorem 3. 1.

$$\dim C = \begin{cases} 0 \dots g = 1, r = 0 \text{ and } \psi^0 = 0 \\ g \dots \text{otherwise.} \end{cases}$$

If $g=1$ and $r=0$, then $W=\mathbb{C}$, $N=\mathbb{Z}^2$ and V is a complex torus. We give a condition for ψ^0 to be zero in this case. First, if $W=\mathbb{C}$, then $B=W \times \mathbb{C}^*$ and $H^1(N, \mathcal{O}_W^*) \simeq H^1(N, H^0(W, \mathcal{O}_W^*))$. Therefore, the element m defining (N, B) can be also represented by a crossed homomorphism $m: N \rightarrow H^0(W, \mathcal{O}_W^*)$. Set $m(w; \alpha) = m(\alpha)(w)$ and $M(w, \alpha) = \frac{1}{2\pi i} \log m(w; \alpha)$. Then for each $\alpha \in N$, $\frac{dM}{dw}(w; \alpha)$ is a single valued holomorphic function on W . The map $\frac{dM}{dw}$ which assigns $\frac{dM}{dw}(w; \alpha)$ to each α is a crossed homomorphism from N into $H^0(W, \mathcal{O}_W)$.

Proposition 3. 2. When $g=1$ and $r=0$, ψ^0 is non-zero if and only if $\frac{dM}{dw}$ is a principal crossed homomorphism.

Using

Lemma 3. 3. $H^p(N, \mathcal{O}_W) = 0$ when $p \geq 2$,

we get

Theorem 3. 4.

$$\dim F = \dim H^1(V, \theta_{V|d}) = \begin{cases} 0 & \dots & g = 0 \text{ and } r \leq 3, \\ 1 & \dots & g = 1 \text{ and } r = 0, \\ 3g-3+r & \dots & \text{otherwise.} \end{cases}$$

§5. Deformations of isolated singularities with $\mathbb{C}^*$ action.

Let X be an affine algebraic variety over $\mathbb{C}$ with an isolated singular point p and a $\mathbb{C}^*$ action. In this section we consider the relationship between deformations of (X, p) and deformations of the $\mathbb{C}^*$ -Seifert fibration $X - p \rightarrow X - p/\mathbb{C}^*$. For simplicity, we assume that X is a homogeneous cone with vertex p . Thus for

a suitable embedding $(X, p) \subset (\mathbb{P}^{n+1}, 0)$, the action $(\mathbb{P}^*, X)$ is given by $t(z_0, z_1, \dots, z_n) = (tz_0, tz_1, \dots, tz_n)$. Also we assume that p is a normal point of X and that $\dim X \geq 2$. Let I be the ideal of X and let $N_X = \mathcal{H}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$ be the normal sheaf of X in $\mathbb{P}^{n+1}$. The isomorphism classes of first order deformations of (X, p) is given by the Schlessinger's T_X^1 , which is defined by the exact sequence ([1], [6], [7])

$$(4.1) \quad H^0(X, \theta_{\mathbb{P}^{n+1}|_X}) \rightarrow H^0(X, N_X) \rightarrow T_X^1 \rightarrow 0.$$

Setting $B = X - p$, we get the diagram

$$\begin{array}{ccc} B \subset \mathbb{P}^{n+1} & \xrightarrow{\quad} & 0 \\ \tilde{\omega} \downarrow & & \downarrow p \\ W \subset \mathbb{P}^n & & \end{array},$$

where W is defined by I in $\mathbb{P}^n$, p is the projection of the universal $\mathbb{P}^*$ -bundle and $\tilde{\omega}$ is the restriction of p to B . Note that

$$H^0(X, \theta_X) \simeq H^0(B, \theta_B), \quad H^0(X, \theta_{\mathbb{P}^{n+1}|_X}) \simeq H^0(B, \theta_{\mathbb{P}^{n+1}|_B})$$

and

$$H^0(X, N_X) \simeq H^0(B, N_B)$$

([6], [7]). If we denote by N_W the normal sheaf of W in $\mathbb{P}^n$, we get $N_B = \tilde{\omega}^* N_W$. Hence we have $H^0(B, N_B) = H^0(B, \tilde{\omega}^* N_W) = \sum_{v=-\infty}^{\infty} H^0(W, N_W(v))$.

Also from $\theta_{\mathbb{P}^{n+1}|_B} \simeq \tilde{\omega}^* \mathcal{O}_W(1)^{n+1}$, we get $H^0(B, \theta_{\mathbb{P}^{n+1}|_B}) = \sum_{v=-\infty}^{\infty} H^0(W, \mathcal{O}_W(v+1))^{n+1}$. Thus we get a grading $T_X^1 = \sum_{v=-\infty}^{\infty} T_X^1(v)$ ([5], [6], [7]). $T_X^1(0)$ is defined by the exact sequence

$$(4.2) \quad H^0(W, \mathcal{O}_W(1))^{n+1} \rightarrow H^0(W, N_W) \rightarrow T_W^1(0) \rightarrow 0.$$

On the other hand, dividing the sheaf exact sequence

$$0 \rightarrow \mathcal{O}_B \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}|_B} \rightarrow N_B \rightarrow 0$$

over B by $\mathbb{P}^*$, we get the sheaf exact sequence

$$0 \rightarrow \Sigma_W \rightarrow \mathcal{O}_W(1)^{n+1} \rightarrow N_W \rightarrow 0$$

over W . From this we get the exact sequence

$$(4.3) \quad \begin{aligned} \dots \rightarrow H^0(W, \mathcal{O}_W(1))^{n+1} &\rightarrow H^0(W, N_W) \rightarrow H^1(W, \Sigma_W) \\ &\rightarrow H^1(W, \mathcal{O}_W(1))^{n+1} \rightarrow \dots \end{aligned}$$

Comparing (4.2) and (4.3), we get

$$T_X^1(0) = H^1(W, \Sigma_W),$$

if $H^1(W, \mathcal{O}_W(1))=0$. In this case, the set of first order infinitesimal deformations of $B \rightarrow W$ coincides with the set of first order infinitesimal deformations of (X, p) in which the $\mathbb{P}^*$ action on X is stable (extendible). The condition is satisfied, for example, if W is a complete intersection of dimension greater than one or if W is a plane curve of degree less than four.

The non-homogeneous case can be dealt with by taking a suitable covering of X .

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